Non-Kähler Hamiltonian torus actions

Brad Hannigan-Daley
Department of Mathematics, University of Toronto
brad.hannigan.daley@utoronto.ca

May 22, 2010

Abstract

We discuss conditions under which a symplectic manifold equipped with a Hamiltonian torus action admits an invariant compatible complex structure, and we describe Tolman's construction [6] of a compact six-dimensional symplectic manifold with a Hamiltonian 2-torus action which admits no invariant Kähler structure.

1 Introduction

Compact Kähler manifolds have many nice properties. For example, their cohomology groups have a Hodge decomposition, and they satisfy the hard Lefschetz property. It is therefore of interest to know what conditions may force a symplectic manifold to admit a Kähler structure. In particular, we may wish to know under what conditions a symplectic manifold equipped with a Hamiltonian torus action admits a compatible complex structure which is invariant under the action.

It is not the case that every symplectic manifold admits a Kähler structure (or, indeed, any complex structure [3]). In particular, Lerman [5] constructed a compact twelve-dimensional simply connected symplectic manifold with a Hamiltonian circle action on it, with an isolated fixed point, which does not admit any Kähler structure. This can be seen because one of its odd-degree Betti numbers is odd, which is impossible for a Kähler manifold due to the Hodge decomposition. In this case, the dimension of the torus is small relative to the dimension of the manifold acted upon.

However, the existence of sufficiently large Hamiltonian symmetries may force the existence of an invariant compatible complex structure. For example, Delzant [2] showed that every symplectic toric manifold (that is, a compact $2n$-dimensional symplectic manifold equipped with an effective Hamiltonian action of an n-dimensional torus) admits a compatible complex structure which is invariant under the action of the torus. Similarly, Karshon [4] showed that any compact four-dimensional symplectic manifold equipped with an effective Hamiltonian circle action admits an invariant compatible complex structure. Taking these two results together, the lowest possible n for which there exists a non-Kähler effective Hamiltonian torus action on an n-dimensional symplectic manifold is six.

Tolman [6] constructed a compact six-dimensional symplectic manifold M with an effective Hamiltonian 2-torus action such that there exists no invariant compatible complex structure on M. This example is sharp in the sense that, as indicated above, there is no lower-dimensional M for which such a construction exists, and the construction is not possible for the action of a higher-dimensional torus (i.e. a 3-torus) on a six-dimensional manifold by Delzant's result. It is also notable for being the first such example all of whose fixed points are isolated, and for being the first such that all of its reduced spaces are Kähler. Moreover, M is simply connected, and the action is quasi-free (the stabilizer of each point is connected).

Given a compact symplectic manifold with a Hamiltonian torus action, Tolman uses a theorem of Atiyah to formulate an "extension criterion" that the x-ray of the action must satisfy if there exists an invariant compatible complex structure. She then constructs M by taking two Kähler 6-manifolds with Hamiltonian 2-torus actions and gluing pieces of them together. The x-ray of M is then easily seen to fail the extension criterion, and so there exists no invariant compatible complex structure on M. Tolman then shows that this manifold does not admit any invariant Kähler structure (even if the symplectic form is changed). However, M does satisfy the hard Lefschetz property and its odd-degree Betti numbers are even; it is thought that M does admit a Kähler structure.

2 The x-ray and the extension criterion

Let T be a torus acting on a compact symplectic manifold M, with moment map $\phi : M \to \mathfrak{t}^*$. For a subgroup $K \subset T$, we denote by M^K the set of points fixed by K, and by $\mathfrak{X}_K$ the set of connected components of M^K. The **closed orbit type stratification** of M is the set

$$\mathfrak{X} = \bigcup_{K \subset T} \mathfrak{X}_K$$

which is partially ordered by inclusion. Each $X \in \mathfrak{X}$ is itself a symplectic T-invariant manifold with moment map $\phi|_X$, so by the convexity theorem $\phi(X)$ is a convex polytope in $\mathfrak{t}^*$.

The **x-ray** of (M, ω, ϕ) is the closed orbit type stratification $\mathfrak{X}$ together with the convex polytope $\phi(X)$ for each $X \in \mathfrak{X}$. This set of convex polytopes is partially ordered since it is indexed by $\mathfrak{X}$.

Tolman gives two examples $\widetilde{M}$ and $\widehat{M}$ of six-dimensional Kähler manifolds equipped with Hamiltonian actions of T^2, whose respective x-rays look like the following in $\mathfrak{t}^* \cong \mathbb{R}^2$:

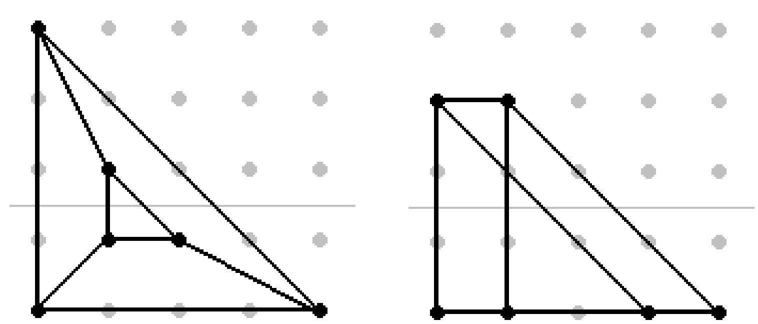

Figure 1: The x-rays of $\widetilde{M}$ (left) and $\widehat{M}$

Each of the black dots is the image under the moment map of a connected component of the set of fixed points, and each of the black edges is the image of a connected component of the set of points with a given one-dimensional stabilizer.

If T is a torus acting smoothly on a compact Kähler manifold M in a way that preserves the complex structure of M, then this action extends to a holomorphic action of the complexification $T_{\mathbb{C}}$ on M. (The complexification of $(S^1)^n$ is the **complex torus** $(\mathbb{C}^\times)^n$.) Furthermore, if the action is Hamiltonian, each orbit of $T_{\mathbb{C}}$ is contained in one of the orbit type strata.

Tolman's notion of **compatibility** was motivated by the following result of Atiyah [1]:

Theorem 2.1. *Let T act by holomorphic symplectomorphisms on a compact Kähler manifold (M, ω, J) with moment map $\phi : M \to \mathfrak{t}^*$. Let $y \in M$, and let $T_{\mathbb{C}} \cdot y$ denote the orbit of y under the action of the complex torus $T_{\mathbb{C}}$. Let $Z_1, \ldots, Z_p$ be the components of $M^T \cap \overline{T_{\mathbb{C}} \cdot y}$ and let $c_j = \phi(Z_j)$. Then $\phi(\overline{T_{\mathbb{C}} \cdot y})$ is the convex polytope with vertices $c_1, \ldots, c_p$, and for each of its open faces σ, the inverse image $\phi^{-1}(\sigma)$ consists of a single $T_{\mathbb{C}}$-orbit.*

Hence, given an open face σ of $\phi(\overline{T_{\mathbb{C}} \cdot y})$, the preimage is a $T_{\mathbb{C}}$-orbit and by a fact cited above, this is contained in an orbit type stratum X_σ, so that $\sigma \subset \phi(X_\sigma)$. Moreover, $\dim \phi(X_\sigma) = \dim \sigma$.

Definition 2.2. Given a convex polytope $\Delta \subset \mathfrak{t}^*$, we say that it is **compatible** with the x-ray $(\mathfrak{X}, \phi)$ if

1. for each face σ of Δ, there exists $X_\sigma \in \mathfrak{X}$ such that $\dim(\phi(X_\sigma)) = \dim(\sigma)$ and $\sigma \subset \phi(X_\sigma)$.
2. if σ and σ' are faces of Δ such that $\sigma \subset \sigma'$, then $X_\sigma \subset X_{\sigma'}$.

Note that, in particular, this implies that each vertex of Δ must be the image of a connected component of the set of fixed points. With this definition, Atiyah's result says that $\phi(\overline{T_{\mathbb{C}} \cdot y})$ is compatible with the x-ray of (M, ω, ϕ).

A **convex cone** in a real vector space is a subset which can be written as the intersection of finitely many half-planes; equivalently, it is the set of positive linear combinations of a finite set. More generally, we allow any translation of such a cone (so it need not contain 0). A convex cone is *strictly convex* if it doesn't contain a line.

Definition 2.3. A convex cone $C \subset \mathfrak{t}^*$ is **compatible** with $(\mathfrak{X}, \phi)$ if there is a neighbourhood U of the vertex of C such that

1. for each face σ of C, there exists $X_\sigma \in \mathfrak{X}$ such that $\dim(\phi(X_\sigma)) = \dim(\sigma)$ and $\sigma \cap U \subset \phi(X_\sigma)$, and
2. if σ and σ' are faces of C such that $\sigma \subset \sigma'$, then $X_\sigma \subset X'_\sigma$.

We say that a convex polytope Δ is an **extension** of a convex cone C if there is a neighbourhood U of the vertex of C such that $C \cap U = \Delta \cap U$. (But Δ need not contain C as a subset.)

Now we can state Tolman's extension criterion.

Definition 2.4. An x-ray satisfies the **extension criterion** if every compatible strictly convex cone has an extension to a compatible convex polytope.

As noted above, Atiyah's theorem shows that for any $y \in M$, $\phi(\overline{T_{\mathbb{C}} \cdot y})$ is a convex polytope compatible with the x-ray. Tolman proves the following lemma:

Lemma 2.5. *Let a torus T act by holomorphic symplectomorphisms on a compact Kähler manifold (M, ω, J) with a moment map ϕ. Let C be a strictly convex cone which is compatible with the x-ray of (M, ω, ϕ). Then there exists $y \in M$ such that $\phi(\overline{T_{\mathbb{C}} \cdot y})$ is an extension of C.*

Putting this together with Atiyah's result gives the following:

Theorem 2.6. *Let a torus T act on a compact symplectic manifold (M, ω) with a moment map $\phi : M \to \mathfrak{t}^*$. If the x-ray of (M, ω, ϕ) does not satisfy the extension criterion, then there exists no compatible T-invariant complex structure on (M, ω).*

3 Sketch of the construction

Tolman proceeds to construct a Hamiltonian 2-torus action on a six-dimensional compact symplectic manifold which does not satisfy the extension criterion, and therefore admits no compatible invariant complex structure. The idea is as follows. Tolman begins with the manifolds $\widetilde{M}$ and $\widehat{M}$ as above. The x-rays of the two manifolds look the same along the grey line in figure 1; if we take the bottom half of the x-ray of $\widetilde{M}$ and the top half of that of $\widehat{M}$, we obtain the following x-ray:

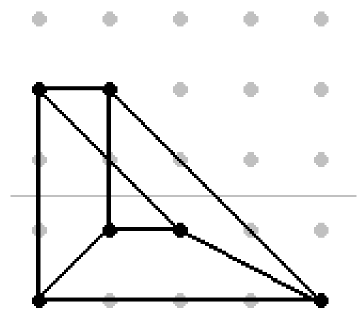

Figure 2: The x-ray of M

It is shown that this gluing can be done on the level of manifolds, in such a way that the resulting manifold M inherits a symplectic structure and Hamiltonian action of T^2, and the x-ray is indeed the one depicted above. This x-ray does not satisfy the extension criterion, for the convex cone

$$\{(s,t) \in \mathbb{R}^2 : t \geq 1, s + t \leq 3\}$$

is compatible with the x-ray, but is easily seen to admit no extension to a compatible convex polytope:

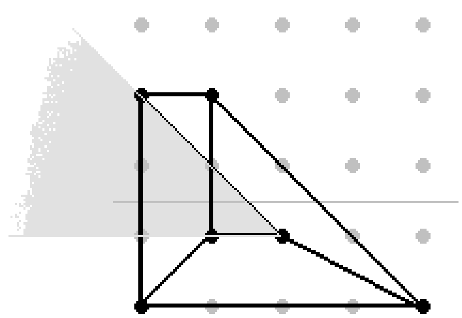

Figure 3: A compatible convex cone which does not extend to a compatible convex polytope

Hence M admits no T^2-invariant compatible complex structure. Tolman continues by enumerating the possible shapes of x-ray that could conceivably result by varying the symplectic form on M, and noting that each is either not convex or does not satisfy the extension property, and so there exists no T^2-invariant Kähler structure on M.

References

[1] M.F. Atiyah. Convexity and commuting Hamiltonians. *Bull. London Math. Soc.*, 14:1–15, 1982.

[2] T. Delzant. Hamiltoniens périodiques et images convexes de l'application moment. *Bull. Soc. math. France*, 116:315-339, 1988.

[3] R. Gompf. A new construction of symplectic manifolds. *Ann. of Math.*, 142:526-595, 1995.

[4] Y. Karshon. Periodic Hamiltonian flows on 4-manifolds. *Mem. Amer. Math. Soc.*, 141(672):viii+71 pp, 1999.

[5] E.Lerman. A compact symmetric symplectic non-Kaehler manifold. *Math. Res. Lett.*, 3(5):587–590, 1996.

[6] S. Tolman. Examples of non-Kähler Hamiltonian torus actions. *Invent. math.*, 131:299–310, 1998.