Hypertoric varieties and wall-crossing

by

Brad Hannigan-Daley

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Abstract

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A hypertoric variety is a quaternionic analogue of a toric variety, constructed as an algebraic symplectic quotient of $T^*\mathbb{C}^n$ by the action of an algebraic torus K, dependent on a choice of character of K. The real Lie coalgebra of K contains a hyperplane arrangement called the discriminantal arrangement, with the property that the hypertoric variety corresponding to a given character η depends only on which face of the discriminantal arrangement contains η. We prove two descriptions of the η-semistability condition in terms of a hyperplane arrangement associated to K, and using these we give a new proof of a theorem of Hiroshi Konno that, given two regular characters separated by a single wall of the discriminantal arrangement, the corresponding hypertoric varieties are related by a Mukai flop. By modifying an argument due to Yoshinori Namikawa, we use the latter result to construct an equivalence between the bounded derived categories of coherent sheaves of these two hypertoric varieties. We end with a conjecture that these equivalences give rise to a representation of the Deligne groupoid of the complexified discriminantal arrangement.

Contents

Chapter 1

Introduction

1.1 Executive summary

Let V be a finite-dimensional complex vector space. The cotangent bundle to the projective space $\mathbb{P}(V)$ can then be described as

$$T^*\mathbb{P}(V) = \left\{(L, X) \in \mathbb{P}(V) \times \operatorname{End} V : X^2 = 0, \operatorname{im} X \subset L\right\}.$$

Similarly, identifying the dual projective space $\mathbb{P}(V^*)$ with the space of hyperplanes in V, we have

$$T^*\mathbb{P}(V^*) = \left\{(H, X) \in \mathbb{P}(V^*) \times \operatorname{End} V : X^2 = 0, H \subset \ker X\right\}.$$

Of course V and V^* are isomorphic and so the two varieties above are isomorphic, but not canonically so. In particular, we do not have a canonical equivalence between their respective categories of coherent sheaves. However, we do have a canonical equivalence once we pass to the bounded derived categories of coherent sheaves $D^b(T^*\mathbb{P}(V))$ and

$D^b(T^*\mathbb{P}(V^*))$, as follows: the varieties $T^*\mathbb{P}(V)$ and $T^*\mathbb{P}(V^*)$ have the common affinization

$$A(V) = \left\{X \in \operatorname{End} V : X^2 = 0, \operatorname{rank} X \leq 1\right\}$$

with affinization maps given by forgetting L, respectively H. We can then define the fibre product

$$Z = T^*\mathbb{P}(V) \times_{A(V)} T^*\mathbb{P}(V^*),$$

a closed subvariety of $T^*\mathbb{P}(V) \times T^*\mathbb{P}(V^*)$. As discovered by Yujiro Kawamata [11] and Yoshinori Namikawa [17], the Fourier-Mukai transform $\Phi_Z : D^b(T^*\mathbb{P}(V)) \to D^b(T^*\mathbb{P}(V^*))$ with kernel $\mathcal{O}_Z$ is an exact equivalence of triangulated categories.

The variety $T^*\mathbb{P}(V)$ is perhaps the simplest example of a smooth hypertoric variety, which can be thought of as a quaternionic analogue of a toric variety. The data necessary to define a hypertoric variety are a subtorus K of the standard complex n-torus $(\mathbb{C}^\times)^n$ and a character $\eta : K \to \mathbb{C}^\times$, the latter of which we consider as an integral element of the Lie coalgebra $\mathfrak{k}^*$ of K. We denote by M_η the associated hypertoric variety. In the real form $\mathfrak{k}^*_{\mathbb{R}}$ of the Lie coalgebra there is a central hyperplane arrangement, called the discriminantal arrangement, such that M_η depends only, up to canonical isomorphism, on which face of the discriminantal arrangement contains η. In the case of $T^*\mathbb{P}(V)$, the discriminantal arrangement consists of the single point 0 on the line $\mathbb{R}$; we have $M_\eta = T^*\mathbb{P}(V)$ for $\eta > 0$, $M_{\eta'} = T^*\mathbb{P}(V^*)$ for $\eta' < 0$, and $M_0 = A(V)$. The equivalence Φ_Z constructed above can hence be thought of as a kind of wall-crossing phenomenon.

In this thesis, we generalize this construction to define equivalences between the derived categories of other smooth hypertoric varieties. The diagram of affinizations

$$T^*\mathbb{P}(V) \to A(V) \leftarrow T^*\mathbb{P}(V^*)$$

is a basic example of a kind of birational map known as a Mukai flop. Given characters

η and η' which lie in the complement of the discriminantal arrangement and which are separated by a single wall, we construct a fibre product Z of the hypertoric varieties M_η and $M_{\eta'}$ and use it to show that they are related by a Mukai flop. We then use this construction to conclude that the Fourier-Mukai transform $\Phi_\eta^{\eta'}$ with kernel $\mathcal{O}_Z$ is an equivalence of categories between $D^b(M_\eta)$ and $D^b(M_{\eta'})$.

In Chapter 2 we begin by recalling some necessary notions in the theory of hypertoric varieties, defining the latter as geometric invariant theory (GIT) quotients. Next, we prove a description of the locus of semistable points for such a GIT quotient in terms of an associated hyperplane arrangement. We conclude the chapter by defining the discriminantal arrangement and proving a combinatorial description of the semistable locus, in terms of the circuits of the matroid of the associated hyperplane arrangement.

Chapter 3 begins with a review of the definitions of Fourier-Mukai transforms and Mukai flops. We modify an argument due to Namikawa to show that every Mukai flop of holomorphic symplectic varieties gives rise to a Fourier-Mukai transform which is an equivalence of derived categories.

In Chapter 4, we fix regular characters η and η' separated by a single wall as above, and we use the description of the semistable loci from Chapter 2 to show that there are natural morphisms

$$M_\eta \to M_\theta \leftarrow M_{\eta'}$$

where θ is a character on the separating wall, and we show that this diagram is a Mukai flop. This result appears in the literature already, and is originally due to Hiroshi Konno [13, Theorem 6.3] [14, Theorem 6.6], who approaches the problem from a differential-geometric perspective and uses a different characterization of the semistable loci. The major difference between his approach and ours is that we proceed by giving an explicit description of the structure of the fibre product $M_\eta \times_{M_\theta} M_{\eta'}$. In contrast, Konno's original proof proceeds by defining certain open subvarieties W^+ and W^- of M_η and $M_{\eta'}$, respectively, such that the restriction of the maps in the diagram to these subvarieties is more easily seen to be a Mukai flop – however, we were not able to make sense of his definition of W^+ and W^- as subvarieties of M_η and $M_{\eta'}$, as detailed in Chapter 4.

In addition to the cotangent bundle $T^*\mathbb{P}(V)$, another well-understood example of a hypertoric variety is the minimal resolution $X = \widetilde{\mathbb{C}^2/\mathbb{Z}_{m+1}}$ of the type-A_m Kleinian singularity. Paul Seidel and Richard Thomas [23] construct an action by Fourier-Mukai transforms of the braid group $\mathcal{B}_{m+1}$ on the derived category $D^b(X)$, and in particular this gives an action of the pure braid group $\mathcal{PB}_{m+1} \subset \mathcal{B}_{m+1}$ on $D^b(X)$. This pure braid group is isomorphic to the fundamental group of the complement of the complexified discriminantal arrangement in this case. We therefore expect that, in general, the Fourier-Mukai transforms we construct by wall-crossing should generalize the construction of Seidel-Thomas, giving rise to an action on each category $D^b(M_\eta)$ of the fundamental group of the complement $\Upsilon_\mathbb{C}$ of the complexified discriminantal arrangement. More generally, we conjecture that the Fourier-Mukai transforms $\Phi_\eta^{\eta'}$ form a representation of the Deligne groupoid of the discriminantal arrangement, which is a certain subcategory of the fundamental groupoid of $\Upsilon_\mathbb{C}$. Chapter 5 consists of a precise formulation and discussion of this conjecture, as well as a discussion of $\mathbb{P}^n$-functors and how they are expected to relate to our Fourier-Mukai transforms.

1.2 Acknowledgments

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1.3 Dedication

This thesis is dedicated to the loving memory of Dr. Dan Daley.

Chapter 2

Semistability criteria for hypertoric varieties

2.1 Review of hypertoric varieties

Let $T = (\mathbb{C}^\times)^n$, the standard complex torus of dimension n, with Lie algebra $\mathfrak{t}$ and coweight lattice $\mathfrak{t}_\mathbb{Z}$. Fix a connected algebraic subtorus $K \subset T$, thus giving a faithful representation of K on $\mathbb{C}^n$. We then have an induced action of K on the cotangent bundle $T^*\mathbb{C}^n = \mathbb{C}^n \times (\mathbb{C}^n)^*$ defined by $t \cdot (z, w) = (tz, t^{-1}w)$. Let $\mathfrak{k} \subset \mathfrak{t}$ be the Lie algebra of K.

Assumption 2.1. We shall assume that none of the standard basis elements e_i of $\mathfrak{t} \cong \mathbb{C}^n$ lie in $\mathfrak{k}$.

The action of K on $T^*\mathbb{C}^n$ is hamiltonian with respect to the natural symplectic structure on $T^*\mathbb{C}^n$, with moment map $\mu : T^*\mathbb{C}^n \to \mathfrak{k}^*$ defined by

$$\mu(z, w)(x_1, \ldots, x_n) = \sum_{i=1}^{n} z_i w_i x_i.$$

For each $\lambda \in \mathfrak{k}^*$, the level set $\mu^{-1}(\lambda)$ is a K-invariant affine subvariety of $T^*\mathbb{C}^n$. A hypertoric variety is by definition a symplectic quotient of $T^*\mathbb{C}^n$ by K, or equivalently a geometric invariant theory (GIT) quotient of a level set $\mu^{-1}(\lambda)$ by K.

Definition 2.2. Let $\eta : K \to \mathbb{C}^\times$ be a multiplicative character of K and let $\lambda \in \mathfrak{k}^*$. The **hypertoric variety** $M_{\eta,\lambda}$ is the projective GIT quotient

$$M_{\eta,\lambda} := \mu^{-1}(\lambda) /\!\!/_\eta K.$$

Equivalently,

$$M_{\eta,\lambda} := \operatorname{Proj} \bigoplus_{m=0}^{\infty} \left\{ f \in \mathcal{O}(\mu^{-1}(\lambda)) : f(t^{-1}x) = \eta(t)^m f(x) \textit{ for all } t \in K \right\}.$$

As $M_{\eta,\lambda}$ is a symplectic quotient of $T^*\mathbb{C}^n$ by K, its dimension is $2(n-k)$, where k is the rank of K.

We can describe this construction more geometrically using the locus of *semistable* points, as follows. The choice of character η defines a lift of the action of K on $\mu^{-1}(\lambda)$ to the trivial line bundle $\mu^{-1}(\lambda) \times \mathbb{C}$ by the equation

$$t \cdot (p, x) = (t \cdot p, \eta(t)^{-1} x).$$

Definition 2.3. A point $p \in \mu^{-1}(\lambda)$ is η**-semistable** if the closure of the K-orbit through $(p, 1)$ in $\mu^{-1}(\lambda) \times \mathbb{C}$ does not intersect the zero section $\mu^{-1}(\lambda) \times \{0\}$. A point which is not η-semistable is said to be η**-unstable**. We denote the locus of η-semistable points by $\mu^{-1}(\lambda)^\eta$.

In other words, p is η-semistable if, whenever $\{t_n\}_{n=1}^\infty$ is a sequence of elements of K such that $\lim_{n\to\infty} \eta(t_n) = \infty$, the sequence $\{t_n \cdot p\}_{n=1}^\infty$ does not converge in $\mu^{-1}(\lambda)$.

There is a surjective morphism of varieties

$$\varphi_\eta : \mu^{-1}(\lambda)^\eta \to M_{\eta,\lambda}$$

characterized by the property that two points $p, q \in \mu^{-1}(\lambda)^\eta$ have the same image under φ_η if and only if the closures of their K-orbits have nontrivial intersection in $\mu^{-1}(\lambda)^\eta$ (not just in the larger set $\mu^{-1}(\lambda)$). Instead of $\varphi_\eta(p)$ we may write $[p]_\eta$ or simply $[p]$ if this would cause no confusion.

Definition 2.4. The pair (η, λ) is **regular** if every K-orbit in $\mu^{-1}(\lambda)^\eta$ is closed.

Thus, if (η, λ) is regular, the fibres of φ_η are precisely the K-orbits in $\mu^{-1}(\lambda)$, and so $M_{\eta,\lambda}$ is the geometric quotient $\mu^{-1}(\lambda)^\eta/K$.

In this thesis we will be exclusively concerned with the case where $\lambda = 0$, and we shall write M_η instead of $M_{\eta,0}$. Likewise, we will say that η is **regular** if $(\eta, 0)$ is regular.

Note that the semistable locus $\mu^{-1}(0)^0$ for the trivial character is simply $\mu^{-1}(0)$. The associated hypertoric variety

$$M_0 = \operatorname{Spec} \mathcal{O}(\mu^{-1}(0))^K$$

is the affinization of each M_η; the affinization map $M_\eta \to M_0$ is induced by the inclusion $\mu^{-1}(0)^\eta \subset \mu^{-1}(0)$.

Definition 2.5. Let $\{e_1, \ldots, e_n\}$ be the standard basis of $\mathfrak{t} = \mathbb{C}^n$, and let $\mathfrak{k}_\mathbb{Z} \subset \mathfrak{t}_\mathbb{Z}$ be the coweight lattice of K. For $1 \leq i \leq n$, let a_i denote the image of e_i under the quotient map $\mathfrak{t} \to \mathfrak{t}/\mathfrak{k}$. We say that K is **unimodular** if every linearly independent collection of $n - k$ elements of $\{a_1, \ldots, a_n\}$ generates the lattice $\mathfrak{t}_\mathbb{Z}/\mathfrak{k}_\mathbb{Z}$.

Remark 2.6. Since we assume (2.1) that $e_i \notin \mathfrak{k}$ for each i, we have $a_i \neq 0$ for each i.

Proposition 2.7. [13] Assuming K is unimodular, the following conditions on η are equivalent:

1. The hypertoric variety M_η is smooth.

2. η is regular.

3. The action of K on the semistable locus $\mu^{-1}(0)^\eta$ is free (*i.e.* each stabilizer is trivial).

Assumption 2.8. We shall henceforth assume that K is unimodular.

Example 2.9. Let

$$K = \left\{(t, \cdots, t) \in (\mathbb{C}^\times)^n : t \in \mathbb{C}^\times\right\}.$$

We then have

$$\mu^{-1}(0) = \left\{(z, w) \in T^*\mathbb{C}^n : \sum_{i=1}^n z_i w_i = 0\right\}.$$

A character $\eta : K \to \mathbb{C}^\times$ is of the form $\eta(t, \ldots, t) = t^r$ for some $r \in \mathbb{Z}$. For $r > 0$, we have

$$\mu^{-1}(0)^\eta = \left\{(z, w) \in \mu^{-1}(0) : z \neq 0\right\}.$$

We recall that for V a finite-dimensional complex vector space with projectivization $\mathbb{P}(V)$, the cotangent bundle $T^*\mathbb{P}(V)$ can be described as

$$T^*\mathbb{P}(V) = \left\{(L, X) \in \mathbb{P}(V) \times \operatorname{End} V : X^2 = 0, \operatorname{im} X \subset L\right\}.$$

We then see that the hypertoric variety $M_\eta = \mu^{-1}(0)^\eta / K$ is isomorphic to $T^*\mathbb{P}(\mathbb{C}^n)$ by identifying the orbit of (z, w) with the pair $(\operatorname{span}(z), w \otimes v)$, using the natural isomorphism $\operatorname{End} V = V^* \otimes V$. If $r < 0$, the semistability condition is instead given by $w \neq 0$, and the resulting hypertoric variety is identified with $T^*\mathbb{P}((\mathbb{C}^n)^*)$.

Example 2.10. Let

$$K = \left\{(t_1, \ldots, t_{m+1}) \in (\mathbb{C}^\times)^{m+1} : t_1 \cdots t_{m+1} = 1\right\},$$

acting on $T^*\mathbb{C}^{m+1}$. We then have

$$\mu^{-1}(0) = \left\{(z, w) \in T^*\mathbb{C}^{m+1} : z_1 w_1 = z_2 w_2 = \cdots = z_{m+1} w_{m+1}\right\}.$$

The affine hypertoric variety M_0 is isomorphic to the type-A_m Kleinian singularity

$$\mathbb{C}^2/\mathbb{Z}_{m+1} = \left\{(x,u,v) \in \mathbb{C}^3 : x^{m+1} + uv = 0\right\},$$

and the GIT quotient map $\mu^{-1}(0) \to \mathbb{C}^2/\mathbb{Z}_{m+1}$ is given by

$$(z,w) \mapsto (z_1 w_1, z_1 \cdots z_{m+1}, w_1 \cdots w_{m+1}).$$

For η a regular character, the affinization $M_\eta \to M_0$ is the minimal resolution

$$\widetilde{\mathbb{C}^2/\mathbb{Z}_{m+1}} \to \mathbb{C}^2/\mathbb{Z}_{m+1}.$$

2.2 Review of real hyperplane arrangements

We review some of the terminology of real hyperplane arrangements to be used in the sequel. Let $\mathcal{A} = \{H_i\}_{i\in I}$ be a hyperplane arrangement, which is to say a finite collection of affine hyperplanes in a finite-dimensional real vector space V. If each H_i contains the origin 0, we say that $\mathcal{A}$ is a **central** arrangement. For each $i \in I$, the open set $V \setminus H_i$ has two connected components H_i^+ and H_i^-. (Here we label these components arbitrarily, but in the sequel our hyperplanes will be equipped with normal vectors.) These can be described as the loci $\varphi_i > 0$ and $\varphi_i < 0$ where $\varphi_i : V \to \mathbb{R}$ is an affine functional such that $H_i = \varphi_i^{-1}(0)$.

Definition 2.11. A (relatively open) **face** of $\mathcal{A}$ is a nonempty subset of V of the form $F = \bigcap_{i\in I} H_i^{\sigma_i}$ where $\sigma_i \in \{-,0,+\}$ and $H_i^0 := H_i$. The collection $\sigma = (\sigma_i)_{i\in I}$ is called the **sign sequence** of F.

The vector space V is then partitioned by the faces.

Definition 2.12. A **chamber** of $\mathcal{A}$ is a relatively open face as defined above, with $\sigma_i \neq 0$ for all $i \in I$.

In other words, the chambers of $\mathcal{A}$ are the connected components of $V \setminus \bigcup_{i \in I} H_i$.

2.3 The hyperplane arrangement associated to a hypertoric variety

Just as the data defining a projective toric variety can be encoded by a convex rational polytope, the data defining a hypertoric variety M_η can be encoded by an oriented real hyperplane arrangement. We identify the group of multiplicative characters $K \to \mathbb{C}^\times$ with the weight lattice $\mathfrak{k}^*_\mathbb{Z}$ by taking derivatives at the identity element of K. Recall that a_i is the image of the generator e_i under the quotient map $\mathfrak{t} \to \mathfrak{t}/\mathfrak{k}$.

Definition 2.13. Let $(\eta_1, \ldots, \eta_n)$ be a lift of $\eta \in \mathfrak{k}^*_\mathbb{Z}$ to $\mathfrak{t}^*_\mathbb{Z} = \mathbb{Z}^n$, so that $\eta(t_1, \ldots, t_n) = t_1^{\eta_1} \cdots t_n^{\eta_n}$ for each $(t_1, \ldots, t_n) \in K$. Let $(\mathfrak{t}/\mathfrak{k})_\mathbb{R} = (\mathfrak{t}_\mathbb{Z}/\mathfrak{k}_\mathbb{Z}) \otimes_\mathbb{Z} \mathbb{R}$, with dual $(\mathfrak{t}/\mathfrak{k})^*_\mathbb{R}$. For $1 \leq i \leq n$, define the real affine hyperplane

$$H_{\eta,i} = \{x \in (\mathfrak{t}/\mathfrak{k})^*_\mathbb{R} : \langle x, a_i \rangle + \eta_i = 0\}$$

where $\langle -, - \rangle$ denotes the pairing between $\mathfrak{t}/\mathfrak{k}$ and its dual. The **associated hyperplane arrangement** is the collection

$$\mathcal{H}_\eta = \{H_{\eta,1}, \ldots, H_{\eta,n}\}.$$

This arrangement is independent of the choice of lift $(\eta_1, \ldots, \eta_n)$ up to simultaneous translation of the constituent hyperplanes. We note that the hyperplanes in the arrangement need not be distinct.

We can, in fact, reverse this construction: given affine hyperplanes $H_1, \ldots, H_n$ in $\mathbb{R}^d$, with $d \leq n$, together with integer vectors $a_1, \ldots, a_n \in \mathbb{Z}^d$ and integers $\eta_1, \ldots, \eta_n \in \mathbb{Z}$

such that

$$H_i = \left\{ x \in \mathbb{R}^d : \langle x, a_i \rangle + \eta_i = 0 \right\},$$

we recover the Lie algebra $\mathfrak{k}$ as the kernel of the linear map $\mathbb{C}^n \to \mathbb{C}^d$ defined by sending the i^{th} basis vector e_i to a_i, we recover K as the image of $\mathfrak{k}$ under the exponential map, and the character η is defined by sending $(t_1, \ldots, t_n) \in K$ to $t_1^{\eta_1} \cdots t_n^{\eta_n}$.

Remark 2.14. Recall that we have an affinization map $M_\eta \to M_0$. The fibre of this map over the point $[0]$ is called the **core** of M_η, and it is a union of compact toric varieties. The action of $T = (\mathbb{C}^\times)^n$ on $T^*\mathbb{C}^n$ descends to an action of the quotient torus T/K on M_η, the compact form of which acts in a hamiltonian way, giving a moment map

$$\overline{\mu}_{\mathbb{R}} : M_\eta \to (\mathfrak{t}/\mathfrak{k})^*_{\mathbb{R}}.$$

The closures of the maximal bounded faces of the associated hyperplane arrangement are precisely the moment polytopes of the components of the core of M_η with respect to this action.

Example 2.15. Let K be as in Example 2.9. Then we have

$$(\mathfrak{t}/\mathfrak{k})^* = \left\{ (x_1, \ldots, x_n) \in \mathbb{C}^n : \sum_i x_i = 0 \right\}$$

and the associated central arrangement $\mathcal{H}_0$ consists of the n hyperplanes $x_1 = 0, \ldots, x_n = 0$. For regular (i.e. nonzero) η, the arrangement $\mathcal{H}_\eta$ is in general position with precisely one bounded chamber, the closure of which is an $(n-1)$-dimensional simplex. This is the moment polytope for the core $\mathbb{P}(\mathbb{C}^n)$ of $M_\eta = T^*\mathbb{P}(\mathbb{C}^n)$.

Example 2.16. We return to Example 2.10, where K is the determinant-1 subtorus of $(\mathbb{C}^\times)^{m+1}$. Then $(\mathfrak{t}/\mathfrak{k})^*$ is a line, so the associated arrangement $\mathcal{H}_\eta$ consists of $m+1$ points, which are all distinct precisely when η is regular. In this latter case, the core of M_η is an A_m-chain of $\mathbb{P}^1$s – that is, its components $X_1, \ldots, X_m$ are each isomorphic to

$\mathbb{P}^1$, and $X_i \cap X_j$ is empty when $|i-j|>1$ and is a single point when $|i-j|=1$. The chambers of the moment polytopes are line segments, which are the moment polytopes for the curves $X_1,\ldots,X_m$.

Example 2.17. Let

$$K = \left\{(s, st^{-1}, t, s^{-1}) : s,t \in \mathbb{C}^\times\right\} \subset (\mathbb{C}^\times)^4.$$

This has Lie algebra

$$\mathfrak{k} = \{(a, a-b, b, -a) : a,b \in \mathbb{C}\}$$

and coalgebra

$$\mathfrak{k}^* = \operatorname{span}(f_1,f_2,f_3,f_4)/\operatorname{span}(f_1+f_4, f_1-f_2-f_3).$$

The ambient space for the associated hyperplane arrangement is

$$\begin{aligned}(\mathfrak{t}/\mathfrak{k})^*_{\mathbb{R}} &= \left\{(x_1,x_2,x_3,x_4)\in\mathbb{R}^4 : x_4 = x_1+x_2, x_2 = x_3\right\}\\ &= \{(x_1,x_2,x_2,x_1+x_2) : x_1,x_2\in\mathbb{R}\},\end{aligned}$$

which we identify with $\mathbb{R}^2$ by projecting onto the first two coordinates. We choose $\eta = f_1+f_2$, which lifts to $(1,1,0,0)\in\mathfrak{t}^*=\mathbb{C}^4$. The associated arrangement $\mathcal{H}_\eta$ then consists of the hyperplanes

$$\begin{aligned}H_{\eta,1} &= \{(x_1,x_2) : x_1+1=0\}\\ H_{\eta,2} &= \{(x_1,x_2) : x_2+1=0\}\\ H_{\eta,3} &= \{(x_1,x_2) : x_2+0=0\}\\ H_{\eta,4} &= \{(x_1,x_2) : x_1+x_2+0=0\},\end{aligned}$$

as shown in Figure 2.1.

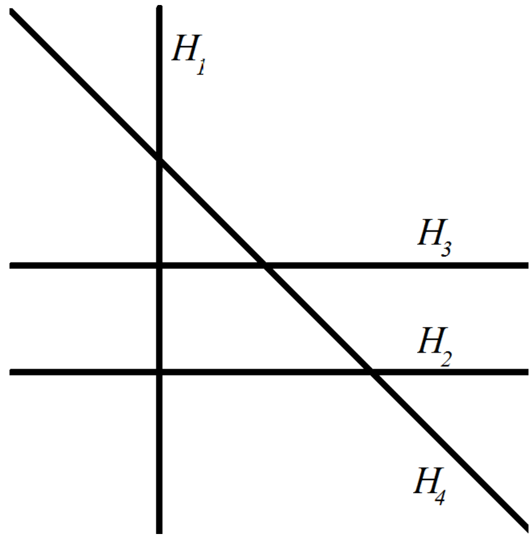


Figure 2.1: The associated arrangement $\mathcal{H}_\eta$ for Example 2.17.

2.4 A semistability criterion in terms of half-spaces

The hyperplanes $H_{\eta,i} \subset (\mathfrak{t}/\mathfrak{k})^*_{\mathbb{R}}$ come equipped with normal vectors defined by the generators a_i of $(\mathfrak{t}/\mathfrak{k})_{\mathbb{Z}}$; we denote the corresponding half-spaces by

$$H^+_{\eta,i} = \{x \in (\mathfrak{t}/\mathfrak{k})^*_{\mathbb{R}} : \langle x, a_i\rangle + \eta_i \geq 0\}\,,$$

$$H^-_{\eta,i} = \{x \in (\mathfrak{t}/\mathfrak{k})^*_{\mathbb{R}} : \langle x, a_i\rangle + \eta_i \leq 0\}\,.$$

We give a description of the semistable locus $\mu^{-1}(0)^\eta$ in terms of these half-spaces. First, we recall a well-known characterization of the semistable points for the action of a torus $H \subset G = (\mathbb{C}^\times)^N$ on $\mathbb{C}^N$. Define a lift of this action to the trivial line bundle on $\mathbb{C}^N$ by the character $\alpha = (\alpha_1, \ldots, \alpha_N)$. Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of G and H respectively, and let b_i denote the image of the i^{th} standard basis element of $\mathfrak{g} = \mathbb{C}^N$ in $\mathfrak{g}/\mathfrak{h}$. Define the polyhedron

$$\Delta = \{p \in (\mathfrak{g}/\mathfrak{h})^*_{\mathbb{R}} : \langle p, b_i\rangle + \alpha_i \geq 0 \text{ for } 1 \leq i \leq N\}\,.$$

For $1 \leq i \leq n$, define the face

$$F_i = \{p \in \Delta : \langle p, b_i \rangle + \alpha_i = 0\}.$$

Proposition 2.18. $x \in \mathbb{C}^N$ is semistable with respect to α if and only if the intersection $\bigcap_{x_i=0} F_i$ is nonempty.

See, for example, [19, Theorem 2.3]. We now use this result to prove the following characterization of the semistable locus.

Proposition 2.19. Let $(z, w) \in \mu^{-1}(0)$ and

$$R_{z,w} = \bigcap_{z_i=0} H^-_{\eta,i} \cap \bigcap_{w_i=0} H^+_{\eta,i}.$$

Then (z, w) is η-semistable if and only if $R_{z,w} \neq \emptyset$.

Proof. We apply Proposition 2.18. Take $N = 2n$, identifying $T^*\mathbb{C}^n$ with $\mathbb{C}^N = \mathbb{C}^n \times \mathbb{C}^n$, and let

$$G = T \times T,$$

$$H = \left\{(t, t^{-1}) : t \in K\right\},$$

$$\alpha = (\eta_1, \dots, \eta_n, 0, \dots, 0).$$

Then

$$(\mathfrak{g}/\mathfrak{h})^*_{\mathbb{R}} = \{(f, g) \in \mathfrak{t}^*_{\mathbb{R}} \times \mathfrak{t}^*_{\mathbb{R}} : f - g \in (\mathfrak{t}/\mathfrak{k})^*_{\mathbb{R}}\}$$

and

$$\Delta = \{(f, g) \in (\mathfrak{g}/\mathfrak{h})^*_{\mathbb{R}} : f_i + \eta_i \geq 0 \text{ and } g_i \geq 0 \text{ for } 1 \leq i \leq n\}.$$

For $1 \leq i \leq n$,

$$F_i = \{(f, g) \in (\mathfrak{g}/\mathfrak{h})^*_{\mathbb{R}} : f_i + \eta_i = 0, f_j + \eta_j \geq 0 \text{ and } g_j \geq 0 \text{ for all } 1 \leq j \leq n\}$$

and

$$F_{i+n} = \{(f,g) \in (\mathfrak{g}/\mathfrak{h})^*_{\mathbb{R}} : g_i = 0, f_j + \eta_j \geq 0 \text{ and } g_j \geq 0 \text{ for all } 1 \leq j \leq n\}.$$

Then a point (z, w) is η-semistable if and only if

$$Q := \bigcap_{i \in I} F_i \cap \bigcap_{i \in J} F_{i+n} \neq \emptyset$$

where $I = \{i : z_i = 0\}$ and $J = \{i : w_i = 0\}$. It therefore suffices to show that Q is nonempty if and only if $R_{z,w}$ is nonempty. Given $(f, g) \in Q$, it is easy to see that $f - g \in R_{z,w}$. Conversely, suppose $x \in R_{z,w}$. Let $g = (g_1, \dots, g_n)$ where

$$g_i = \begin{cases} -(x_i + \eta_i) & \text{if } i \in I \\ 0 & \text{if } i \in J \\ \max(0, -(x_i + \eta_i)) & \text{otherwise} \end{cases}.$$

(Note that $x_i + \eta_i = 0$ for $i \in I \cap J$ since $x \in H^+_{\eta,i} \cap H^-_{\eta,i}$.) Then we have $(g + x, g) \in Q$. □

2.5 Circuits and the discriminantal arrangement

The discriminantal arrangement is a real central hyperplane arrangement in $\mathfrak{k}^*_{\mathbb{R}} := \mathfrak{k}^*_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{R}$ with the property that, for each $\eta \in \mathfrak{k}^*_{\mathbb{Z}}$, the semistable locus $\mu^{-1}(0)^\eta$ depends only on which face of this arrangement η lies on. In particular, the semistability conditions are constant on each chamber of the discriminantal arrangement. The discriminantal hyperplanes are indexed by distinguished subsets of $\{1, \dots, n\}$ called the circuits of the action of K on $T^*\mathbb{C}^n$.

Definition 2.20. For each subset $C \subset \{1, \dots, n\}$, let $\mathfrak{k}_C = \mathfrak{k} \cap \operatorname{span}(e_i : i \in C)$. Then C is a **circuit** if $\mathfrak{k}_C \neq 0$ and C is minimal for this property. (In particular, $\dim \mathfrak{k}_C = 1$.)

This terminology comes from the theory of matroids: C is a circuit if and only if

$\{a_i : i \in C\}$ is a minimal linearly dependent subset of $\mathfrak{t}/\mathfrak{k}$, corresponding by definition to the circuits of the linear matroid defined by $\{a_1, \ldots, a_n\} \subset \mathfrak{t}/\mathfrak{k}$.

Let $\{e_i^\vee\}_{i=1}^n$ denote the dual of the standard basis for $\mathfrak{t}$, and for $1 \leq i \leq n$ let f_i denote the restriction of $e_i^\vee$ to $\mathfrak{k}$. Then $\{f_i\}_{i=1}^n$ generates the character lattice $\mathfrak{k}^*_{\mathbb{Z}}$, though it is of course not linearly independent unless $\mathfrak{k} = \mathfrak{t}$.

Definition 2.21. For each circuit C, the associated **discriminantal hyperplane** is $P_C := (\mathfrak{k}_C)^\perp_{\mathbb{R}} \subset \mathfrak{k}^*_{\mathbb{R}}$. The **discriminantal arrangement** is the collection of all discriminantal hyperplanes.

We note that P_C is spanned over $\mathbb{R}$ by $\{f_i : i \notin C\}$.

Proposition 2.22. [12] A character $\eta \in \mathfrak{k}^*_{\mathbb{Z}}$ is regular if and only if it does not lie on any discriminantal hyperplane.

For C a circuit, the lattice $(\mathfrak{k}_C)_{\mathbb{Z}}$ is isomorphic to $\mathbb{Z}$ and so it has two generators, each the negative of the other. These correspond to the co-orientations of P_C.

Definition 2.23. Let C be a circuit and $\eta \in \mathfrak{k}^*_{\mathbb{Z}}$ a character with $\eta \notin P_C$. Let β_C^η be the generator of $(\mathfrak{k}_C)_{\mathbb{Z}}$ such that $\langle \eta, \beta_C^\eta \rangle > 0$, where $\langle -, - \rangle$ denotes the pairing between $\mathfrak{k}^*$ and $\mathfrak{k}$. We then define

$$C_+^\eta = \{i \in C : \langle f_i, \beta_C^\eta \rangle > 0\}$$

and

$$C_-^\eta = \{i \in C : \langle f_i, \beta_C^\eta \rangle < 0\}.$$

We refer to the partition $C = C_+^\eta \sqcup C_-^\eta$ as an **orientation** of C.

In other words, $i \in C_+^\eta$ if f_i and η are in the same connected component of $\mathfrak{k}^*_{\mathbb{R}} \setminus P_C$, and $i \in C_-^\eta$ if they are in different components. Since we assumed that K is unimodular (2.5), the generator β_C^η of $(\mathfrak{k}_C)_{\mathbb{Z}}$ can then be written as

$$\beta_C^\eta = \sum_{i \in C_+^\eta} e_i - \sum_{i \in C_-^\eta} e_i.$$

Of course, if η and η' are on opposite sides of P_C, then $C_\pm^{\eta'} = C_\mp^\eta$ and $\beta_C^{\eta'} = -\beta_C^\eta$.

Proposition 2.24. Let $\eta \in \mathfrak{k}_\mathbb{Z}^*$ be a regular character and $\mu : T^*\mathbb{C}^n \to \mathfrak{k}^*$ the moment map for the action of K on $T^*\mathbb{C}^n$. Then

$$\mu^{-1}(0) = \left\{ (z,w) \in T^*\mathbb{C}^n : \sum_{i \in C_+^\eta} z_i w_i = \sum_{i \in C_-^\eta} z_i w_i \text{ for all circuits } C \right\}.$$

Proof. Recall that

$$\mu(z,w)(x_1, \dots, x_n) = \sum_{i=1}^n z_i w_i x_i$$

for $(x_1, \dots, x_n) \in \mathfrak{k}$. Then we have $\mu(z,w) = 0$ if and only if

$$\sum_{i=1}^n z_i w_i e_i^\vee(x) = 0$$

for all $x \in \mathfrak{k}$. From the definition of circuit, $\mathfrak{k}$ is generated over $\mathbb{Z}$ by the subtori $\mathfrak{k}_C$. It follows that $\mu(z,w) = 0$ if and only if

$$\sum_{i=1}^n z_i w_i e_i^\vee(\beta_C^\eta) = 0$$

for all circuits C, which gives the claim immediately. □

Example 2.25. We return again to Example 2.10. Here we have

$$\mathfrak{k} = \left\{ (x_1, \dots, x_{m+1}) \in \mathbb{C}^{m+1} : \sum_i x_i = 0 \right\}$$

and

$$\mathfrak{k}^* = \text{span}(f_1, \dots, f_{m+1})/\text{span}(\sum_i f_i).$$

The circuits are precisely the unordered pairs $\{i, j\}$ with $1 \le i < j \le m+1$. The discriminantal hyperplanes are

$$P_{i,j} = \left\{ \sum_i \lambda_i f_i : \lambda_i = \lambda_j \right\}.$$

For $\eta = \sum_i \eta_i f_i$ a regular character, the generator $\beta^{\eta}_{i,j}$ of $\mathfrak{k}_{i,j}$ is equal to $e_i - e_j$ if $\eta_i > \eta_j$, and $e_j - e_i$ otherwise.

2.6 Subtorus and quotient associated to a circuit

Definition 2.26. Let C be a circuit for the action of K on $T^*\mathbb{C}^n$. Recall that $\mathfrak{k}_C = \mathfrak{k} \cap \operatorname{span}(e_i : i \in C)$ is one-dimensional. Let K_C be the rank-1 subtorus of K whose Lie algebra is $\mathfrak{k}_C$. We further denote by $\overline{K}_C$ the quotient torus K/K_C, and by $\bar{\mathfrak{k}}_C$ its Lie algebra $\mathfrak{k}/\mathfrak{k}_C$.

Note that $(\bar{\mathfrak{k}}_C)^*_{\mathbb{R}}$ is precisely the discriminantal hyperplane $(\mathfrak{k}_C)^{\perp}_{\mathbb{R}} = P_C$, and so the character lattice of $\overline{K}_C$ is $P_C \cap \mathfrak{k}^*_{\mathbb{Z}}$. The torus $\overline{K}_C$ does not naturally act on $\mathbb{C}^n$, but since K_C acts trivially on the coordinates z_i and w_i for $i \notin C$, we do have an action of $\overline{K}_C$ on the subspace defined by the vanishing of the coordinates in C.

Definition 2.27. Let $E_C = \operatorname{span}(e_i : i \notin C) \subset \mathbb{C}^n$.

Then we have an action of $\overline{K}_C$ on E_C, hence on $T^*E_C \subset T^*\mathbb{C}^n$. The hypertoric varieties arising from the action of $\overline{K}_C$ on T^*E_C will come into play later on in this thesis, and so we will need to understand the circuits of this action.

Definition 2.28. A character of K is said to be **subregular** if it lies on exactly one discriminantal hyperplane.

Lemma 2.29. 1. The set of circuits of the action of $\overline{K}_C$ on T^*E_C is

$$\{S \setminus C : S \text{ a circuit of } K, S \neq C\}.$$

2. For S a circuit with $S \neq C$ and $\eta \in P_C \setminus P_S$, we have $(S \setminus C)^\eta_\pm = S^\eta_\pm \setminus C$.

3. If $\eta \in P_C$ is subregular as a character of K, then it is regular as a character of $\overline{K}_C$.

Proof. We have a commutative diagram

$$\begin{array}{ccccc} \mathfrak{k} & \longrightarrow & \mathfrak{t} & \longrightarrow & \mathfrak{t}/\mathfrak{k} \\ \downarrow & & \downarrow & & \downarrow \\ \overline{\mathfrak{k}}_C & \longrightarrow & \mathfrak{t}/\mathfrak{t}_C & \longrightarrow & (\mathfrak{t}/\mathfrak{t}_C)/\overline{\mathfrak{k}}_C \end{array}$$

A circuit of $\overline{K}_C$ is a subset R of $\{1,\dots,n\} \setminus C$ such that $\dim(\overline{\mathfrak{k}}_C)_R = 1$, where $(\overline{\mathfrak{k}}_C)_R$ is the image of $\mathfrak{k}_R$ under the map $\mathfrak{k} \to \mathfrak{t}/\mathfrak{t}_C$ above, which has kernel $\mathfrak{k}_C$. We also note that $(\overline{\mathfrak{k}}_C)_R = (\overline{\mathfrak{k}}_C)_{R \setminus C}$ since e_i is annihilated by this map for each $i \in C$. Hence if S is a circuit of K with $S \neq C$, this map is injective when restricted to the line $\mathfrak{k}_S \neq \mathfrak{k}_C$ and so $\dim(\overline{\mathfrak{k}}_C)_{S \setminus C} = 1$. Thus $S \setminus C$ is a circuit of $\overline{K}_C$.

Conversely, let R be a circuit of $\overline{K}_C$. Then $\{\overline{a}_i : i \in R\}$ is linearly dependent, where a_i is the image of e_i under the map $\mathfrak{t} \to (\mathfrak{t}/\mathfrak{t}_C)/\overline{\mathfrak{k}}_C$ above. Then we have scalars m_i such that

$$\sum_{i \in R} m_i e_i + \sum_{i \in C} m_i e_i \in \mathfrak{k}$$

with not all m_i zero for $i \in R$. Then there is some circuit S of K such that $S \subset R \sqcup C$ and $S \cap R \neq \emptyset$, so for some $n_i \in \{-1, 1\}$ we have

$$\sum_{i \in S \cap R} n_i e_i + \sum_{i \in S \cap C} n_i e_i \in \mathfrak{k}.$$

In particular, $\{\overline{a}_i : i \in S \cap R\}$ is linearly dependent in $(\mathfrak{t}/\mathfrak{t}_C)/\overline{\mathfrak{k}}_C$. By minimality of R, then, $S \cap R = R$ and so $R = S \setminus C$.

If $\eta \notin P_S$ then the second claim above follows from the fact that the image of

$$\beta^\eta_S = \sum_{i \in S^\eta_+} e_i - \sum_{i \in S^\eta_-} e_i$$

in $(\bar{\mathfrak{k}}_C)_{S\setminus C}$ is

$$\sum_{i\in S_+^\eta\setminus C} e_i - \sum_{i\in S_-^\eta\setminus C} e_i$$

and generates its coweight lattice.

The third claim above follows from the observation that the discriminantal hyperplane in $\bar{\mathfrak{k}}_C^* = P_C$ corresponding to the circuit $S\setminus C$ is precisely $P_S \cap P_C$. □

2.7 A semistability criterion in terms of circuits

Konno [13, Theorem 5.10] proved that if η is a *regular* character of K then the semistable locus $\mu^{-1}(0)^\eta$ consists of precisely those points $(z,w)\in\mu^{-1}(0)$ such that, for each circuit C, we have $z_i \neq 0$ for some $i\in C_+^\eta$ or $w_i\neq 0$ for some $i \in C_-^\eta$. Motivated by this result, we define the following coordinate functions.

Definition 2.30. Let η be a character of K. For each circuit C such that $\eta\notin P_C$, define the coordinate function

$$x_C^\eta : T^*\mathbb{C}^n \to \mathbb{C}^{|C|}$$

by

$$x_C^\eta(z,w) = (z_i : i\in C_+^\eta; w_i : i \in C_-^\eta).$$

Observe that the coweight

$$\beta_C^\eta = \sum_{i\in C_+^\eta} e_i - \sum_{i\in C_-^\eta} e_i \in \mathfrak{k}_\mathbb{Z}$$

defines an isomorphism $\mathbb{C}^\times \cong K_C$, and for $t\in\mathbb{C}^\times$ we have

$$x_C^\eta(\beta_C^\eta(t)\cdot(z,w)) = t x_C^\eta(z,w).$$

We also note that if η and η' are characters on opposite sides of the hyperplane P_C, then

$C^{\eta'}_{\pm} = C^{\eta}_{\mp}$ and so

$$x^{\eta'}_C(z,w) = x^{\eta}_C(w,z).$$

Using the notation of these coordinate functions, Konno's semistability criterion can be expressed as follows:

Theorem 2.31. [13, Theorem 5.10] Let $\eta \in \mathfrak{k}^*_{\mathbb{Z}}$ be a regular character. Then the η-semistable locus in $\mu^{-1}(0)$ is

$$\mu^{-1}(0)^{\eta} = \left\{ p \in \mu^{-1}(0) : x^{\eta}_C(p) \neq 0 \text{ for all circuits } C \right\}.$$

Using Proposition 2.19, we give a new proof of this result and generalize it to arbitrary (*i.e.* possibly non-regular) η.

Theorem 2.32. Let $\eta \in \mathfrak{k}^*_{\mathbb{Z}}$. Then a point $(z,w) \in T^*\mathbb{C}^n$ is η-semistable if and only if $x^{\eta}_C(p) \neq 0$ for all circuits C such that $\eta \notin P_C$.

Proof. Suppose $p = (z,w)$ with $x^{\eta}_C(z,w) = 0$ for some circuit C with $\eta \notin P_C$. Let

$$\beta_C = \sum_{i \in C^{\eta}_{+}} e_i - \sum_{j \in C^{\eta}_{-}} e_j$$

so that $\langle \eta, \beta_C \rangle > 0$. Then

$$\lim_{t \to \infty} \eta(\beta^{\eta}_C(t)) = \infty$$

and

$$\lim_{t \to \infty} \beta^{\eta}_C(t) \cdot p = p$$

since the image K_C of β^{η}_C fixes p, so p is η-unstable. Conversely, suppose

$$\bigcap_{i \in I} H^{-}_{\eta,i} \cap \bigcap_{j \in J} H^{+}_{\eta,j} = \emptyset,$$

where $I = \{i : z_i = 0\}$ and $J = \{j : w_j = 0\}$. We then wish to show that there exists a circuit C with $\eta \notin P_C$, $C_+^\eta \subset I$, and $C_-^\eta \subset J$.

We recall a form of Farkas's Lemma: given a finite-dimensional real vector space V and $\alpha_1, \dots, \alpha_m \in V^*$ and $y_1, \dots, y_m \in \mathbb{R}$, then

$$\bigcap_{i=1}^{m} \{v \in V : \langle \alpha_i, v \rangle \geq y_i\} = \emptyset$$

if and only if there exist $r_1, \dots, r_m \geq 0$ with $\sum_i r_i \alpha_i = 0$ and $\sum_i r_i y_i > 0$.

We write

$$H_{\eta,i}^- = \{x \in (\mathfrak{t}/\mathfrak{k})_{\mathbb{R}}^* : \langle -a_i, x \rangle \geq \eta_i\}$$

$$H_{j,+}^\eta = \{x \in (\mathfrak{t}/\mathfrak{k})_{\mathbb{R}}^* : \langle a_j, x \rangle \geq -\eta_j\}$$

and use Farkas's Lemma to conclude that there exist $r_i \geq 0$ for $i \in I$ and $s_j \geq 0$ for $j \in J$ such that

$$\sum_{i \in I} r_i(-a_i) + \sum_{j \in J} s_j a_j = 0,$$

hence

$$\lambda := \sum_{i \in I} r_i e_i - \sum_{j \in J} s_j e_j \in \mathfrak{k},$$

and

$$\sum_{i \in I} r_i \eta_i - \sum_{j \in J} s_j \eta_j > 0,$$

i.e. $\langle \lambda, \eta \rangle > 0$.

Note that I and J are not necessarily disjoint. Let

$$(I \cap J)^+ = \{i \in I \cap J : r_i - s_i \geq 0\}$$

and

$$(I \cap J)^- = (I \cap J) \setminus (I \cap J)^+.$$

For $i \in (I \cap J)^+$, let $u_i = r_i - s_i$, and for $j \in (I \cap J)^-$ let $u_j = s_j - r_j$. Then

$$\lambda = \left(\sum_{i \in I \setminus J} r_i e_i + \sum_{i \in (I \cap J)^+} u_i e_i \right) - \left(\sum_{j \in J \setminus I} s_j e_j + \sum_{j \in (I \cap J)^+} u_j e_j \right)$$

with all of the coefficients in these sums being nonnegative. Then since $\langle \lambda, \eta \rangle > 0$, using [4, Theorem 3.7.2] there exists a circuit C with $\eta \notin P_C$ and such that

$$C_+^\eta \subset (I \setminus J) \cup (I \cap J)^+ \subset I$$

and

$$C_-^\eta \subset (J \setminus I) \cup (I \cap J)^- \subset J,$$

as required. □

Corollary 2.33. For $\eta \in \mathfrak{k}_\mathbb{Z}^*$, the semistable locus $\mu^{-1}(0)^\eta$ depends only on which face of the discriminantal arrangement contains η.

Proof. By the above, the semistable locus depends only on which circuits C satisfy $\eta \notin P_C$ and on the orientation $C = C_+^\eta \sqcup C_-^\eta$ for each such C. The latter orientation is determined by which component of the complement of P_C contains η. All of this is determined by the face containing η. □

Chapter 3

Fourier-Mukai transforms and Mukai flops

3.1 Fourier-Mukai transforms

For X a complex variety, we denote by $D^b(X)$ the bounded derived category of coherent sheaves on X.

Definition 3.1. Let X and Y be smooth complex varieties, and let

$$\pi_X : X \times Y \to X, \pi_Y : X \times Y \to Y$$

be the natural projections. Let $\mathcal{P}$ be an object of $D^b(X \times Y)$ whose support is proper over X and over Y. The **Fourier-Mukai transform** with **kernel** $\mathcal{P}$ is the functor

$$\Phi_{\mathcal{P}} : D^b(X) \to D^b(Y)$$

defined by

$$\Phi_{\mathcal{P}}(\mathcal{E}^\bullet) = (\pi_Y)_*(\pi_X^* \mathcal{E}^\bullet \otimes \mathcal{P})$$

where $(\pi_Y)_*, \pi_X^*$, and $- \otimes \mathcal{P}$ are the derived pushforward, pullback and tensor functors between the derived categories.

Fourier-Mukai transforms are ubiquitous: derived pushforward and pullback functors, the shift functor on $D^b(X)$, and many other naturally occurring functors can be expressed as Fourier-Mukai transforms (see, for example, [8]). Indeed, it is a deep theorem of D. Orlov [18] that if X and Y are smooth projective varieties, then every fully faithful exact functor $D^b(X) \to D^b(Y)$ is isomorphic to a Fourier-Mukai transform $\Phi_{\mathcal{P}}$ for an object $\mathcal{P}$ of $D^b(X \times Y)$ which is unique up to isomorphism.

Remark 3.2. The right and left adjoints of $\Phi_{\mathcal{P}}$ are the Fourier-Mukai transforms with respective kernels

$$\mathcal{P}_R := \mathcal{P}^\vee \otimes \pi_X^* \omega_X[\dim X], \mathcal{P}_L := \mathcal{P}^\vee \otimes \pi_Y^* \omega_Y[\dim Y]$$

where $\mathcal{P}^\vee$ is the dual of $\mathcal{P}$, viewed as a complex of sheaves on $Y \times X$, and ω_X, ω_Y are the canonical bundles of X, Y respectively.

3.2 Mukai flops

A Mukai flop, or elementary transform, is a type of birational surgery which, given a holomorphic symplectic variety containing a projective bundle as a subvariety, produces a new variety by removing that bundle and replacing it by its dual.

More precisely, suppose M is a $2m$-dimensional holomorphic symplectic variety containing a closed subvariety P isomorphic to $\mathbb{P}^m$, and $\nu : M \to \overline{M}$ is a projective birational morphism which contracts P to a point and is an isomorphism away from P. Let $\mathcal{N} = \mathcal{N}_{P/M}$ be the normal bundle of P in M. Since P is a Lagrangian subvariety of M, the bundle $\mathcal{N} \to P$ is isomorphic to the cotangent bundle T^*P. Let us fix an $(m+1)-$dimensional vector space V and an isomorphism $P \cong \mathbb{P}(V)$. From the Euler sequence, we have an embedding of vector bundles $T^*\mathbb{P}(V) \subset V^* \otimes \mathcal{O}(-1)$ which embeds the projectivization $\mathbb{P}(T^*\mathbb{P}(V))$ in $\mathbb{P}(V) \times \mathbb{P}(V^*)$ as the incidence variety

$$\{(L, H) \in \mathbb{P}(V) \times \mathbb{P}(V^*) : L \subset H\}.$$

Here we identify $\mathbb{P}(V^*)$ with the variety of hyperplanes in V. Blowing up M along P gives a projective morphism $\widetilde{M} \to M$ with exceptional divisor $E = \mathbb{P}(\mathcal{N})$ which we hence identify with this incidence variety. Mukai [16] showed that there is a variety M' and a birational morphism $\widetilde{M} \to M'$ with exceptional divisor E, such that the restriction to E is the second projection

$$E \subset \mathbb{P}(V) \times \mathbb{P}(V^*) \to \mathbb{P}(V^*).$$

We then have a birational morphism $\nu' : M' \to \overline{M}$ contracting the image $\mathbb{P}(V^*)$ of E to a point, and a commutative diagram

$$\begin{array}{ccc} \widetilde{M} & \longrightarrow & M' \\ \downarrow & & \downarrow \nu' \\ M & \xrightarrow{\nu} & \overline{M} \end{array}$$

Definition 3.3. The diagram $M \xrightarrow{\nu} \overline{M} \xleftarrow{\nu'} M'$ is the **Mukai flop** of M along P.

More generally, suppose M is a $2m$-dimensional holomorphic symplectic variety, $P \subset M$ is an m-dimensional closed subvariety, $\nu : M \to \overline{M}$ is a proper birational morphism with exceptional locus P such that the image $Y = \nu(P)$ is a smooth closed subvariety of $\overline{M}$, and the restriction $\nu : P \to Y$ is the projectivization $\mathbb{P}(\mathcal{V})$ of a rank-$(\operatorname{codim} P + 1)$ vector bundle $\mathcal{V}$ over Y. It can then be shown [9, Section 3] that the normal bundle $\mathcal{N}_{P/M}$ is isomorphic to the relative cotangent bundle of ν, *i.e.* its restriction to each fibre of ν is the cotangent bundle of that fibre. Performing Mukai flops in a family then yields a commutative diagram of birational morphisms as above, which we also refer to as a **Mukai flop**. So here, M' has the dual bundle $\mathbb{P}(\mathcal{V}^*) \to Y$ as a subvariety.

Let $Z = M \times_{\overline{M}} M'$, and let $Z_0 = \mathbb{P}(\mathcal{V}) \times_Y \mathbb{P}(\mathcal{V}^*) \subset Z$. The maps in the above diagram restrict to isomorphisms

$$\begin{array}{ccc} \widetilde{M} \setminus E & \longrightarrow & M' \setminus \mathbb{P}(\mathcal{V}^*) \\ \downarrow & & \downarrow \nu' \\ M \setminus \mathbb{P}(\mathcal{V}) & \xrightarrow{\nu} & \overline{M} \setminus Y \end{array}$$

and so we see that the induced morphism $i : \widetilde{M} \to Z$ identifies $\widetilde{M} \setminus E$ with $Z \setminus Z_0$. Let Z_1 denote the closure in Z of $Z \setminus Z_0$. Then i identifies E with $Z_0 \cap Z_1$, and indeed $\widetilde{M}$ with Z_1. To summarize, the fibre product Z has two components

$$Z_0 = \mathbb{P}(\mathcal{V}) \times_Y \mathbb{P}(\mathcal{V}^*)$$

and

$$Z_1 = \widetilde{M},$$

with

$$Z_0 \cap Z_1 = \{(L, H) \in \mathbb{P}(\mathcal{V}) \times_Y \mathbb{P}(\mathcal{V}^*) : L \subset H\}.$$

Given regular characters η and η' of the torus K which are separated by a single wall of the discriminantal arrangement, we show in the next chapter that the hypertoric varieties M_η and $M_{\eta'}$ are related by a Mukai flop, with the role of $\overline{M}$ played by M_θ where θ is a subregular character on the wall separating η from η' such that θ lies in the closure of each of the chambers containing η and η' respectively.

In the above definition of Mukai flop, we assumed that M and M' are holomorphic symplectic varieties. The same definition has been made for M and M' smooth and projective, but not necessarily equipped with a symplectic form (see, for example, [8, 11.4]). In this context, it is not automatic that the normal bundle $\mathcal{N}_{P/M}$ is isomorphic to the relative cotangent bundle of ν, and this is imposed as a separate condition in the definition of a Mukai flop of projective varieties. Suppose that M and M' are smooth and projective and related by a Mukai flop. As found by Y. Namikawa and Y. Kawamata independently, the fibre product Z defines an equivalence between the bounded derived categories of M and M':

Theorem 3.4. [17], [11] Let M and M' be smooth projective varieties, let $M \dashrightarrow M'$ be a Mukai flop and define the fibre product Z as above. Then the Fourier-Mukai transform $\Phi_Z : D^b(M) \to D^b(M')$ with kernel $\mathcal{O}_Z$ is an equivalence of triangulated categories.

As written, this theorem cannot be directly applied to the situation of a Mukai flop of hypertoric varieties, as these are generally not projective (over $\operatorname{Spec} \mathbb{C}$). However, its conclusion is still valid in the symplectic context. Namikawa's argument in [17, Section 4] applies here to show that Φ_Z is fully faithful, as this part of the proof does not rely on M and M' being projective. It then remains to show that Φ_Z is essentially surjective. We have $\dim M = \dim M'$ since M and M' are birationally equivalent, and their canonical bundles are trivial since they are each equipped with a holomorphic symplectic form. Then by Remark 3.2, the left and right adjoints of Φ_Z coincide: regarding $\mathcal{O}_Z^\vee$ as a sheaf on $M' \times M$, these adjoints are isomorphic to the Fourier-Mukai transform with kernel $\mathcal{O}_Z^\vee[\dim M]$. Since Φ_Z is fully faithful and its left and right adjoints coincide, we conclude by [5, Theorem 3.3] that Φ_Z is an equivalence. We summarize this in the following theorem.

Theorem 3.5. Let M and M' be holomorphic symplectic varieties, let $M \dashrightarrow M'$ be a Mukai flop and define the fibre product Z as above. Then the Fourier-Mukai transform $\Phi_Z : D^b(M) \to D^b(M')$ with kernel $\mathcal{O}_Z$ is an equivalence of triangulated categories.

Chapter 4

Wall-crossing

4.1 Partial affinization

Throughout this chapter, we fix two regular characters $\eta, \eta' \in \mathfrak{k}^*_{\mathbb{Z}}$ separated by a single discriminantal hyperplane P_C, and a subregular character $\theta \in \mathfrak{k}^*_{\mathbb{Z}} \cap P_C$ which lies in the closures of the chambers containing η and η'. Thus P_C is the only discriminantal hyperplane containing θ. For $\alpha \in \mathfrak{k}^*_{\mathbb{Z}}$ and S a circuit with $\alpha \notin P_S$, recall that we defined the coordinate function

$$x^\alpha_S(z, w) = (z_i : i \in S^\alpha_+; w_i : i \in S^\alpha_-).$$

Lemma 4.1.

$$\mu^{-1}(0)^\eta = \Big\{(z, w) \in \mu^{-1}(0)^\theta : x^\eta_C(z, w) \neq 0\Big\}$$

Proof. For each circuit $S \neq C$, the characters η and θ are on the same side of the discriminantal hyperplane P_S, and so $x^\eta_S = x^\theta_S$. The result follows immediately from Theorem 2.32. □

We therefore have inclusions $\mu^{-1}(0)^\eta \subset \mu^{-1}(0)^\theta \supset \mu^{-1}(0)^{\eta'}$.

Definition 4.2. Let $M_\eta \xrightarrow{\nu} M_\eta \xleftarrow{\nu'} M_{\eta'}$ denote the morphisms of varieties induced by the above inclusions. We call these **partial affinizations**.

The reason we call these "partial affinizations" is that they are compatible with the affinization morphisms $M_\eta \to M_0$ and $M_\theta \to M_0$, which are induced by the inclusions of the respective semistable loci into $\mu^{-1}(0)$.

We begin by showing that $\nu : M_\eta \to M_\theta$ contracts a closed subvariety $B_\theta^\eta \subset M_\eta$ to a subvariety $B_\theta \subset M_\theta$, and that the restriction $\nu : B_\theta^\eta \to B_\theta$ is the projectivization of a rank-$|C|$ vector bundle. Recall that $E_C = \operatorname{span}(e_i : i \notin C) \subset \mathbb{C}^n$.

Definition 4.3. Let

$$B_\theta := \varphi_\theta(T^*E_C \cap \mu^{-1}(0)^\theta)$$

where $\varphi_\theta : \mu^{-1}(0)^\theta \to M_\theta$ is the GIT quotient map.

Proposition 4.4. B_θ is a smooth hypertoric variety.

Proof. Recall from Section 2.6 that we have an action of the quotient torus $\overline{K}_C$ on T^*E_C. The θ-semistable locus for this action is precisely $T^*E_C \cap \mu^{-1}(0)^\theta$, in which the $\overline{K}_C$-orbits are closed since θ is regular as a character of $\overline{K}_C$. The associated hypertoric variety is therefore the geometric quotient $(T^*E_C \cap \mu^{-1}(0)^\theta)/\overline{K}_C$, which is smooth again by regularity of θ. But φ_θ also realizes B_θ as this geometric quotient, as the K-orbits in $T^*E_C \cap \mu^{-1}(0)^\theta$ are the same as the $\overline{K}_C$-orbits. □

Lemma 4.5. For each $p \in \mu^{-1}(0)^\eta \cap \mu^{-1}(0)^{\eta'}$, the orbit Kp is closed in $\mu^{-1}(0)^\theta$.

Proof. Let $p \in \mu^{-1}(0)^\eta \cap \mu^{-1}(0)^{\eta'}$ and $q \in \overline{Kp} \cap \mu^{-1}(0)^\theta$. By the Hilbert-Mumford criterion for tori due to Richardson [3], there is a one-parameter subgroup $\lambda \in \mathfrak{k}_\mathbb{Z}$ with

$$\lim_{t\to\infty} \lambda(t)p \in Kq.$$

It suffices to show that $\lambda = 0$. Suppose otherwise, for contradiction. Write $\lambda = (\lambda_1, \ldots, \lambda_n)$, and define

$$I_+ = \{i : \lambda_i > 0\},$$

$$I_{-} = \{i : \lambda_i < 0\}.$$

Choose a circuit S such that, orienting S by η, we have

$$S_{+} \subset I_{+}, S_{-} \subset I_{-}$$

or

$$S_{-} \subset I_{+}, S_{+} \subset I_{-}.$$

In the former case, or in the latter case with $S = C$ we obtain a contradiction as

$$\lim_{t \to \infty} x_S^{\eta}(\lambda(t)p) = \infty;$$

finally if $S_{-} \subset I_{+}$ and $S_{+} \subset I_{-}$ with $S \neq C$ then

$$\lim_{t \to \infty} x_S^{\theta}(\lambda(t)p) = 0,$$

which contradicts $q \in \mu^{-1}(0)^{\theta}$. Thus $Kp = Kq$. □

Lemma 4.6. The complement $B_\theta^c := M_\theta \setminus B_\theta$ is equal to $\varphi_\theta(\mu^{-1}(0)^{\eta} \cap \mu^{-1}(0)^{\eta'})$, and ν is an isomorphism over B_θ^c.

Proof. We note that the second claim follows from the first by Lemma 4.5, which implies that B_θ^c and $\nu^{-1}(B_\theta^c)$ are both given by the geometric quotient $(\mu^{-1}(0)^{\eta} \cap \mu^{-1}(0)^{\eta'})/K$.

Given $(z, w) \in \mu^{-1}(0)^{\theta} \setminus (\mu^{-1}(0)^{\eta} \cap \mu^{-1}(0)^{\eta'})$, we have $x_C^{\eta}(z, w) = 0$ or $x_C^{\eta}(w, z) = 0$. Orienting C according to η in the former case or to η' in the latter case, we have

$$\lim_{t \to \infty} \beta_C(t)(z, w) \in T^* E_C \cap \mu^{-1}(0)^{\theta}$$

and so $\varphi_\theta(z, w) \in B_\theta$. Then $B_\theta^c \subset \varphi_\theta(\mu^{-1}(0)^{\eta} \cap \mu^{-1}(0)^{\eta'})$.

Conversely, if $p \in \mu^{-1}(0)^{\eta} \cap \mu^{-1}(0)^{\eta'}$ and $q \in T^* E_C \cap \mu^{-1}(0)^{\theta}$, the orbits of p and

q are closed in $\mu^{-1}(0)^\theta$ by Lemma 4.5 and by subregularity of θ, respectively. Since $Kp \cap T^*E_C = \emptyset$, it follows that $\varphi_\theta(p) \notin B_\theta$. $\square$

Definition 4.7. Let

$$V_C = \operatorname{span}(e_i : i \in C),$$

$$V_C^\eta = \operatorname{span}(e_i : i \in C_+^\eta) \oplus \operatorname{span}(e_i^\vee : i \in C_-^\eta),$$

and

$$V_C^{\eta'} = \operatorname{span}(e_i : i \in C_+^{\eta'}) \oplus \operatorname{span}(e_i^\vee : i \in C_-^{\eta'}),$$

each of which is a $|C|$-dimensional linear subspace of $T^*\mathbb{C}^n$, with

$$T^*V_C = V_C^\eta \oplus V_C^{\eta'}.$$

Observe that we have a natural symplectic form ω on T^*V_C given by $\omega(e_i, e_j) = \omega(e_i^\vee, e_j^\vee) = 0$ and $\omega(e_i, e_j^\vee) = \delta_{ij}$, and that V_C^η and $V_C^{\eta'}$ are complementary Lagrangian subspaces with respect to ω. The pairing ω thus identifies V_C^η and $V_C^{\eta'}$ as dual to each other.

Lemma 4.8.

$$(T^*E_C \oplus V_C^\eta) \cap \mu^{-1}(0)^\eta = \left\{p + v : p \in T^*E_C \cap \mu^{-1}(0)^\theta, v \in V_C^\eta \setminus 0\right\}.$$

Proof. Given $p \in T^*E_C \cap \mu^{-1}(0)^\theta$ and $v \in V_C^\eta \setminus 0$, we have $x_C^\eta(p+v) = v \neq 0$ and for $S \neq C$, $x_S^\eta(p+v) \neq 0$ since $x_S^\eta(p) = x_S^\theta(p) \neq 0$, and so $p+v$ is η-semistable.

Conversely, suppose $p \in T^*E_C$ and $v \in V_C^\eta$ with $p+v \in \mu^{-1}(0)^\eta$. Then immediately $v = x_C^\eta(p+v)$ is nonzero, so it suffices to show that p is θ-semistable. Assume for contradiction that p is θ-unstable. Then there exist $t_n \in K$ such that

$$\lim_{n\to\infty} \theta(t_n) = \infty$$

and such that the limit

$$q := \lim_{n\to\infty} t_n \cdot p$$

exists. Recall that f_j is the restriction of the character $e_i^\vee$ to K. For each $j \in C$, let

$$c_j = \begin{cases} f_j & \text{if } j \in C_+^\eta \\ -f_j & \text{if } j \in C_-^\eta \end{cases}.$$

Choose $i \in C$ such that

$$\lim_{n\to\infty} c_i(t_n)^{-1} c_j(t_n)$$

exists for each $j \in C$. Let

$$u_n = \beta_C^\eta(c_i(t_n))^{-1}$$

so that $u_n \in K_C$ and $c_j(u_n) = c_i(t_n)^{-1}$ for all $j \in C$.

Recall that, by Corollary 2.33, semistability conditions are constant on the faces of the discriminantal arrangement. We may then assume without loss of generality that

$$\eta = \begin{cases} \theta + f_i & \text{if } i \in C_+^\eta \\ \theta - f_i & \text{if } i \in C_-^\eta \end{cases}.$$

Then

$$\begin{aligned} \eta(t_n u_n) &= \theta(t_n u_n) c_i(t_n u_n) \\ &= \theta(t_n)\theta(u_n) c_i(t_n) c_i(u_n) \\ &= \theta(t_n) \end{aligned}$$

since θ is trivial on K_C and $c_i(u_n) = c_i(t_n)^{-1}$, and so

$$\lim_{n\to\infty} \eta(t_n u_n) = \infty.$$

Next, since $p \in T^*E_C$ is fixed by K_C, we have

$$\lim_{n\to\infty} t_n u_n \cdot p = \lim_{n\to\infty} t_n \cdot p = q.$$

Finally, writing z_i and w_j for the coordinates of v,

$$\begin{aligned} t_n u_n \cdot v &= (c_j(t_n)c_j(u_n)z_j : j \in C_+^\eta; c_j(t_n)c_j(u_n)w_j : j \in C_-^\eta) \\ &= (c_j(t_n)c_i(t_n)^{-1}z_j : j \in C_+^\eta; c_j(t_n)c_i(t_n)^{-1}w_j : j \in C_-^\eta) \end{aligned}$$

which converges as $n \to \infty$ by our choice of i. Hence

$$\lim_{n\to\infty} t_n u_n \cdot (p+v)$$

converges and

$$\lim_{n\to\infty} \eta(t_n u_n) = \infty,$$

which contradicts the assumption that $p+v$ is η-semistable. □

We note that the above lemma is generally not true if θ is not in the closure of the chamber containing η, which we relied on in the above proof when we assumed that $\eta = \theta \pm f_i$ for some $i \in C$.

Proposition 4.9. Let $B_\theta^\eta = \nu^{-1}(B_\theta)$. Then $\nu : B_\theta^\eta \to B_\theta$ is the projectivization of a rank-$|C|$ vector bundle $\mathcal{V}$ over B_θ.

Proof. By Lemma 4.6,

$$\begin{aligned} B_\theta^\eta &= (\mu^{-1}(0)^\eta \setminus \mu^{-1}(0)^{\eta'})/K \\ &= ((T^*E_C \oplus V_C^\eta) \cap \mu^{-1}(0)^\eta)/K. \end{aligned}$$

Let $X = T^*E_C \cap \mu^{-1}(0)^\theta$, so that the quotient map $X \to B_\theta$ is a principal $\overline{K}_C$-bundle. We remark that the 1-parameter subgroup $K_C \subset K$ acts trivially on X and acts on V_C^η by scaling, so that $(V_C^\eta \setminus 0)/K_C = \mathbb{P}(V_C^\eta)$. Then by Lemma 4.8 we have

$$\begin{aligned} B_\theta^\eta &= (X \times (V_C^\eta \setminus 0))/K \\ &= B_\theta \times_{\overline{K}_C} \mathbb{P}(V_C^\eta). \end{aligned}$$

Now let G be a complement to K_C in K, so that $K = K_C \times G$ and $G \cong \overline{K}_C$. Then we can instead write

$$B_\theta^\eta = B_\theta \times_G \mathbb{P}(V_C^\eta),$$

the projectivization of the vector bundle $\mathcal{V} := B_\theta \times_G V_C^\eta$. □

Example 4.10. We return to Example 2.17, with

$$K = \left\{ (s, st^{-1}, t, s^{-1}) : s, t \in \mathbb{C}^\times \right\} \subset (\mathbb{C}^\times)^4.$$

The circuits of the action of K on $T^*\mathbb{C}^4$ are $\{1,2,4\}, \{1,3,4\}$, and $\{2,3\}$, with corresponding discriminantal hyperplanes

$$P_{124} = \mathrm{span}(f_3), P_{134} = \mathrm{span}(f_2), P_{23} = \mathrm{span}(f_1, f_4) = \mathrm{span}(f_1).$$

We choose the regular character $\eta = f_1 + f_2$ and subregular character $\theta = f_1$, the latter of which lies on the wall P_{23}.

The associated arrangement $\mathcal{H}_\theta$ is obtained from $\mathcal{H}_\eta$ by translating the hyperplane H_2 until it coincides with H_3:

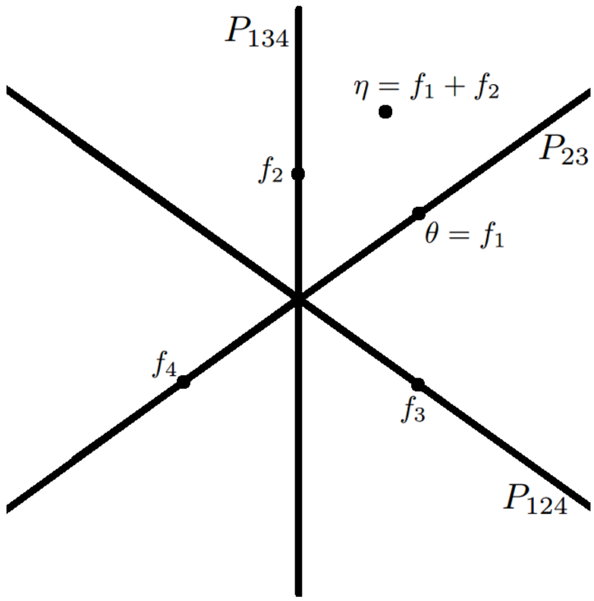


Figure 4.1: The discriminantal arrangement for Example 4.10.

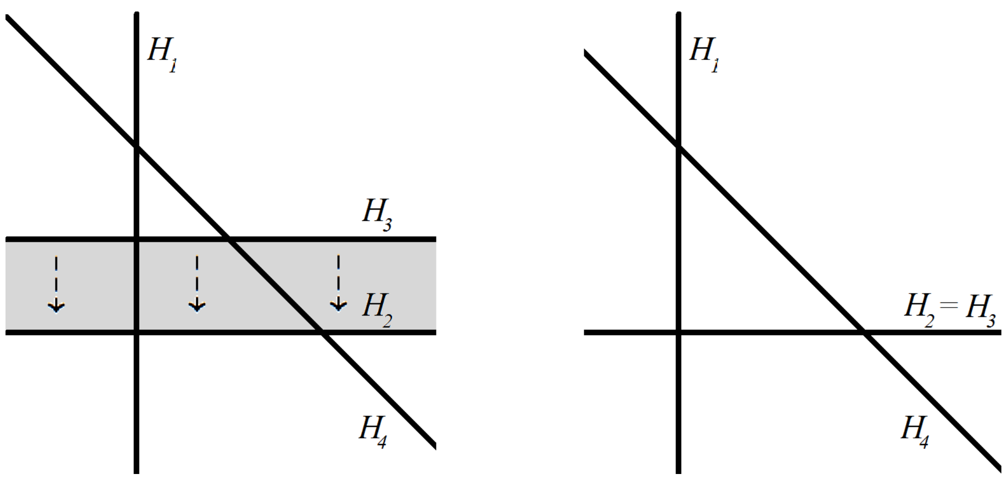


Figure 4.2: $\mathcal{H}_\eta$ (left) and $\mathcal{H}_\theta$.

The associated arrangement for the hypertoric variety $B_\theta \subset M_\theta$ can be seen on the right as the pair of points $H_1 \cap H_2$ and $H_4 \cap H_2$ in the ambient space $H_2 = H_3$, identifying $B_\theta \cong T^*\mathbb{P}^1$. The partial affinization $\nu : M_\eta \to M_\theta$ collapses a component, isomorphic to a Hirzebruch surface, of the core of M_η to the core $\mathbb{P}^1$ of B_θ. This is reflected on the level of moment polytopes above by the collapse of the trapezoidal chamber of $\mathcal{H}^\eta$ to a line segment in $\mathcal{H}^\theta$. The restriction $\nu : B^\eta_\theta \to B_\theta$ is a $\mathbb{P}^1$-bundle, and so the moment polytopes of its fibres are line segments. These are the vertical line segments joining H_2

to H_3 in the diagram on the left.

By the same token, the morphism $\nu' : M_{\eta'} \to M_\theta$ is birational, and its exceptional locus $B_\theta^{\eta'}$ is the projectivization of the vector bundle $B_\theta \times_G V_C^{\eta'}$. Recall that the symplectic pairing ω on V_C identifies V_C^η and $V_C^{\eta'}$ as dual to each other, thereby identifying $B_\theta^{\eta'}$ with the dual projective bundle $\mathbb{P}(\mathcal{V}^*)$.

Our goal is to show that the diagram

$$M_\eta \xrightarrow{\nu} M_\theta \xleftarrow{\nu'} M_{\eta'}$$

is the Mukai flop of M_η (resp. $M_{\eta'}$) along B_θ^η (resp. $B_\theta^{\eta'}$). That is, we need to show that there is a common blowup

$$M_\eta \leftarrow \widetilde{M} \to M_{\eta'}$$

along B_θ^η and $B_\theta^{\eta'}$ respectively, such that these maps are given on the exceptional locus by restricting the projections

$$\mathbb{P}(\mathcal{V}) \leftarrow \mathbb{P}(\mathcal{V}) \times \mathbb{P}(\mathcal{V}^*) \to \mathbb{P}(\mathcal{V}^*).$$

We recall that if $M \to \overline{M} \leftarrow M'$ is a Mukai flop, then this common blowup is one of the two irreducible components of the fibre product $M \times_{\overline{M}} M'$, the other component being the fibre product of the projective bundles along which M and M' are blown up. We will then proceed by analysing the fibre product of this diagram and demonstrating that it has two components, one being the fibre product

$$B_\theta^\eta \times_{B_\theta} B_\theta^{\eta'} = \mathbb{P}(\mathcal{V}) \times_{B_\theta} \mathbb{P}(\mathcal{V}^*)$$

and the other realising the blowup $\widetilde{M}$.

4.2 The fibre product Z

Definition 4.11. Let

$$Z = M_\eta \times_{M_\theta} M_{\eta'}$$

and

$$Z_0 = B_\theta^\eta \times_{B_\theta} B_\theta^{\eta'}.$$

That is,

$$Z_0 = \mathbb{P}(\mathcal{V}) \times_{B_\theta} \mathbb{P}(\mathcal{V}^*),$$

a $\mathbb{P}^{|C|-1} \times \mathbb{P}^{|C|-1}$-bundle over B_θ.

Definition 4.12. Let $Z_1^o = Z \setminus Z_0$, and let Z_1 be the closure of Z_1^o in Z.

Remark 4.13. Recall that the partial affinizations ν and ν' are isomorphisms away from B_θ. It follows that we have a diagram of isomorphisms

$$\begin{array}{ccc} Z_1^o & \longrightarrow & M_{\eta'} \setminus B_\theta^{\eta'} \\ \big\downarrow & & \big\downarrow \nu' \\ M_\eta \setminus B_\theta^\eta & \xrightarrow{\nu} & M_\theta \setminus B_\theta \end{array}$$

with each of these varieties isomorphic to the geometric quotient $(\mu^{-1}(0)^\eta \cap \mu^{-1}(0)^{\eta'})/K$. Explicitly,

$$Z_1^o = \left\{([p+u+v]_\eta, [p+u+v]_{\eta'}) : p+u+v \in \mu^{-1}(0)^\eta \cap \mu^{-1}(0)^{\eta'}\right\}.$$

Lemma 4.14. Let $y = [p]_\theta \in B_\theta$ for $p \in \mu^{-1}(0)^\theta \cap T^*E_C$, and let $(Z_0)_y$ denote the fibre of $Z_0 \to B_\theta$. Then $(Z_0)_y \cap Z_1 \neq \emptyset$.

Proof. We assumed (2.1) that $e_i \notin \mathfrak{k}$ for each i, and so $|C| \geq 2$. Let $k, \ell \in C$ with $k \neq \ell$. We assume without loss of generality that $k \in C_+^\eta$ and $\ell \in C_-^\eta$; the other three cases are similar. Given $u \in V_C^\eta$ and $v \in V_C^{\eta'}$, the coweight $\beta_C^\eta : \mathbb{C}^\times \to K$ acts on $p + u + v$ by

$$\beta_C^\eta(s) \cdot (p + u + v) = p + su + s^{-1}v$$

for $s \in \mathbb{C}^\times$. Then given $p + u + v \in \mu^{-1}(0)^\eta \cap \mu^{-1}(0)^{\eta'}$, we have

$$[p + u + v]_\eta = [p + su + s^{-1}v]_\eta \tag{4.1}$$

and similarly for η'. For $t \in \mathbb{C}^\times$ define $u_t \in V_C^\eta$ by setting $z_k = t$ and all other coordinates to 0, and define $v_t \in V_C^{\eta'}$ by setting $z_\ell = 1$ and all other coordinates to 0. We claim that $p + u_t + v_t \in \mu^{-1}(0)$ for each t. By Proposition 2.24 this is equivalent to satisfying the equation

$$\sum_{i \in S_+^\eta} z_i w_i = \sum_{i \in S_-^\eta} z_i w_i$$

for each circuit S. Since $p \in \mu^{-1}(0)$ we have

$$\sum_{i \in S_+^\eta \setminus C} z_i w_i = \sum_{i \in S_-^\eta \setminus C} z_i w_i$$

and we clearly have $z_i w_i = 0$ for each $i \in C$, and the claim follows. Moreover, since p is θ-semistable, $u_t \neq 0$ and $v_t \neq 0$, we have $p + u + v \in \mu^{-1}(0)^\eta \cap \mu^{-1}(0)^{\eta'}$ by Theorem 2.32, and so

$$([p + u_t + v_t]_\eta, [p + u_t + v_t]_{\eta'}) \in Z_1^o.$$

But by the equation (4.1),

$$([p + u_t + v_t]_\eta, [p + u_t + v_t]_{\eta'}) = ([p + u_1 + v_{t^2}]_\eta, [p + u_{t^2} + v_1]_{\eta'})$$

which, as $t \to 0$, tends to

$$([p + u_1 + 0]_\eta, [p + 0 + v_1]_{\eta'}) \in (Z_0)_y.$$

□

We will show that Z_1 is the simultaneous blowup $\widetilde{M}$ of M_η and $M_{\eta'}$ from the previous section. The key to doing so is the following proposition:

Proposition 4.15. Let

$$I = \{(L, H) \in \mathbb{P}(\mathcal{V}) \times_{B_\theta} \mathbb{P}(\mathcal{V}^*) : L \subset H\},$$

a smooth divisor in Z_0. Then $Z_0 \cap Z_1 = I$.

Proof. For a given point $(z, w) \in T^*\mathbb{C}^n$, we write

$$(z, w) = p + u + v$$

according to the decomposition

$$T^*\mathbb{C}^n = T^*E_C \oplus V_C^\eta \oplus V_C^{\eta'},$$

and similarly $(z', w') = p' + u' + v'$. We then have

$$Z_0 = \left\{([p + u + 0]_\eta, [p' + 0 + v']_{\eta'} : p = p' \in \mu^{-1}(0)^\theta, u \neq 0, v' \neq 0\right\}.$$

Recall that the identification $V_C^{\eta'} = (V_C^\eta)^*$ is given by the symplectic form

$$\omega((z, w), (z', w')) = \sum_{i \in C} z_i w_i' - \sum_{i \in C} z_i' w_i$$

on T^*V_C. We then have

$$I = \{([p+u+0]_\eta, [p+0+v']_{\eta'} \in Z_0 : \omega(u,v') = 0\}.$$

Note that since $z_i = w_i' = 0$ for $i \in C_-^\eta$ and $w_i = z_i' = 0$ for $i \in C_+^\eta$, we actually have

$$\omega(u,v') = \sum_{i \in C_+^\eta} z_i w_i' - \sum_{i \in C_-^\eta} z_i' w_i.$$

We shall first show that $Z_0 \cap Z_1 \subset I$. Choose a complement G to K_C in K, so that $K = K_C \times G$. Projection onto G gives an isomorphism $G \cong \overline{K}_C$ which respects the actions of these tori on T^*E_C. Define the set

$$\widetilde{W} = \left\{(p+u+v, p'+u'+v') \in \mu^{-1}(0)^\eta \times \mu^{-1}(0)^{\eta'} : p, p' \text{ are } \theta\text{-stable}, Gp = Gp'\right\}$$

and let

$$W = \widetilde{W}/(K \times K) \subset M_\eta \times M_{\eta'}.$$

This contains Z_0, which is cut out by the equations $v = 0$ and $u' = 0$. Recall that the hypertoric variety B_θ, defined by the action of $\overline{K}_C$ on T^*E_C, is smooth. It follows by the work of Hausel and Sturmfels [7] that the Lawrence toric variety $T^*E_C /\!\!/_\theta \overline{K}_C$ is smooth, and so the θ-stable points of T^*E_C have trivial stabilizers in $\overline{K}_C$, hence in K. We therefore see that W can alternatively be described as the geometric quotient of

$$\widetilde{W}_1 = \left\{(p+u+v, p'+u'+v') \in \widetilde{W} : p = p'\right\}$$

by the subtorus

$$H = \{(g \cdot t_1, g \cdot t_2 : g \in G, t_1, t_2 \in K_C\}$$

of $K \times K$. The condition

$$\sum_{i \in C_+^\eta} z_i w_i' - \sum_{i \in C_-^\eta} z_i' w_i = 0$$

is closed in $\widetilde{W}$ and invariant under the action of H, and so

$$D := \{([p+u+v]_\eta, [p'+u'+v']_{\eta'}) \in W : \omega(u, v') = 0\}$$

is closed in W.

Now let

$$Z_1^{oo} = \left\{([p+u+v]_\eta, [p+u+v]_{\eta'}) : p+u+v \in \mu^{-1}(0)^\eta \cap \mu^{-1}(0)^{\eta'}, p \text{ is } \theta\text{-stable}\right\},$$

an open subset of Z_1^o. Immediately we have $Z_1^{oo} \subset W$. For $(z, w) = p+u+v \in \mu^{-1}(0)$, we have

$$\sum_{i \in C_+^\eta} z_i w_i' - \sum_{i \in C_-^\eta} z_i' w_i$$

by Proposition 2.24, and so $Z_1^{oo} \subset D$. Hence we have

$$Z_1 \cap W = \overline{Z_1^{oo}} \cap W \subset D$$

since D is closed in W. In particular, $Z_0 \cap Z_1 \subset Z_0 \cap D = I$.

Finally, to see that $Z_0 \cap Z_1 = I$, consider the action of $\mathrm{GL}(V_C^\eta)$ on W given by

$$g \cdot ([p+u+v]_\eta, [p+u'+v']_{\eta'}) = ([p+gu+gv]_\eta, [p+gu'+gv']_{\eta'}),$$

using the usual action on V_C^η and $V_C^{\eta'} = (V_C^\eta)^*$. This is well-defined since p has trivial stabilizer in K. It is clear that Z_0, I and Z_1^{oo} are invariant subsets of W under this action, and hence so is $Z_1 \cap W$, the closure of Z_1^{oo} in W. Thus the intersection $Z_0 \cap Z_1$ is a $\mathrm{GL}(V_C^\eta)$-invariant subset of I. Moreover, the map $Z_0 \to B_\theta$, which we recall is defined by sending $([p+u+0]_\eta, [p+0+v']_{\eta'})$ to $[p]_\theta$, is invariant under this action. Hence for each $y \in B_\theta$, the fibres $(Z_0 \cap Z_1)_y$ and I_y of the restrictions of this map to $Z_0 \cap Z_1$ and I, respectively, are $\mathrm{GL}(V_C^\eta)$-invariant. The action of $\mathrm{GL}(V_C^\eta)$ on I_y is transitive: this corresponds to the fact that, given a finite-dimensional vector space V, the action of $\mathrm{GL}(V)$ on the incidence variety

$$\{(L, H) \in \mathbb{P}(V) \times \mathbb{P}(V^*) : L \subset H\}$$

is transitive. Since $(Z_0 \cap Z_1)_y$ is nonempty by Lemma 4.14, we must then have

$$(Z_0 \cap Z_1)_y = I_y$$

and so $Z_0 \cap Z_1 = I$. □

Proposition 4.16. The maps $M_\eta \leftarrow Z_1 \rightarrow M_{\eta'}$ are the blowups of M_η and $M_{\eta'}$ along B_θ^η and $B_\theta^{\eta'}$, respectively.

Proof. We demonstrate that the projection $\pi : Z_1 \rightarrow M_\eta$ is the blowup of M_η along B_θ^η; the case of $M_{\eta'}$ is similar. It suffices to show that π is an isomorphism away from B_θ^η and that its fibre over each point of B_θ^η is isomorphic to $\mathbb{P}^{\dim M_\eta - \dim B_\theta^\eta - 1}$. The fact that π is an isomorphism away from B_θ^η follows from Remark 4.13.

Let k denote the rank of the torus K. Since M_η is a symplectic quotient of $T^*\mathbb{C}^n$ by K, we have $\dim M_\eta = 2(n-k)$. Recall that B_θ is given by a symplectic quotient of T^*E_C by the rank-$(k-1)$ torus $\overline{K}_C$, and so $\dim B_\theta = 2(n-|C|-(k-1))$ since $\dim E_C = n-|C|$. As B_θ^η is a $\mathbb{P}^{|C|-1}$-bundle over B_θ, we have $\dim B_\theta^\eta = \dim B_\theta + |C| - 1 = 2(n-k) - |C| + 1$. To complete the proof, then, it suffices to show that the fibre of π over each point L of $B_\theta^\eta = \mathbb{P}(\mathcal{V})$ is a projective space of dimension

$$\dim M_\eta - \dim B_\theta^\eta - 1 = |C| - 2.$$

Let $y = \nu(L) \in B_\theta$, so that L is a line in the $|C|-$dimensional vector space $\mathcal{V}_y$. By Proposition 4.15, the fibre $\pi^{-1}(L)$ is isomorphic to

$$\left\{H \in \mathbb{P}(\mathcal{V}_y^*) : L \subset H\right\} \cong \mathbb{P}(\mathcal{V}_y/L) \cong \mathbb{P}^{|C|-2}.$$

□

Theorem 4.17. The diagram $M_\eta \xrightarrow{\nu} M_\theta \xleftarrow{\nu'} M_{\eta'}$ is the Mukai flop of M_η along B_θ^η.

Proof. The hypertoric variety M_η is equipped with an algebraic symplectic form and the codimension of B_θ^η in M_η is $|C|-1$, which equals the dimension of the fibre $\mathbb{P}^{|C|-1}$ of $B_\theta^\eta \to B_\theta$. It follows [9, Section 3] that the normal bundle of B_θ^η in M_η restricts to the cotangent bundle of each fibre of the projective bundle $B_\theta^\eta \to B_\theta$. By Proposition 4.16, the map $Z_1 \to M_\eta$ is the blowup of M_η along B_θ^η, with exceptional divisor $Z_0 \cap Z_1$. By Propostion 4.15, this exceptional divisor is precisely the incidence variety in $\mathbb{P}(\mathcal{V}) \times \mathbb{P}(\mathcal{V}^*)$, and the restrictions of the blowup maps $M_\eta \leftarrow Z_1 \to M_{\eta'}$ are given by projection onto the factors $\mathbb{P}(\mathcal{V})$, $\mathbb{P}(\mathcal{V}^*)$ respectively. □

Corollary 4.18. Let Z denote the fibre product $M_\eta \times_{M_\theta} M_{\eta'}$. Then the Fourier-Mukai transform $\Phi_\eta^{\eta'} : D^b(M_\eta) \to D^b(M_{\eta'})$ with kernel $\mathcal{O}_Z$ is an equivalence of triangulated categories.

Proof. This follows immediately from Theorem 4.17 and Theorem 3.5. □

Remark 4.19. Theorem 4.17 appears originally in a paper of Konno [13, Theorem 6.3]. The strategy taken there is to define a neighbourhood W^+ of B_θ^η, equipped with a map $W^+ \to B_\theta^\eta$ whose restriction to each fibre $\mathbb{P}(\mathcal{V}_y)$ of the projective bundle $B_\theta^\eta \to B_\theta$ is isomorphic to the cotangent bundle $T^*\mathbb{P}(\mathcal{V}_y)$. Similarly a neighbourhood W^- of $B_\theta^{\eta'}$ is defined, and the diagram $M_\eta \to M_\theta \leftarrow M_{\eta'}$ is shown to be a Mukai flop by restricting these maps to the subvarieties $W^\pm$ and comparing to the standard Mukai flop

$$T^*\mathbb{P}(V) \to A(V) \leftarrow T^*\mathbb{P}(V^*)$$

as described in the introduction to this thesis. However, we were not able to make sense of the definition of these varieties W^+ and W^-. Konno appears to define a subset $\widetilde{W}^+ \subset T^*\mathbb{C}^n$ by

$$\widetilde{W}^+ = \left\{ p + u + v : p \in \mu^{-1}(0)^\theta \cap T^*E_C, u \neq 0, \sum_{i \in C} z_i w_i = 0 \right\},$$

which is claimed to be a subset of $\mu^{-1}(0)^\eta$, and W^+ is then defined to be the geometric quotient of $\widetilde{W}^+$ by K. The quotient $\widetilde{W}^+/K$ does indeed have a map to B^η_θ with the property claimed above, defined on the level of $\widetilde{W}^+$ by setting v to 0. But in general the set $\widetilde{W}^+$ as defined above is not contained in $\mu^{-1}(0)^\eta$, and so it is not clear that the quotient $\widetilde{W}^+/K$ embeds in M_η (though it does naturally embed in the Lawrence toric variety $T^*\mathbb{C}^n /\!\!/_\eta K$ and is a neighbourhood of B^η_θ there). We demonstrate this using Example 4.10, using the same values of η and θ, and taking $\eta' = f_1 + f_3$. We have $C = \{2, 3\}$, whence

$$T^*E_C = \{(z_1, z_4, w_1, w_4) : z_1, z_4, w_1, w_4 \in \mathbb{C}\},$$

$$V^\eta_C = \{(z_2, w_3) : z_2, w_3 \in \mathbb{C}\},$$

and

$$V^{\eta'}_C = \{(z_3, w_2) : z_3, w_2 \in \mathbb{C}\}.$$

By Proposition 2.24, we have

$$\mu^{-1}(0) = \left\{ (z, w) \in T^*\mathbb{C}^4 : z_1 w_1 + z_2 w_2 = z_4 w_4, z_2 w_2 = z_3 w_3 \right\},$$

and by Theorem 2.32

$$\mu^{-1}(0)^\eta = \left\{ (z, w) \in \mu^{-1}(0) : (z_1, z_2, w_4) \neq 0, (z_1, z_3, w_4) \neq 0, (z_2, w_3) \neq 0 \right\},$$

$$\mu^{-1}(0)^\theta = \left\{(z,w) \in \mu^{-1}(0) : (z_1, z_2, w_4) \neq 0, (z_1, z_3, w_4) \neq 0\right\}.$$

Then

$$\mu^{-1}(0)^\theta \cap T^*E_C = \{(z_1, z_4, w_1, w_4) : z_1 w_1 = z_4 w_4, (z_1, w_4) \neq 0\}.$$

Choose any $p \in \mu^{-1}(0)^\theta \cap T^*E_C$ and let $u = (1,1) \in V_C^\eta, v = (1,-1) \in V_C^{\eta'}$. Then $p+u+v \in \widetilde{W}^+$, but it does not satisfy the equation $z_1w_1 + z_2w_2 = z_4w_4$ and so is not a point of $\mu^{-1}(0)$.

Chapter 5

Future directions

5.1 Spherical twists

Here we recall some definitions and results of P. Seidel and R.P. Thomas on spherical twists. In this section X is a smooth complex variety and $D^b(X)$ is the bounded derived category of coherent sheaves on X, and $\mathrm{Auteq}(D^b(X))$ is the group of exact autoequivalences of $D^b(X)$.

Definition 5.1. [23, 2.14] An object $\mathcal{E} \in D^b(X)$ is called n**-spherical** ($n > 0$) if

1. $\mathrm{Ext}^*(\mathcal{E}, \mathcal{F})$ and $\mathrm{Ext}^*(\mathcal{F}, \mathcal{E})$ are finite-dimensional for each $\mathcal{F} \in D^b(X)$,

2. $\mathrm{Ext}^i(\mathcal{E}, \mathcal{E}) \cong H^*(S^n, \mathbb{C})$ (the cohomology of the n-sphere), and

3. There is an isomorphism $\mathrm{Hom}(\mathcal{E}, \mathcal{F}) \cong Ext^n(\mathcal{F}, \mathcal{E})^\vee$ which is natural in $\mathcal{F} \in D^b(X)$.

Remark 5.2. If X is assumed to be projective (over $\mathrm{Spec}\,\mathbb{C}$) and $\dim X = n$, this definition is equivalent to the conditions $\mathrm{Ext}^*(\mathcal{E}, \mathcal{E}) \cong H^*(S^n, \mathbb{C})$ and $\mathcal{E} \otimes \omega_X \cong \mathcal{E}$, where ω_X is the canonical bundle of X. We note, however, that hypertoric varieties are not projective over $\mathrm{Spec}\,\mathbb{C}$.

Definition 5.3. For $m \geq 1$, an A_m-configuration in $D^b(X)$ is a collection $\mathcal{E}_1, \ldots, \mathcal{E}_m$ of n-spherical objects such that

$$\dim \operatorname{Ext}^*(\mathcal{E}_i, \mathcal{E}_j) = \begin{cases} 1 & \text{if } |i-j| = 1 \\ 0 & \text{if } |i-j| > 1 \end{cases}.$$

Example 5.4. [23, Example 3.5] Suppose X is a surface and $C_1, \ldots, C_m$ are smooth rational curves in X such that each C_i has self-intersection number -2, $C_i \cap C_j = \emptyset$ for $|i-j| > 1$, and C_i intersects C_{i+1} transversely in a single point for $1 \leq i < m$. Then $\mathcal{O}_{C_1}, \ldots, \mathcal{O}_{C_m}$ is an A_m-configuration in $D^b(X)$.

Definition 5.5. [23] Let $\mathcal{E} \in D^b(X)$. The **twist** around $\mathcal{E}$ is the Fourier-Mukai transform $T_{\mathcal{E}} : D^b(X) \to D^b(X)$ whose kernel is the cone of the evaluation morphism $\mathcal{E}^\vee \boxtimes \mathcal{E} \to \mathcal{O}_\Delta$ where $\Delta \subset X \times X$ is the diagonal of X.

We recall that the **braid group** $\mathcal{B}_{m+1}$ has the Artin presentation

$$\mathcal{B}_{m+1} = \langle \sigma_1, \ldots, \sigma_m | \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \text{ for } 1 \leq i < m,\ \sigma_i \sigma_j = \sigma_j \sigma_i \text{ for } |i-j| > 1 \rangle.$$

Theorem 5.6. (Seidel-Thomas [23, Theorem 2.17, Theorem 2.18])

(i) If $\mathcal{E}$ is an n-spherical object in $D^b(X)$ then $T_{\mathcal{E}}$ is an equivalence.

(ii) If $\mathcal{E}_1, \ldots, \mathcal{E}_m$ is an A_m-configuration in $D^b(X)$, then the twists $T_{\mathcal{E}_i}$ satisfy the braid relations

$$\begin{aligned} T_{\mathcal{E}_i} T_{\mathcal{E}_{i+1}} T_{\mathcal{E}_i} &\cong T_{\mathcal{E}_{i+1}} T_{\mathcal{E}_i} T_{\mathcal{E}_{i+1}} && \text{for } 1 \leq i < m \\ T_{\mathcal{E}_i} T_{\mathcal{E}_j} &\cong T_{\mathcal{E}_j} T_{\mathcal{E}_i} && \text{for } |i-j| > 1 \end{aligned}$$

and so there is a well-defined group homomorphism $\mathcal{B}_{m+1} \to \operatorname{Auteq}(D^b(X))$ mapping the generator σ_i to $T_{\mathcal{E}_i}$. Moreover, this homomorphism is injective.

5.2 The type-A_m Kleinian singularity

We recall that the type-A_m Kleinian singularity is the quotient of $\mathbb{C}^2$ by the cyclic group $\mathbb{Z}_{m+1} \subset \mathrm{SL}_2(\mathbb{C})$, which embeds in $\mathbb{C}^3$ as

$$\mathbb{C}^2/\mathbb{Z}_{m+1} = \left\{(x,u,v) \in \mathbb{C}^3 : x^{m+1} + uv = 0\right\}.$$

The origin 0 is the unique singular point of $\mathbb{C}^2/\mathbb{Z}_{m+1}$. As originally shown by du Val [6], the exceptional fibre of the minimal resolution of singularities $\widetilde{\mathbb{C}^2/\mathbb{Z}_{m+1}} \to \mathbb{C}^2/\mathbb{Z}_{m+1}$ is a union $C_1 \cup \cdots \cup C_m$ of smooth rational curves $C_i \cong \mathbb{P}^1$ which satisfy the hypothesis of Example 5.4. Then where $\mathcal{E}_i = \mathcal{O}_{C_i}$, Theorem 5.6 tells us that the spherical twists $T_{\mathcal{E}_i}$ give a faithful action of the braid group on $D^b(\widetilde{\mathbb{C}^2/\mathbb{Z}_{m+1}})$.

Recall from Example 2.10 that $\widetilde{\mathbb{C}^2/\mathbb{Z}_{m+1}}$ can be constructed as a hypertoric variety M_η, where

$$K = \left\{(t_1,\ldots,t_{m+1}) \in (\mathbb{C}^\times)^{m+1} : t_1 \cdots t_{m+1} = 1\right\}$$

and η is any regular character; the resolution $\widetilde{\mathbb{C}^2/\mathbb{Z}_{m+1}} \to \mathbb{C}^2/\mathbb{Z}_{m+1}$ is the affinization map $M_\eta \to M_0$. We will fix $\eta = f_1 + 2f_2 + \cdots + (m+1)f_{m+1}$ throughout this section. For this choice of η, the discriminantal hyperplanes bounding the chamber containing η are $P_{i,i+1}$ for $1 \le i \le m$. We fix subregular characters $\theta_i \in P_{i,i+1}$ in the closure of that chamber. For each i, we have

$$B^\eta_{\theta_i} = \{[z,w]_\eta : w_i = z_{i+1} = 0\}$$

which can be seen to be isomorphic to $\mathbb{P}^1$ by the projective coordinates $[z_i, w_{i+1}]$. Indeed, these subvarieties $B^\eta_{\theta_i}$ are precisely the curves C_i above, and the partial affinization $M_\eta \to M_{\theta_i}$ contracts C_i to a point.

Let us fix $k \in \{1,\ldots,m\}$ and let η' be the reflection of η in $P_{k,k+1}$. We have

$$\mu^{-1}(0) = \left\{(z, w) \in T^*\mathbb{C}^{m+1} : z_1 w_1 = \cdots = z_{m+1} w_{m+1}\right\},$$

$$\mu^{-1}(0)^\eta = \left\{(z, w) \in \mu^{-1}(0) : (z_i, w_j) \neq 0 \text{ for } i < j\right\},$$

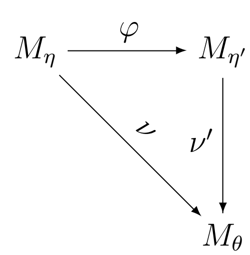
and

$$\mu^{-1}(0)^{\eta'} = \left\{(z, w) \in \mu^{-1}(0) : (z_{k+1}, w_k) \neq 0 \text{ and } (z_i, w_j) \neq 0 \text{ for } i < j, (i, j) \neq (k, k+1)\right\}.$$

We define a map $\widetilde{\varphi} : \mu^{-1}(0)^\eta \to \mu^{-1}(0)^{\eta'}$ by interchanging z_k with z_{k+1} and w_k with w_{k+1}. This map $\widetilde{\varphi}$ is not K-invariant, but it does descend to a morphism $\varphi : M_\eta \to M_{\eta'}$ since for $t = (t_1, \ldots, t_{m+1}) \in K$, we have

$$\widetilde{\varphi}(t \cdot (z, w)) = \sigma_k(t)\widetilde{\varphi}(z, w)$$

where σ_k is the automorphism of K which interchanges t_k and t_{k+1}, and so points in the same K-orbit are sent by $\widetilde{\varphi}$ to points in the same K-orbit. This morphism φ is an isomorphism since we can clearly define an inverse in a similar way, and we have a commutative diagram



Such an isomorphism φ exists in the general situation of a Mukai flop of hypertoric varieties whenever the relevant circuit has exactly two elements; see [14, 6.6 (2)]. Recall that the fibre product $Z = M_\eta \times_{M_{\theta_k}} M_{\eta'}$ has two irreducible components Z_0 and Z_1. Identifying $M_\eta \times M_{\eta'}$ with $M_\eta \times M_\eta$ by way of φ, the component Z_1 becomes the diagonal copy of M_η, and Z_0 becomes the product $B^\eta_{\theta_k} \times B^\eta_{\theta_k}$.

Recall that we denote by $\Phi_\eta^{\eta'} : D^b(M_\eta) \to D^b(M_{\eta'})$ the Fourier-Mukai transform with kernel $\mathcal{O}_Z$. It should be straightforward to show that $\Phi_\eta^{\eta'} \cong \varphi_* \circ T_{\mathcal{E}_k}$, where $\mathcal{E}_k$ denotes the structure sheaf of $B^\eta_{\theta_k}$. In particular, this would give an alternate proof that $\Phi_\eta^{\eta'}$ is an equivalence.

5.3 $\mathbb{P}^n$-objects and $\mathbb{P}^n$-functors

For a general Mukai flop $M_\eta \to M_\theta \leftarrow M_{\eta'}$ of smooth hypertoric varieties, the structure sheaf of B^η_θ is not always a spherical object of $D^b(M_\eta)$ and so there is no well-defined spherical twist along this object. However, there is an analogous kind of autoequivalence called a $\mathbb{P}^n$-twist, and we conjecture that the projective bundle $B^\eta_\theta \to B_\theta$ can be used to construct such an autoequivalence.

Recall that when X is a smooth projective complex variety, Remark 5.2 gives us a simple definition of a spherical object of $D^b(X)$ in terms of the cohomology of the sphere. As articulated by Huybrechts and Thomas [10], there is an analogous definition of a $\mathbb{P}^n$-**object** of $D^b(X)$ when X is projective: namely, that $\mathcal{E}$ is a $\mathbb{P}^n$-object if $\mathcal{E} \otimes \omega_X \cong \mathcal{E}$ and $\operatorname{Ext}^*(\mathcal{E}, \mathcal{E})$ is isomorphic to $H^*(\mathbb{P}^n, \mathbb{C})$ as a graded ring, where $\dim X = 2n$.

More generally, for possibly non-projective varieties X, starting from the observation that an object $\mathcal{E}$ of $D^b(X)$ can be identified with the functor

$$\mathcal{E} \otimes - : D^b(\text{point}) \to D^b(X),$$

Addington defined a relative version of $\mathbb{P}^n$-object called a $\mathbb{P}^n$-functor.

Definition 5.7. [1, 3.1] Let $\mathcal{A}$ and $\mathcal{B}$ be triangulated categories. A $\mathbb{P}^n$**-functor** is a functor $F : \mathcal{A} \to \mathcal{B}$ with left and right adjoints L and R such that

1. There is an autoequivalence H of $\mathcal{A}$ such that

$$RF \cong \mathrm{id} \oplus H \oplus H^2 \oplus \cdots \oplus H^n.$$

2. Let $\epsilon : FR \to 1$ be the counit of the adjunction. The map

$$HRF \hookrightarrow RFRF \overset{R\epsilon F}{\to} RF,$$

when written in components

$$H \oplus H^2 \oplus \cdots \oplus H^n \oplus H^{n+1} \to \mathrm{id} \oplus H \oplus H^2 \oplus \cdots \oplus H^n,$$

is of the form

$$\begin{pmatrix} * & * & \cdots & * & * \\ 1 & * & \cdots & * & * \\ 0 & 1 & \cdots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & * \end{pmatrix}.$$

3. $R \cong H^n L$.

Remark 5.8. We can then say that $\mathcal{E}$ is a $\mathbb{P}^n$**-object** of $D^b(X)$ if $\mathcal{E} \otimes -$ is a $\mathbb{P}^n$-functor. For example, the structure sheaf of the zero section of $T^*\mathbb{P}^n$ is a $\mathbb{P}^n$-object in $D^b(T^*\mathbb{P}^n)$. More generally, the structure sheaf of a Lagrangian $\mathbb{P}^n$ in a holomorphic symplectic variety is a $\mathbb{P}^n$-object [1, 3.1].

Remark 5.9. In the above definition, typically each of the categories $\mathcal{A}$ and $\mathcal{B}$ is the derived category of coherent sheaves on a variety, and $H = [-2]$.

Remark 5.10. Just as this is a relative version of "$\mathbb{P}^n$-object," there is a relative version of "spherical object" known as a spherical functor [21], [2].

Given a $\mathbb{P}^n$-functor $F : \mathcal{A} \to B$, Addington [1, 3.3] constructs an autoequivalence of $\mathcal{B}$ called the **($\mathbb{P}^n$-)twist** along F. This is analogous to the spherical twist as defined above.

Suppose $q : E \to Y$ is a $\mathbb{P}^n$-bundle and $i : E \to \Omega^1_q$ is the zero section of the relative cotangent bundle of q. As shown by Addington [1, 3.2.4], the composition $i_* q^* : D^b(Y) \to D^b(\Omega^1_q)$ is a $\mathbb{P}^n$-functor. It should be easy to modify his proof to establish that, if $q : B^\eta_\theta \to B_\theta$ is the projective bundle defined in Proposition 4.9 and $i : B^\eta_\theta \to M_\eta$ is the inclusion, then $i_* q^* : D^b(B_\theta) \to D^b(M_\eta)$ is a $\mathbb{P}^n$-functor. This is certainly true in the case that B_θ is a point, as B^η_θ is then a Lagrangian $\mathbb{P}^n$ in M_η and its structure sheaf is therefore a $\mathbb{P}^n$-object. We expect the twist along the $\mathbb{P}^n$-functor $i_* q^*$ to be isomorphic to the composition $\Phi^\eta_{\eta'} \circ \Phi^{\eta'}_\eta$.

5.4 The pure braid group

Let S_{m+1} denote the symmetric group on $\{1, \ldots, m+1\}$, generated by the simple transpositions $s_i = (i\ \ i+1)$ for $1 \leq i \leq m$. Then we have a homomorphism $\mathcal{B}_{m+1} \to S_{m+1}$ defined by $\sigma_i \to s_i$, the kernel of which is the **pure braid group** $\mathcal{PB}_{m+1}$. The homomorphism $\mathcal{B}_{m+1} \to \operatorname{Auteq}(D^b(M_\eta))$, defined by mapping σ_i to the twist $T_{\mathcal{E}_i}$, then restricts to an action of $\mathcal{PB}_{m+1}$ on $D^b(M_\eta)$.

This pure braid group $\mathcal{PB}_{m+1}$ arises naturally from the hypertoric perspective. We first note that $\mathcal{PB}_{m+1}$ can be realized as the fundamental group of the complement of the **braid arrangement**

$$\mathcal{A} = \bigcup_{1 \leq i < j \leq m+1} H_{ij}$$

where

$$H_{ij} = \left\{ (x_1, \ldots, x_{m+1}) \in \mathbb{C}^{m+1} : x_i = x_j \right\}.$$

We denote the complement of the braid arrangement by $\mathcal{A}^c = \mathbb{C}^{m+1} \setminus \mathcal{A}$. Recall that the circuits of the action of K on $T^*\mathbb{C}^{m+1}$ are the unordered pairs $\{i, j\}$ with $1 \leq i < j \leq$

$m+1$, and the complexified discriminantal hyperplanes are

$$P_{ij} = \left\{ \sum_{i=1}^{m+1} \lambda_i f_i : \lambda_i = \lambda_j \right\}$$

in the ambient space

$$\mathfrak{k}^* = \mathrm{span}_{\mathbb{C}}(f_1, \ldots, f_{m+1}) / \mathrm{span}_{\mathbb{C}}(\sum_i f_i).$$

Let $\pi : \mathbb{C}^{m+1} \to \mathfrak{k}^*$ be the linear projection

$$\pi(x_1, \ldots, x_{m+1}) = \sum_{i=1}^{m+1} x_i f_i.$$

Then where

$$\Upsilon_{\mathbb{C}} := \mathfrak{k}^* \setminus \bigcup_{i \neq j} P_{ij}$$

is the complement to the complexified discriminantal arrangement, the restriction of π to $\mathcal{A}^c$ has image $\Upsilon_{\mathbb{C}}$, and indeed $\pi : \mathcal{A}^c \to \Upsilon_{\mathbb{C}}$ is a trivial line bundle and, in particular, a homotopy equivalence. We can hence realize the pure braid group $\mathcal{PB}_{m+1}$ as the fundamental group of $\Upsilon_{\mathbb{C}}$. We expect that, in general, the Fourier-Mukai transforms $\Phi_\eta^{\eta'}$ satisfy the appropriate relations to give rise to an action of the fundamental group of the complement $\Upsilon_{\mathbb{C}}$ of the complexified discriminantal arrangement. More generally, we expect to obtain an action of the Deligne groupoid of the discriminantal arrangement on the categories $D^b(M_\eta)$. We discuss this hope in more detail in the final section.

5.5 Representation of the Deligne groupoid

For each chamber Y of the real discriminantal arrangement, fix a character $\eta_Y \in Y \cap \mathfrak{k}^*_{\mathbb{Z}}$. Let Θ be the set of all such characters η_Y. Recall that the associated hypertoric variety $M_Y := M_{\eta_Y}$ does not depend on the choice of η_Y since the semistability conditions are constant on faces of the discriminantal arrangement.

Definition 5.11. Let $\Upsilon_{\mathbb{C}}$ be the complexification of the complement of the discriminantal arrangement. The **Deligne groupoid** $\mathbb{G} := \Pi_1(\Upsilon_{\mathbb{C}}, \Theta)$ is the full subcategory of the fundamental groupoid of $\Upsilon_{\mathbb{C}}$ on Θ.

That is, the objects of $\mathbb{G}$ are the points of Θ, and for $\eta, \eta' \in \Theta$, the set of morphisms $\mathrm{Hom}_{\mathbb{G}}(\eta, \eta')$ is the set of paths from η to η' in $\Upsilon_{\mathbb{C}}$ up to homotopy. Composition of morphisms is defined by concatenation of paths. As each chamber is simply connected, this groupoid is independent, up to canonical isomorphism, of the choices of η_Y.

Salvetti [22] constructs a CW-complex $X \subset \Upsilon_{\mathbb{C}}$, the inclusion of which is a homotopy equivalence. The 1-skeleton X_1 is the directed graph on $X_0 = \Theta$ with arcs in both directions between η and η' if and only if η and η' lie in adjacent chambers. The inclusion of X into $\Upsilon_{\mathbb{C}}$ then induces an isomorphism $\Pi_1(X, \Theta) \cong \mathbb{G}$. We then have a distinguished set of generators of this groupoid, namely the arcs of X_1. The 2-cells of X, which give the relations on these generators, are indexed by pairs (η, F) where F is a codimension-2 face of the discriminantal arrangement and $\eta \in \Theta$, as follows: let $\overline{\eta}$ be the opposite of η with respect to F and let Γ_1, Γ_2 be the minimal directed paths in X_1 joining η to $\overline{\eta}$. Then the boundary of the corresponding 2-cell is $\Gamma_1 \cup \Gamma_2$.

For each arc in X_1 from η to η', we have previously defined a Fourier-Mukai transform $\Phi_{\eta}^{\eta'} : D(M_\eta) \to D(M_{\eta'})$.

Conjecture 5.12. Let $\mathcal{C}$ be the groupoid whose objects are the categories $D^b(M_\eta)$ for $\eta \in \Theta$ and whose morphisms are the equivalences between these categories, up to natural isomorphism. Then there is a unique functor $\Pi_1(X, \Theta) \to \mathcal{C}$ which assigns to each arc $\eta \to \eta'$ in X_1 the equivalence $\Phi_{\eta}^{\eta'}$.

Given a directed path $\Gamma = (\eta_1, \eta_2, \dots, \eta_m)$ in X_1, define the composition

$$\Phi_\Gamma = \Phi_{\eta_{m-1}}^{\eta_m} \circ \cdots \circ \Phi_{\eta_2}^{\eta_3} \circ \Phi_{\eta_1}^{\eta_2} : D(M_{\eta_1}) \to D(M_{\eta_m}).$$

In light of the above description of the 2-cells of X, to prove Conjecture 5.12 it would suffice to show that if Γ_1 and Γ_2 are the minimal paths in X_1 joining η to $\overline{\eta}$ (so $\eta = \eta_1$ and $\overline{\eta} = \eta_m$), then the functors Φ_{Γ_1} and Φ_{Γ_2} are naturally isomorphic.

Assuming that these functors do indeed form a representation of $\mathbb{G}$ in this way, for each $\eta \in \Theta$ we can then restrict this action to $\pi_1(\Upsilon_{\mathbb{C}}, \eta)$, the fundamental group of $\Upsilon_{\mathbb{C}}$ based at η, and thereby obtain a representation of this group on the category $D^b(M_\eta)$, *i.e.* a homomorphism into the group $\mathrm{Auteq}(D^b(M_\eta))$ of self-equivalences of $D^b(M_\eta)$, thus generalizing the action of $\mathcal{PB}_{m+1}$ obtained by Seidel and Thomas.

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