ON THE KHOVANOV HOMOLOGY OF AN ADEQUATE LINK

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Abstract. We use the concept of alternating planar algebra to study the Khovanov homology of adequate links.

Contents

1. Introduction

The Adequate links was introduced by Lickorish and Thistlethwaite in [Lick] to study the Jones Polynomial. Here we are going to use the the technique introduced in [Bur] and [Bur1] to study the Khovanov homology of this type of links. We call this type of chain complexes *diagonal complexes*. Furthermore, a *coherently diagonal* complex is a diagonal complex whose partial closure is also diagonal. Complexes of this type are the objects in the following theorem

Theorem 1. *Coherently diagonal complexes form an alternating planar algebra (that is, they are closed under "horizontal compositions" in alternating planar diagrams).*

Our second theorem follows from the first; for it reduces that proof to the simple task of verifying that the Khovanov homologies of the one-crossing tangles ($\overcrossing$) and ($\undercrossing$) (which are obviously alternating) are coherently diagonal:

Theorem 2. *Let T be a non-split alternating $2k$-boundary tangle ($k > 0$), then the Khovanov homology $\mathrm{Kh}(T)$ can be interpreted as a coherently diagonal complex.*

In the case of alternating tangles with no boundary, i.e., in the case of alternating links, this result reduces to Lee's theorem on the Khovanov homology of alternating links.

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The work is organized as follows. In section 3, we review Bar-Natan local Khovanov theory and present two additional tools for the proof of theorem 1. These tools are propositions 3.5 and 3.7. Section 4 is devoted to introduce the category $\mathcal{C}ob^3_o$ an give a quick review of some concepts related to alternating planar algebras. In particular we review the concepts of rotation number, alternating planar diagram, associated rotation number, and basic operators.

Section 5 introduces the concepts of diagonal complexes, coherently diagonal complexes, and their partial closures. We state here some results about the complexes obtained when a basic operator is applied to alternating elements, leading to the prove in section **??** of Theorem 1. Finally section **??** is dedicated to the study of non-split alternating tangles. Here, we prove Theorem 2 and derive from it Lee theorem formulated in [Lee].

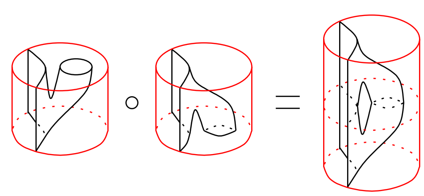
2. Acknowledgement

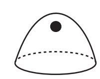
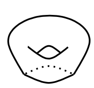
I wish to thank D. Bar-Natan, for many helpful conversations we had at the University of Toronto and for allowing me to use some figures from [BN1, BN2]. I would also like to thank N. Martin for their comments and suggestions.

3. The local Khovanov theory and the alternating planar algebras

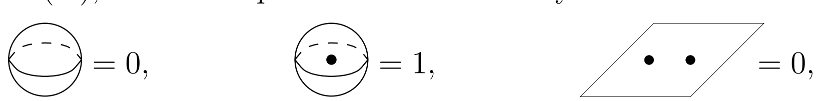
The notation and some results appearing here are treated in more details in [BN1, BN2, Naot]. Given a set B of $2k$ marked points on a circle C, a smoothing with boundary B is a union of strings $a_1, ..., a_n$ embedded in the plane disk for which C is the boundary, such that $\cup_{i=1}^n \partial a_i = B$. These strings are either closed curves, *loops*, or strings whose boundaries are points on B, *strands*. If $B = \emptyset$, the smoothing is a union of circles.

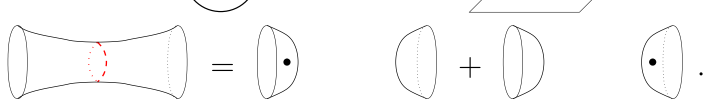
We denote $Cob^3(B)$, the category whose objects are smoothings with boundary B, and whose morphisms are cobordisms between such smoothings, regarded up to boundary preserving isotopy. The composition of morphisms is given by placing the second cobordism atop the other.



Our ground ring is one in which 2^{-1} exists. The dotted figure  is used as an abbreviation of $\frac{1}{2}$  and $\mathcal{C}ob^3_{\bullet/l}(B)$ represents the category with the same objects and morphisms as $Cob^3(B)$, whose morphisms are mod out by the local relations:

(1)



and 

We will use the notation $\mathcal{C}ob^3$ and $\mathcal{C}ob^3_{\bullet/l}$ as a generic reference, namely, $\mathcal{C}ob^3 = \bigcup_B \mathcal{C}ob^3(B)$ and $\mathcal{C}ob^3_{\bullet/l} = \bigcup_B \mathcal{C}ob^3_{\bullet/l}(B)$. If B has $2k$ elements, we usually write $\mathcal{C}ob^3_{\bullet/l}(k)$ instead of

$\mathcal{C}ob^3_{\bullet/l}(B)$. If $\mathcal{C}$ is any category, $\mathrm{Mat}(\mathcal{C})$ will be the additive category whose object are column vectors (formal direct sums) whose elements are formal $\mathbb{Z}$-linear combinations of $\mathcal{C}$. Given two objects in this category,

$$\mathcal{O} = \begin{pmatrix} \mathcal{O}_1 \\ \mathcal{O}_2 \\ \vdots \\ \mathcal{O}_n \end{pmatrix} \qquad \mathcal{O}^1 = \begin{pmatrix} \mathcal{O}^1_1 \\ \mathcal{O}^1_2 \\ \vdots \\ \mathcal{O}^1_m \end{pmatrix},$$

the morphisms between these objects will be matrices whose entries are formal sums of morphisms between them. The morphisms in this additive category are added using the usual matrix addition and the morphism composition is modeled by matrix multiplication, i.e, given two appropriate morphisms $F = (F_{ik})$ and $G = (G_{kj})$ between objects of this category, then $F \circ G$ is given by

$$F \circ G = \sum_k F_{ik} G_{kj} \,,$$

$\mathrm{Kom}(\mathcal{C})$ will be the category of formal complexes over an additive category $\mathcal{C}$. $\mathrm{Kom}_{/h}(\mathcal{C})$ is $\mathrm{Kom}(\mathcal{C})$ modulo homotopy. We also use the abbreviations $\mathrm{Kob}(k)$ and $\mathrm{Kob}_{/h}(k)$ for denoting $\mathrm{Kom}(\mathrm{Mat}(\mathcal{C}ob^3_{\bullet/l}(k)))$ and $\mathrm{Kom}_{/h}(\mathrm{Mat}(\mathcal{C}ob^3_{\bullet/l}(k)))$.

Objects and morphisms of the categories $\mathcal{C}ob^3$, $\mathcal{C}ob^3_{\bullet/l}$, $\mathrm{Mat}(\mathcal{C}ob^3_{\bullet/l})$, $\mathrm{Kob}(k)$, and $\mathrm{Kob}_{/h}(k)$ can be seen as examples of planar algebras, i.e., if D is a n-input planar diagram, it defines an operation among elements of the previously mentioned collections. See [BN1] for specifics of how D defines operations in each of these collections. In particular, if $(\Omega_i, d_i) \in \mathrm{Kob}(k_i)$ are complexes, the complex $(\Omega, d) = D(\Omega_1, \dots, \Omega_n)$ is defined by

$$\begin{aligned} \Omega^r &:= \bigoplus_{r = r_1 + \cdots + r_n} D(\Omega_1^{r_1}, \dots, \Omega_n^{r_n}) \\ d|_{D(\Omega_1^{r_1}, \dots, \Omega_n^{r_n})} &:= \sum_{i=1}^{n} (-1)^{\sum_{j<i} r_j} D(I_{\Omega_1^{r_1}}, \dots, d_i, \dots, I_{\Omega_n^{r_n}}), \end{aligned} \tag{2}$$

$D(\Omega_1, \dots, \Omega_n)$ is used here as an abbreviation of $D((\Omega_1, d_1), \dots, (\Omega_n, d_n))$.

In [BN1] the following very desirable property is also proven. The Khovanov homology is a planar algebra morphism between the planar algebras $\mathcal{T}(s)$ of oriented tangles and $\mathrm{Kob}_{/h}(k)$. That is to say, for an n-input planar diagram D, and suitable tangles $T_1, ..., T_n$, we have

$$Kh(D(T_1, ..., T_n)) = D(Kh(T_1), ..., Kh(T_n)). \tag{3}$$

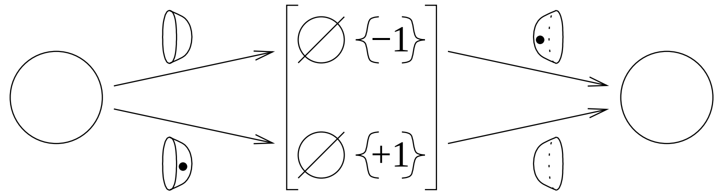
This last property is used in [BN2] to show a local algorithm for computing the Khovanov homology of a link. In that paper, Bar-Natan explained how it is possible to remove the loops in the smoothings, and some terms in the Khovanov complex $Kh(T_i)$ associated to the local tangles $T_1, ..., T_n$, and then combine them together in an n-input planar diagram D obtaining $D(Kh(T_1), ..., Kh(T_n))$, and the Khovanov homology of the original tangle.

The elimination of loops and terms can be done thanks to the following: Lemma 4.1 and Lemma 4.2 in [BN2]. We copy these lemmas verbatim:

Lemma 3.1. *(Delooping) If an object S in $\mathcal{C}ob^3_{\bullet/l}$ contains a closed loop ℓ, then it is isomorphic (in $\mathrm{Mat}(\mathcal{C}ob^3_{\bullet/l})$) to the direct sum of two copies $S'\{+1\}$ and $S'\{-1\}$ of S in which ℓ is* *removed, one taken with a degree shift of* $+1$ *and one with a degree shift of* -1*. Symbolically, this reads* $\bigcirc \equiv \emptyset\{+1\} \oplus \emptyset\{-1\}$.

The isomorphisms for the proof can be seen in:

(4)



using all the relations in (1).

Lemma 3.2. *(Gaussian elimination, made abstract) If* $\phi : b_1 \to b_2$ *is an isomorphism (in some additive category* $\mathcal{C}$*), then the four term complex segment in* $\mathrm{Mat}(\mathcal{C})$

$$\cdots\ [C] \xrightarrow{\begin{pmatrix}\alpha\\ \beta\end{pmatrix}} \begin{bmatrix} b_1\\ D\end{bmatrix} \xrightarrow{\begin{pmatrix}\phi & \delta\\ \gamma & \epsilon\end{pmatrix}} \begin{bmatrix} b_2\\ E\end{bmatrix} \xrightarrow{\begin{pmatrix}\mu & \nu\end{pmatrix}} [F]\ \cdots \tag{5}$$

is isomorphic to the (direct sum) complex segment

$$\cdots\ [C] \xrightarrow{\begin{pmatrix}0\\ \beta\end{pmatrix}} \begin{bmatrix} b_1\\ D\end{bmatrix} \xrightarrow{\begin{pmatrix}\phi & 0\\ 0 & \epsilon-\gamma\phi^{-1}\delta\end{pmatrix}} \begin{bmatrix} b_2\\ E\end{bmatrix} \xrightarrow{\begin{pmatrix}0 & \nu\end{pmatrix}} [F]\ \cdots. \tag{6}$$

Both these complexes are homotopy equivalent to the (simpler) complex segment

$$\cdots\ [C] \xrightarrow{(\beta)} [D] \xrightarrow{\left(\epsilon-\gamma\phi^{-1}\delta\right)} [E] \xrightarrow{(\nu)} [F]\ \cdots. \tag{7}$$

Here C*,* D*,* E *and* F *are arbitrary columns of objects in* $\mathcal{C}$ *and all Greek letters (other than* ϕ*) represent arbitrary matrices of morphisms in* $\mathcal{C}$ *(having the appropriate dimensions, domains and ranges); all matrices appearing in these complexes are block-matrices with blocks as specified.* b_1 *and* b_2 *are billed here as individual objects of* $\mathcal{C}$*, but they can equally well be taken to be columns of objects provided (the morphism matrix)* ϕ *remains invertible.*

It will be useful for our purpose to enunciate also the following lemma which is easily demonstrable using the obvious morphism of complexes.

Lemma 3.3. *If* B *is an object of* $\mathrm{Mat}(\mathcal{C})$ *involved in a chain complex* Ω*, then it is possible to interchange the position of two elements* b_i*,* b_j *of* B *obtaining a homotopy equivalent complex. This interchange also changes the position of the* i*-th and* j*-th rows of the morphism pointing at* B *and the* i*-th and* j*-th columns of the morphism coming from* B*. In other words, the* *three term complex segment in* $\mathrm{Mat}(\mathcal{C})$

$$\cdots \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \xrightarrow{\begin{pmatrix} \alpha_{11} & \dots & \alpha_{1n} \\ \vdots & & \vdots \\ \alpha_{i1} & \dots & \alpha_{in} \\ \vdots & & \vdots \\ \alpha_{j1} & \dots & \alpha_{jn} \\ \vdots & & \vdots \\ \alpha_{m1} & \dots & \alpha_{mn} \end{pmatrix}} \begin{bmatrix} b_1 \\ \vdots \\ b_i \\ \vdots \\ b_j \\ \vdots \\ b_m \end{bmatrix} \xrightarrow{\begin{pmatrix} \beta_{11} & \dots & \beta_{1i} & \dots & \beta_{1j} & \dots & \beta_{1m} \\ \vdots & & \vdots & & \vdots & & \vdots \\ \beta_{p1} & \dots & \beta_{pi} & \dots & \beta_{pj} & \dots & \beta_{pm} \end{pmatrix}} \begin{bmatrix} c_1 \\ \vdots \\ c_p \end{bmatrix} \cdots \tag{8}$$

is isomorphic to

$$\cdots \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \xrightarrow{\begin{pmatrix} \alpha_{11} & \dots & \alpha_{1n} \\ \vdots & & \vdots \\ \alpha_{j1} & \dots & \alpha_{jn} \\ \vdots & & \vdots \\ \alpha_{i1} & \dots & \alpha_{in} \\ \vdots & & \vdots \\ \alpha_{m1} & \dots & \alpha_{mn} \end{pmatrix}} \begin{bmatrix} b_1 \\ \vdots \\ b_j \\ \vdots \\ b_i \\ \vdots \\ b_m \end{bmatrix} \xrightarrow{\begin{pmatrix} \beta_{11} & \dots & \beta_{1j} & \dots & \beta_{1i} & \dots & \beta_{1m} \\ \vdots & & \vdots & & \vdots & & \vdots \\ \beta_{p1} & \dots & \beta_{pj} & \dots & \beta_{pi} & \dots & \beta_{pm} \end{pmatrix}} \begin{bmatrix} c_1 \\ \vdots \\ c_p \end{bmatrix} \cdots \tag{9}$$

where every Latin and Greek letter represents respectively a smoothing or a cobordism.

From the three previous lemmas we infer that the Khovanov complex of a tangle is homotopy equivalent to a chain of complex without loops in the smoothings, and in which every differential is a non-invertible cobordism. In other words, if (Ω, d) is a complex in $\mathcal{C}ob^3_{\bullet/l}$, we can use lemmas 3.1, 3.2 and 3.3, and obtain a homotopy equivalent chain complex (Ω', d') with no loop in its smoothings and no invertible cobordism in its differentials. We say that (Ω', d') is a *reduced complex* of (Ω, d) .

For our purposes, it will be useful to recall here the concept of *bounded* chain complex. See [Wei]. A chain complex

$$\Omega : \qquad \dots \Omega^r \xrightarrow{d^r} \Omega^{r+1} \dots$$

is called *bounded* if almost all the Ω^r are zero. If $\Omega^{r_o} \neq 0$, $\Omega^{r_M} \neq 0$ and $\Omega^r = 0$ unless $r_0 \leq r \leq r_M$, we say that (Ω, d) has *amplitude* in $[r_0, r_M]$.

Definition 3.4. Let (Ω, d) be a bounded chain complex in Kob with amplitude in $[r_0, r_M]$. Let $\Omega^r = \begin{bmatrix} \sigma_1^r \\ \vdots \\ \sigma_{n_r}^r \end{bmatrix}$ be the vector in the complex (Ω, d) with homological degree r. Thus the set S^r formed by the elements of this vector has cardinal n_r. Assume that the cardinal of $S = \bigcup_{r=r_0}^{r_M} S^r$ is N, that is to say, there are in total N smoothings in the complex. A *numeration* of (Ω, d) is a map $g : S \longrightarrow \{1, ..., N\}$ defined in this way: $g(\sigma_1^{r_0}) = 1$; $g(\sigma_1^{r+1}) = g(\sigma_{n_r}^r) + 1$, if $r_0 \leq r < r_M$; and $g(\sigma_{i+1}^r) = g(\sigma_i^r) + 1$, if $1 \leq i < n_t$. This numerates the smoothings in (Ω, d), and we can rewrite σ_i^r as $\sigma_{g(\sigma_i^r)}$.

Given a complex (Ω, d) then the component of d connecting σ_j and σ_i is denoted d_{ij}. It is clear from the definition of a numeration in (Ω, d) that if $i \leq j$, then $d_{ij} : \sigma_j \to \sigma_i$ is the zero cobordism. The Figure 1 displays an example of a complex with its numeration.

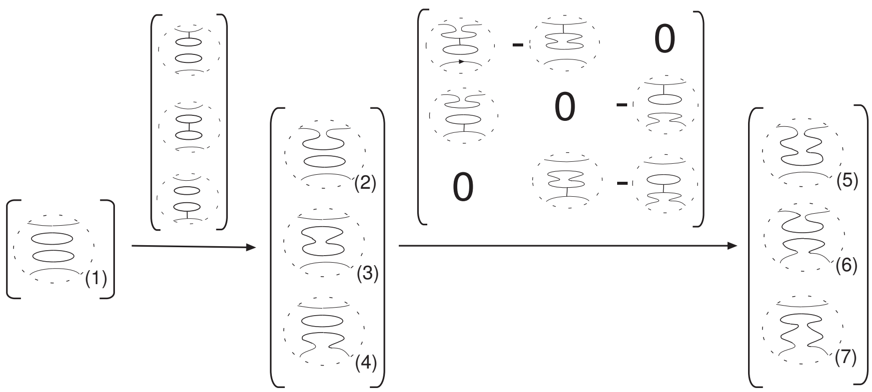


Figure 1. A numeration in a complex, the doted circles around the smoothings represent the discs in which the smoothings are embedded. The subindex in each smoothing is the number assigned to this smoothing by the numeration.

Proposition 3.5. *Let (Ψ, e) and (Φ, f) be chain complexes in* Kob*, Let (Φ, f) be a bounded complex in $[t_0, t_M]$, and D an appropriate 2-input planar arc diagram. Let $\phi_1, \ldots, \phi_N$ be a numeration of (Φ, f) Then $D(\Psi, \Phi)$ is homotopy equivalent to a chain complex (Ω, d) with the following properties:*

(1) *Every vector Ω^r is of the form*

$$\Omega^r = \bigoplus_{\substack{t_0 \leq t \leq t_M \\ s = r - t}} D(\Psi^s, \Phi^t)$$

can be regarded as a block column matrix $\begin{pmatrix} \Omega_1^r \\ \vdots \\ \Omega_N^r \end{pmatrix}$ *in which each block* $\Omega_i^r = D(\Psi^s, \phi_i)$*, where ϕ_i is a smoothing in Φ^t.*

(2) *The differential matrixes d^r can be seen as lower block triangular matrices with blocks $d_{ij}^r : \Omega_j^r \longrightarrow \Omega_i^{r+1}$.*

Proof. The first of these statements follows immediately from the definition of $D(\Psi, \Phi)$, equations (2). Obviously, if $s \leq s_0$ or $s \geq s_M$ we consider $\Psi^s = 0$.

For the second statement we see that given $r = s + t$, the matrix d^r is defined by the second of the equations (2), and is given by

$$d|_{D(\Psi^s, \Phi^t)} = D(e, I_{\Phi^t}) + (-1)^s D(I_{\Psi^s}, f). \tag{10}$$

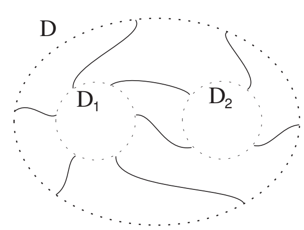
This matrix can be see as a block matrix in which each block d^r_{ij} is a morphism of the form $d^r_{ii} : D(\Psi^s, \phi_i) \longrightarrow D(\Psi^{s+1}, \phi_i)$, and any other block is a morphism of the form $d^r_{ij} : D(\Psi^s, \phi_j) \longrightarrow D(\Psi^{s+1}, \phi_i)$ with $i \neq j$. We conclude from this that $D(e, I_{\Phi^t})$ in the right side of equation (10) is concentrated in the diagonal of blocks. It is clear that the blocks over the diagonal are zero, since they are part of $\pm D(I_{\Psi^s}, f)$ in the right side of equation (10) and if $i < j$, $f_{ij} : \phi_j \to \phi_i$ is the zero cobordism. $\square$

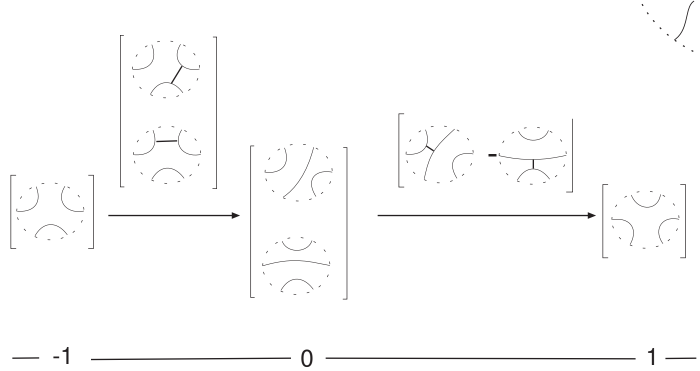
Remark 3.6. The blocks $\Omega^r_i = D(\Psi^s, \phi_i)$ in Ω^r, and the blocks $d^r_{ii} : D(\Psi^s, \phi_i) \longrightarrow D(\Psi^{s+1}, \phi_i)$ in the diagonal of d^r (here $s = r - t$ and ϕ_i is an smoothing in Φ^t), determine the complex

$$D(\Psi, \phi_i) = \cdots \Omega^r_{ii} \xrightarrow{d^r} \Omega^{r+1}_{ii} \cdots$$

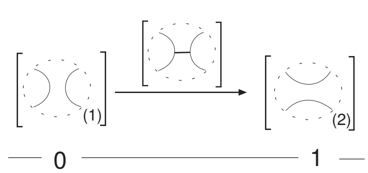
$\square$

We illustrate the previous proposition with an example. Let D be the binary operator defined from the planar arc diagram of the right. If we place the complex

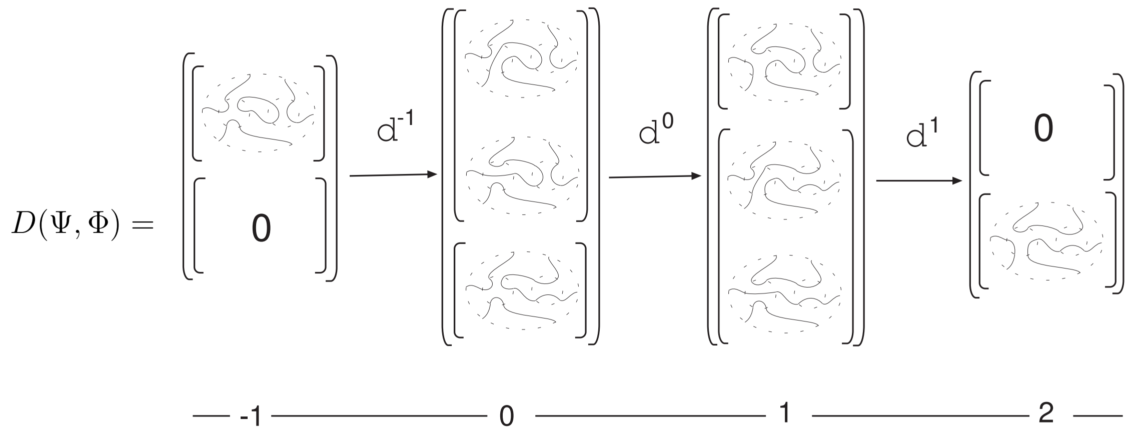


$\Psi =$ 

in the first entry of D and

$\Phi =$ 

in the second entry. Once we have embedded these complexes in D, we obtain a new complex:



The differentials in this complex can be seen as block-lower-triangular matrices, as they are displayed in Figure 2.

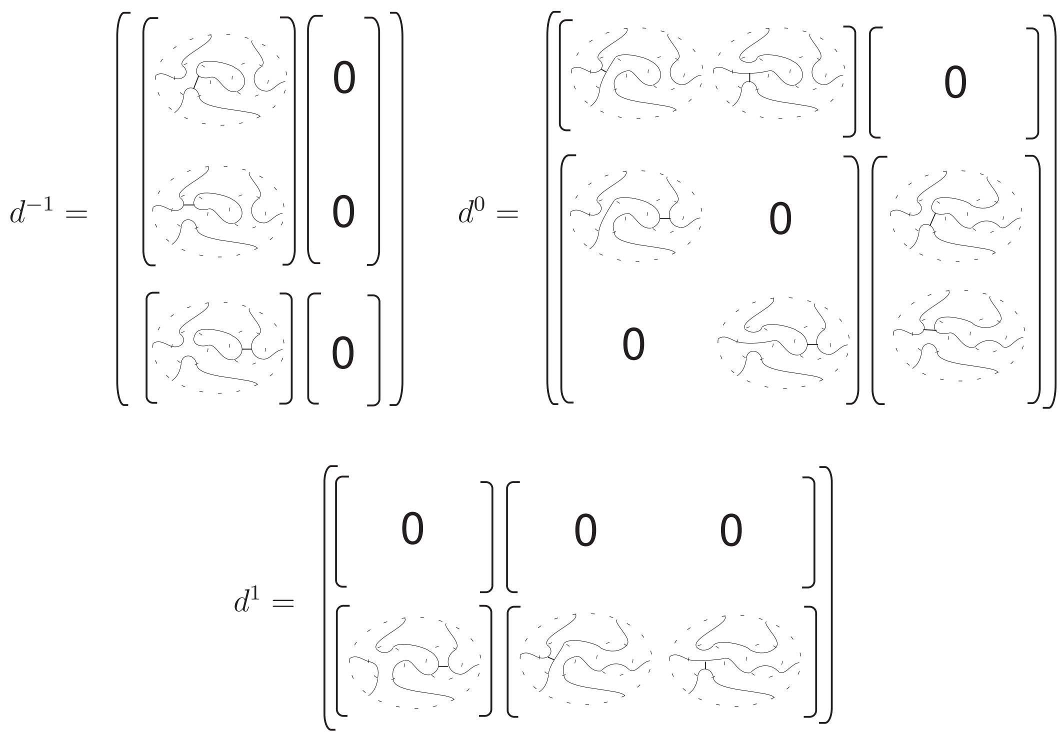


Figure 2. The differentials in the complex $D(\Psi, \Phi)$. The blocks in the diagonal are the differentials $d^r_{ii} : D(\Psi^s, \phi_i) \to D(\Psi^{s+1}, \phi_i)$. the elements in the blocks below the diagonals could have a sign shift. The blocks above the diagonal are blocks of zeros.

Proposition 3.7. *Assume that the three differential matrices in the four term complex segment (5) of lemma 3.2 are block-lower-triangular matrices. After applying gauss elimination, the resulting three differential matrices in the four term complex segment (7) are also block-lower-triangular matrices. Furthermore, the lowest right block of the three initial differential matrices remained unchanged after the Gauss elimination.*

Proof. It is clear that if $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$, $\begin{pmatrix} \phi & \delta \\ \gamma & \epsilon \end{pmatrix}$, and $\begin{pmatrix} \mu & \nu \end{pmatrix}$ are block-lower-triangular matrices, so they are β, ϵ, and ν. Therefore, it is clear that after Gauss elimination, the first and the third of the differential matrices in the form term complex (7) are block-lower-triangular matrices with the same initial lowest-right block.

To prove that the same happens with the second block, we observe that if $\begin{pmatrix} \phi & \delta \\ \gamma & \epsilon \end{pmatrix}$ is a block-lower-triangular matrices then $\begin{pmatrix} \phi & \delta \\ \gamma & \epsilon \end{pmatrix} = \begin{pmatrix} \phi & \delta_1 & 0 \\ \gamma_1 & \epsilon_1 & 0 \\ \gamma_2 & \epsilon_2 & \epsilon_3 \end{pmatrix}$; where $\delta = \begin{pmatrix} \delta_1 & 0 \end{pmatrix}$, $\gamma = \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix}$, and $\epsilon = \begin{pmatrix} \epsilon_1 & 0 \\ \epsilon_2 & \epsilon_3 \end{pmatrix}$. Each 0 in the previous matrices is actually a block of zeros.

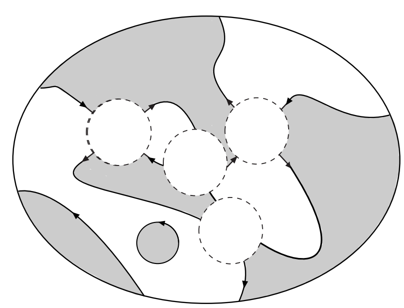
An immediate consequence of the previous paragraph is that the second differential matrix in the four term complex segment (7) is given by

$$\epsilon - \gamma\phi^{-1}\delta = \begin{pmatrix} \epsilon_1 - \gamma_1\phi^{-1}\delta_1 & 0 \\ \epsilon_2 - \gamma_2\phi^{-1}\delta_1 & \epsilon_3 \end{pmatrix}.$$

This completes the proof. □

4. Adequate links

We introduce an alternating orientation in the objects of $\mathcal{C}ob^3_{\bullet/l}(k)$. This orientation induces an orientation in the cobordisms of this category. These oriented k-strand smoothings and cobordisms form the objects and morphisms in a new category. The composition between cobordisms in this oriented category is defined in the standard way, and it is regarded as a graded category, in the sense of [BN1, Section 6]. We subject out the cobordisms in this oriented category to the relations in (1) and denote it as $\mathcal{C}ob^3_o(k)$. Now we can follow [BN1] and define sequentially the categories, $\mathrm{Mat}(\mathcal{C}ob^3_o(k))$, $\mathrm{Kom}(\mathrm{Mat}(\mathcal{C}ob^3_o(k)))$ and $\mathrm{Kom}_{/h}(\mathrm{Mat}(\mathcal{C}ob^3_o(k)))$. This last two categories are what we denote $\mathrm{Kob}_o(k)$, and $\mathrm{Kob}_{o/h}$. As usual, we use Kob_o, and $\mathrm{Kob}_{o/h}$, to denote $\bigcup_k \mathrm{Kob}_o(k)$ and $\bigcup_k \mathrm{Kob}_{o/h}(k)$ respectively.



We denote the class of oriented smoothings as $\mathcal{S}_o$. An alternatively oriented d-input planar diagram, see [Bur], provides a good tool for the horizontal composition of objects in $\mathcal{S}_o$, $\mathcal{C}ob^3_o$, $\mathrm{Mat}(\mathcal{C}ob^3_o(k))$, Kob_o, and $\mathrm{Kob}_{o/h}$. The orientation in the diagrams can be provided as in the figure at the right. For making this text a little more self-contained, we are going to recall briefly some concepts presented before in [Bur]. Given oriented smoothings $\sigma_1, ..., \sigma_d$, a suitable alternating d-input planar diagram D to compose them has the property that the i-th input disc has as many boundary points as σ_i. Moreover placing σ_i in the i-th input disc, The orientation (the coloring) of σ_i and D match.

Given an open strand α of an alternating oriented smoothing σ, possibly with loops, enumerate the boundary points of σ in such a way that α can be denoted by $(0, i)$. The *rotation number* of $(0, i)$, $R(\alpha)$, is $\frac{i-k}{2k}$. If α is a loop, $R(\alpha) = 1$ if α is oriented counterclockwise, and

$R(\alpha) = -1$ if α is oriented clockwise. The rotation number of σ is the sum of the rotation numbers of its strings. See figure 3 We are going to use this alternating diagrams to compute

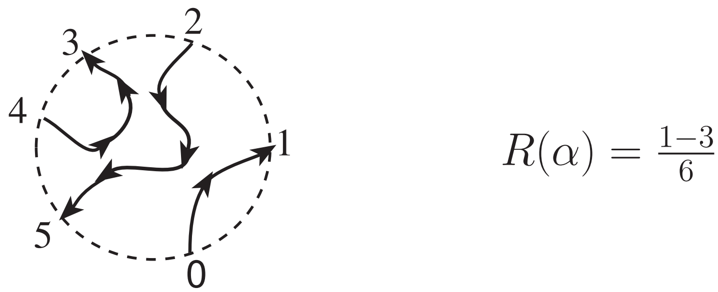
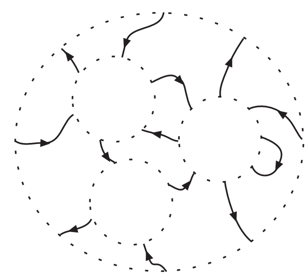


Figure 3. $\alpha = (0, 1)$, $R(\alpha) = -\frac{2}{6} = -\frac{1}{3}$. The rotation number of the complete resolution is 0

non-split alternating tangles, and we want to preserve the non-split property of the tangle. Hence, it will be better if we use d-input type $\mathcal{A}$ diagrams.



A d-input type-$\mathcal{A}$ diagram has an even number of strings ending in each of its boundary components, and every string that begins in the external boundary ends in a boundary of an internal disk. We can classify the strings as: *curls*, if they have its ends in the same input disc; *interconnecting arcs*, if its ends are in different input discs, and *boundary arcs*, if they have one end in an input disc and the other in the external boundary of the output disc. The arcs and the boundaries of the discs divide the surface of the diagram into disjoint regions. Some arcs and regions will be useful in the following definitions and propositions.

Definition 4.1. We assign the following numbers to every d-input planar diagram D:

- i_D: number of interconnecting arcs and curls, i.e., the number of non-boundary arcs.
- w_D: number of negative internal regions. That is, in the checkerboard coloring, the white regions whose boundary does not meet the external boundary of D.
- R_D: the rotation associated number, which is given by the formula

$$R_D = \frac{1}{2}(1 + i_D - d) - w_D$$

.

Proposition 4.2. *Given the smoothings $\sigma_1, ..., \sigma_d$ and a suitable d-input planar diagram D, where every smoothing can be placed, the rotation number of $D(\sigma_1, ..., \sigma_d)$ is:*

$$R(D(\sigma_1, ..., \sigma_d)) = R_D + \sum_{i=1}^{d} R(\sigma_i) \tag{11}$$

Definition 4.3. An alternating planar algebra is a triplet $\{\mathcal{P}, \mathcal{D}, \mathcal{O}\}$ in which $\mathcal{P}$, $\mathcal{D}$, and $\mathcal{O}$ have the same properties as in the definition of a planar algebra but with the collection $\mathcal{D}$ containing only $\mathcal{A}$-type planar diagrams.

Diagrams with only one or two input discs deserves special attention. Operators defined from diagram like these are very important for our purposes since some of them are considered as the generators of the entire collection of operators in a connected alternating planar algebra.

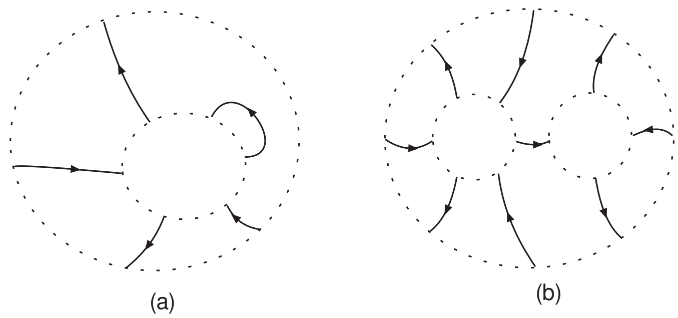


Figure 4. Examples of basic planar diagrams

Definition 4.4. A basic planar diagram is a 1-input alternating planar diagram with a curl in it, or a 2-input alternating planar diagram with only one interconnecting arc. A basic operator is one defined from a basic planar diagram. A negative unary basic operator is one defined from a basic 1-input diagram where the curl completes a negative loop. A positive unary basic operator is one defined from a basic 1-input diagram where the curl completes a positive loop. A binary operator is one defined from a basic 2-input planar diagram.

Proposition 4.5. *The rotation associated number of a planar diagram belongs to $\frac{1}{2}\mathbb{Z}$ and the case when we have a basic planar diagram it is given as follows:*

- *If D is a negative unary basic operator, $R_D = -\frac{1}{2}$*
- *If D is a binary basic operator, $R_D = 0$*
- *If D is a positive unary basic operator, $R_D = \frac{1}{2}$*

Proposition 4.6. *Any operator D in an alternatively oriented planar algebra is the finite composition of basic operators.*

5. Connecting in a wrong way

Once we have applied lemma 3.1 to an element of Kob_o, we obtain a complex (Ω, d), which preserves some properties of the former one, but with a change in the rotation number of the element $\sigma\{q_\sigma\}$, in which we have applied the delooping. In fact, the smoothing has been replaced in the complex by a couple whose rotation number has changed either by -1 or by +1. This shift in the rotation number could be even greater if we continue removing loops in the same smoothing. From lemma 3.1, we know that there is also a change in the grading shift of the smoothings. So it would be a good idea to define a concept that states a relation between the rotation number of σ and its grading shift q_σ.

Definition 5.1. Let (Ω, d) be a class-representative of $\mathrm{Kob}_{o/h}$, and let $\sigma_i\{q_i\}$ be a shifted degree object in Ω^r, then its degree-shifted rotation number is $\overline{R}(\sigma_i\{q_i\}) = R(\sigma_i) + q_i$

Definition 5.2. A diagonal complex is a degree-preserving differential chain complex (Ω, d)

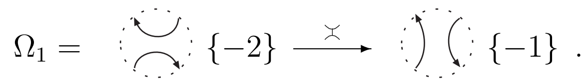
$$\cdots \Omega^r \xrightarrow{d^r} \Omega^{r+1} \cdots$$

in Kob_o, satisfying that for each homological degree r and each shifted degree object $\sigma_i\{q_i\}$ in Ω^r, we have that $2r - \overline{R}(\sigma_i\{q_i\}) = C_\Omega$, where C_Ω is a constant that we call rotation constant of (Ω, d).

Here we have some examples of diagonal complexes in Kob_o.

Example 5.3. As in [BN2], a dotted line represent a dotted curtain, and)(stands for the saddle)($\longrightarrow$ $\asymp$

(1)



This is the Khovanov homology of the negative crossing $\times$, now with orientation in the smoothings. Remember that the first term has homological degree -1. In this example the rotation number in the first term is $-\frac{1}{2}$ and in the second term it is $\frac{1}{2}$. Observe that in each case, the difference between 2 times the homological degree r and the shifted rotation number is $\frac{1}{2}$.

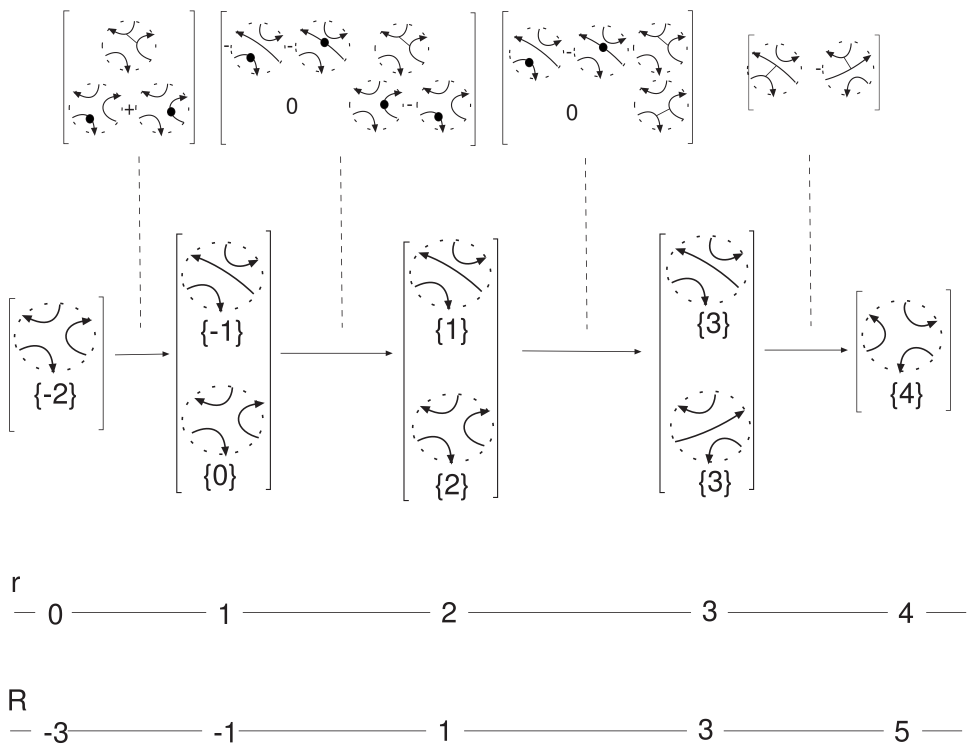


Figure 5. A diagonal complex.

(2) In Figure 5, the number below each smoothing is the grading shift of the smoothing. The upper line below the complex represents the homological degree r, and the lower one represents the degree-shifted rotation number. For instance, the rotation number in the first smoothing with homological degree 1 has rotation number 0 and a grading shift by -1. In the second smoothing of the same vector, the rotation number is -1 and its grading shift is 0, so both term has the same degree-shifted rotation number. We see in this example, that for each r we have that $2r - R = 3$, so this is a diagonal complex.

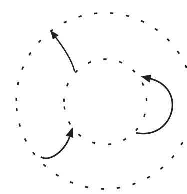
Now, we can establish a parallel between what we did with alternating elements in $\mathcal{M}_k^{(o)}$ and diagonal complexes in Kob_o in such a way that we can obtain similar results as those obtained in section 4 of [Bur].

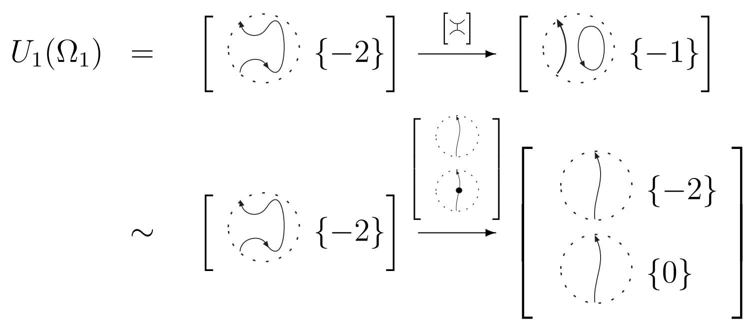
5.1. **Applying unary operators.** The reduced complexes in $\mathrm{Kom}(\mathrm{Mat}(\mathcal{C}ob^3_o))$ can be inserted in appropriate unary basic planar diagrams, and then apply lemmas 3.1, 3.2, and 3.3 to obtain again a reduced complex in Kob_o. This process can be summarized in the following steps:

(1) placing of the complex in the corresponding input disc of the d-input planar arc diagram by using equations (2),
(2) removing the loops obtained by applying lemma 3.1, i.e, replacing each of them by a copy of $\emptyset\{+1\} \oplus \emptyset\{-1\}$, and
(3) applying lemma 3.3 and gaussian elimination (lemma 3.2), and removing in this way each invertible differential in the complex.

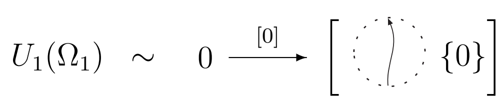
Definition 5.4. Let (Ω, d) be a chain complex in $\mathrm{Kom}(\mathrm{Mat}(\mathcal{C}ob^3_o(k)))$, then a partial closure of (Ω, d) is a chain complex of the form $D_l \circ \cdots \circ D_1(\Omega)$ where $0 \le l < k$ and every D_i $(1 \le i \le l)$ is a unary basic operator

We have diagonal complexes whose partial closures are again diagonal complexes. For instance, embedding Ω_1 of the example 5.3 in a unary basic planar diagram U_1 as the one on the right which has an associated rotation number $R_{U_1} = \frac{1}{2}$, produces the chain complex.





The last complex is the result of applying lemma 3.1. Applying now lemma 3.2 we obtain a homotopy equivalent complex



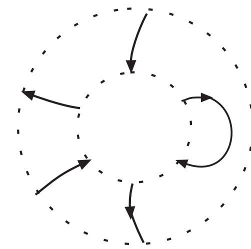
which is also a diagonal complex, but now with rotation constant zero.

Definition 5.5. Let (Ω, d) be a bounded diagonal complex in Kob_o with rotation constant C_R. We say that (Ω, d) is *coherently diagonal* if for any appropriated unary operator with associated rotation number R_U, the closure $U(\Omega, d)$ has a reduced form which is a diagonal complex with rotation constant $C_R - R_U$.

We denote as $\mathcal{D}(k)$ the collection of all coherently diagonal complexes in $\mathrm{Kom}(\mathrm{Mat}(\mathcal{C}ob^3_o(k)))$, and as usual, we write $\mathcal{D}$ to denote $\bigcup_k \mathcal{D}(k)$. It is easy to prove that any coherently diagonal complex satisfies that:

(1) after delooping any of the positive loops obtained in any of its partial closure, by using lemma 3.2, the negative shifted-degree term can be eliminated.

(2) after delooping any of the negative loops obtained in any of its partial closure, by using lemma 3.2, the positive shifted-degree term can be eliminated.



Since the computation of any other of its partial closures produces other diagonal complex, the complex Ω_1 of the example 5.3 is an element of $\mathcal{D}(2)$. Another example of coherently diagonal complex is the complex Ω_2 of the same example. This last complex has $C_R = 3$. All of its partial closures $U(\Omega_2)$ are diagonal complexes with rotation constant given by $C_R - R_U$. Here, we only calculate the one produced by inserting the element in the closure disc U, with $R_U = -\frac{1}{2}$, that appears on the right. It will be easy for the reader to compute the other partial closures. Inserting Ω_2 in U produces the complex of Figure 6, which is also a diagonal complex, but with a loop in some

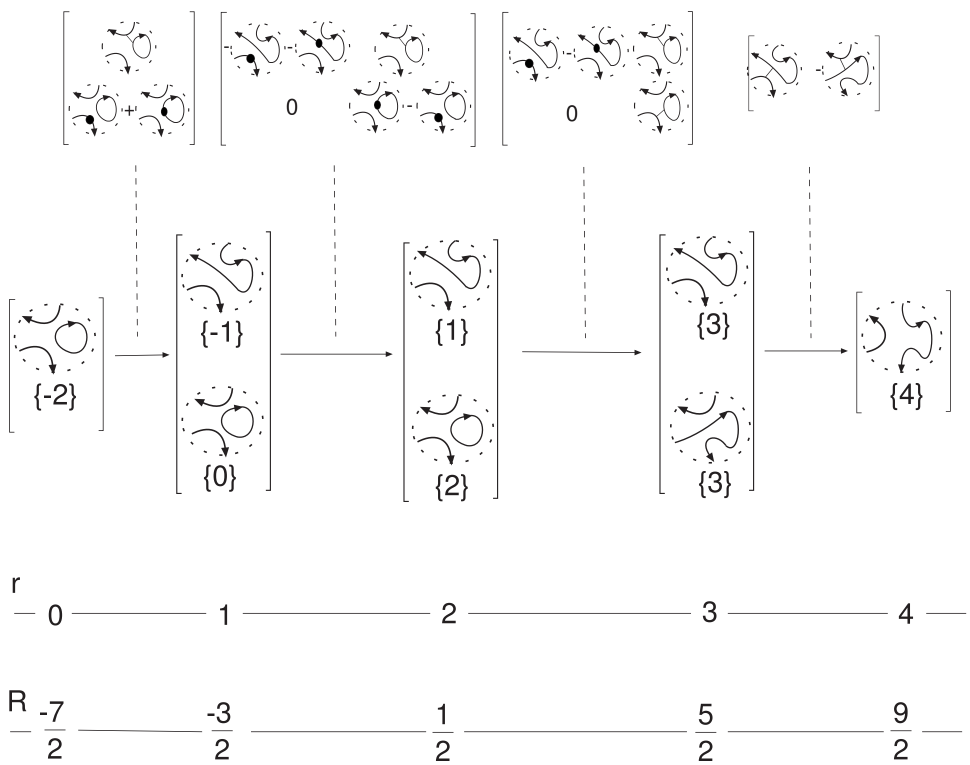


Figure 6. A diagonal complex inserted in a negative unary basic diagram U.

of its smoothings. Observe that the rotation number of the smoothings have decreased in $\frac{1}{2}$ after having been inserted in a negative unary basic diagram.
After applying lemmas 3.1 and 3.2, we obtain the complex in Figure which is also a diagonal complex, but now with rotation constant $\frac{7}{2}$.

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