Aggregate Uncertainty, Money and Banking*

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Abstract

This paper addresses the problem of monitoring the monitor in a model of money and banking with private information and aggregate uncertainty. There is no need to monitor a bank if it requires loans to be repaid partly with money. A market arises at the repayment stage and generates information-revealing prices that perfectly discipline the bank. This mechanism also applies when there exist multiple banks. With multiple banks, competition of private monies improves welfare. A prohibition on private money issue not only eliminates money competition but also triggers free-rider problems among banks, which is detrimental to welfare.

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1 Introduction

The main goal of this paper is to address the problem of monitoring the monitor for banks with undiversifiable risks. The problem of monitoring the monitor refers to the incentive problem of a bank. As recognized by the literature on banking, one of the significant roles of a bank is being the delegated monitor. If the bank does all the work of monitoring borrowers, it avoids the duplication of efforts of lenders monitoring borrowers. Nevertheless, lenders may still find it necessary to monitor the bank with undiversifiable risks. Otherwise, the bank can always claim receiving an adverse aggregate state and default on payments to the lenders. Hence, there remains duplication of monitoring costs unless some mechanism is available to efficiently discipline the bank. Finding such a mechanism becomes the key to solving the problem of monitoring the monitor.

To tackle the incentive problem of banks, I study a model of money and banking with private information and aggregate uncertainty. Banks arise endogenously to offer loans of money and function as the delegated monitor. Banking is competitive. There are strategic interactions between banks and borrowers and among banks themselves. Due to aggregate uncertainty, banks are facing undiversifiable portfolio risks. There are incentive problems caused by private information of both individual borrowers and banks.

I establish the following results: first, there is no need to monitor the bank if it requires loans to be repaid partly with money. Borrowers must trade goods for money to clear debts. As a result, a market arises at the repayment stage and generates information-revealing prices that perfectly discipline the bank. Thus the mechanism costlessly overcomes the incentive problem of a bank. This mechanism can be readily applied to scenarios of multiple banks. If private issue of money is permitted, at the repayment stage there will arise as many markets as the number of existing banks. In each market, banknotes of a specific bank are traded. The market prices reveal the information of the corresponding banks and discipline them respectively. If private money issue is prohibited, only one market will arise at the repayment stage, where outside money is traded. The market price disciplines
Second, a prohibition on private money issue causes inefficient outcomes in the presence of multiple banks. Without the prohibition, banks use loans to support their own banknotes. The competition of private monies drives up the equilibrium returns of banknotes and improves welfare. With the prohibition, however, bank loans are used to support the same kind of money, outside money. The value of outside money is determined by the aggregate of the returns offered by banks. Any value added to outside money by one bank benefits all. Banks get free rides from one another on making outside money more valuable. In this case, the equilibrium return of outside money is inefficiently low as banks make private decisions. Therefore, prohibiting private money issue reduces welfare. If it has to be done, then it is a good idea to make outside money differentiable so that each bank can work with a unique kind of outside money. Then the outcome would be the same as with inside money.

The model of this paper is based on Williamson (1986) and Andolfatto and Nosal (2003). Agents are spatially separated. Money serves as a medium of exchange due to lack of double coincidence of wants and limited communication. Banks lend to borrowers and monitor those who default. In both Williamson (1986) and Andolfatto and Nosal (2003), the cost of monitoring a bank is saved by perfect diversification. In contrast, here an agent’s endowment is the result of an idiosyncratic shock and an aggregate shock. Due to the aggregate risk, one must look for ways other than perfect diversification to tackle the problem of monitoring the monitor. As in Williamson (1986), the optimal contract is proven to be a debt contract. A borrower either makes a fixed level of repayment or defaults and gets monitored by the bank. Nonetheless, there is a twist to this debt contract: some of the repayment must be made in money and the rest in real goods. As is explained before, this twist is the key to solving the incentive problem of a bank. In addition to facilitating trades, here money also has a role in helping generate information for banking. As a result, even outside money can circulate in this model of finite horizons, which is
rarely seen in monetary models.

My work is complementary to the literature that addresses the problem of monitoring the monitor. Diamond (1984) and Williamson (1986) first recognized this problem and show that it can be overcome by perfect diversification. However, one is naturally concerned about the fact that the real-world financial intermediaries cannot perfectly diversify portfolio risks. Krasa and Villamil (1992a,b) and Winton (1995) study finite-sized banks. According to these papers, portfolio diversification and bank capitalization can help reduce the cost of monitoring a bank. My work contributes to this literature by showing that the incentive problem of a bank with undiversifiable risk, can be costlessly overcome if the bank requires loans to be repaid partly with money.

Gorton and Haubrich (1987) analyze issues of banking deregulation in a model focusing on the interaction of banks and markets. As they pointed out, if a secondary market of bank loans can be created, it will reveal bank-specific risks and eliminate the information asymmetry that causes banking panics. The opening of such a secondary market requires improvement in the monitoring technology of the banking system. Their idea is similar to the mechanism studied in this paper. Here as borrowers are required to sell some goods for money, they are essentially trading off the corresponding part of the repayment. It is as if money were traded for claims on bank loans. The market induced at the repayment stage resembles a secondary market of bank loans and it reveals bank-specific information. However, in contrast to Gorton and Haubrich (1987), inside money arises endogenously in my model and the creation of the market at the repayment stage comes naturally with banking and does not hinge on any change of monitoring technology.

The economic audience has seen quite a few examples of the phenomenon that additional trading opportunities can be detrimental to incentive compatibility and reduce welfare. For instance, Hammond (1979, 1987) and Jacklin (1987) all seem to suggest that individual incentive compatibility of a mechanism is not sufficient for truthful revelation of types, provided that there exist frictionless markets. In contrast, my paper presents a
model in which markets are a favorable instrument for incentives of truthful revelation. The creation of a market at the loan repayment stage makes it incentive compatible for the bank not to misrepresent its solvency. Furthermore, the revelation of information through prices is unambiguously welfare-improving.

Shi (1996) has already pointed out that insisting on debt repayments in money instead of goods is beneficial to the economy. In particular, by studying coexistence of money and credit in a search model with divisible commodities, Shi shows that when debt is repaid with only money, the competition of money and credit eliminates the inefficient monetary equilibrium where money has a weak purchasing power. My paper identifies a new dimension of the advantage of nominal repayments over real repayments. That is, as long as debt is repaid partly with money, the incentive problem of a bank will be solved at no cost. This, however, cannot be achieved if repayments are made in goods.

This paper is also related to the literature that examines the functioning of inside money and outside money, e.g. Cavalcanti and Wallace (1999), Williamson (2004) and He, Huang and Wright (2005, 2006). According to these models, inside money has the following advantages: the private issue of money is flexible so that agents are not constrained by trading histories and they can respond to unanticipated shocks better than with outside money. Moreover, bank liabilities are a safer instrument than cash. This paper of mine identifies another advantage of inside money over outside money: it naturally promotes the efficiency of banking. The competition of banks’ private monies improves welfare. The imposition of outside money on banking not only removes the benefit of money competition but also introduces free-rider problems among banks, resulting in a lower level of welfare.

The remainder of the paper is organized as follows. Section 2 describes the environment of the model. Section 3 introduces the bank loan contract. Section 4 characterizes the banking equilibrium and the optimal contract. Section 5 studies alternative monetary regimes in the existence of multiple banks. Section 6 concludes the paper.
2 The Model

There are three islands in the economy, namely $A$, $B$ and $C$. Each island is populated by a $[0,1]$ continuum of agents. The economy lasts for four time periods, $t = 0, 1, 2, 3$. Communication across islands is limited throughout time: at $t = 0$ no communication is available across islands; agents of islands $A$ can visit island $B$ at $t = 1$; agents of islands $B$ can visit island $C$ at $t = 2$; and lastly agents of island $C$ and island $A$ can get in touch at $t = 3$. Traveling agents return to their native island at the end of the period.

Each agent owns a project, which receives endowment of goods. The timing of endowment differs across islands: island $B$ receives endowment at $t = 1$, island $C$ at $t = 2$, and island $A$ at $t = 3$. Endowment is realized at the beginning of a period and prior to the arrival of any traveling agent. Goods are divisible but perishable across time. On both island $B$ and island $C$, the endowment of a project is deterministic at $y$, where $0 < y \leq 1$. The endowment of a type $A$ project is stochastic: $Y = SW$, where $S$ and $W$ are both random variables. Here, $S$ is an aggregate shock, which is common to all projects on island $A$. It is distributed according to the probability density function $f(s)$ and the cumulative distribution function $F(s)$. The variable $W$ is an idiosyncratic shock and is i.i.d. across type $A$ agents, according to PDF $g(w)$ and CDF $G(w)$. Both $f(\cdot)$ and $g(\cdot)$ are differentiable on support $[0,1]$. Let $h(y)$ and $H(y)$ denote the PDF and CDF of $Y$, respectively. By Rohatgi’s well-known result,\footnote{For the distribution of the product of two continuous random variables, see Rohatgi (1976).} $h(y) = \int_{y}^{1} f(s) g\left(\frac{y}{s}\right)^{\frac{1}{s}} ds$. The realization of $y$, not $s$ or $w$ specifically, is costlessly observable only to the agent himself. However, all agents know about $f(s)$, $g(w)$ and hence $h(y)$.

Agents’ preferences are as follows:

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U_A = c_A^1 + \varepsilon c_A^3 - e_A \\
U_B = c_B^2 + c_B^1 \\
U_C = c_C^3 + \ln c_C^2 - e_C
$$
where $0 < \varepsilon < 1$, $c_i^t$ denotes a type $i$ agent's date-$t$ consumption and $e_i$ is the amount of effort expended by the agent. Type $A$ and type $C$ agents are endowed with an unbounded amount of effort. Any type $A$ agent can exert effort to learn other type $A$ agents' project outcomes, each project costing the agent $\mu > 0$ units of effort. The application of type $C$ agents' effort will be specified later. Consumption always takes place at the end of a period, allowing for state verification.

Since $0 < \varepsilon < 1$, type $A$ agents prefer to consume at $t = 1$ rather than at their own "harvesting" date, $t = 3$. However, there is lack of intertemporal double coincidence of wants in this economy. To obtain $t = 1$ goods, type $A$ agents must get type $C$ agents to give up some $t = 2$ endowment for $t = 3$ consumption so that type $B$ agents are willing to trade $t = 1$ goods for $t = 2$ goods. Due to limited communication, there is no way for all three types of agents to arrange multilateral exchanges. Therefore, money is potentially useful to facilitate trades as illustrated by Figure 1.

Further scrutiny, however, makes one realize that personal IOUs cannot circulate in this economy. Here type $A$ agents are the ones who have the incentive to issue personal IOUs. However, because type $C$ agents cannot verify type $A$ project outcomes, type $A$ agents will always default on payment. Knowing that, none of type $B$ or type $C$ agents will value those IOUs. If type $A$ agents are endowed with outside money, they might want to trade outside money for type $B$ goods. Unfortunately, it will not work, either, because of the finite horizons. That being said, money, be it inside or outside, cannot be valued on its own in this environment. However, banking can make money valuable by supporting it with bank loans.

Banking is competitive. At $t = 0$, type $A$ agents each post a contract. By competition, the one who offers the most desirable terms becomes the bank and contracts with all type
A agents. To qualify as the bank, one must yield his project to public surveillance while maintaining the ownership to one’s endowment. The banking profit refers to the utility of the banker netting the utility of him being a project owner. In equilibrium the banking profit is driven down to zero, which implies that the banker has the same expected utility as any other type A agent. The bank issues and lends banknotes to type A agents. Banknotes circulate across islands. In the end, type A agents make payments back to the bank, who uses these payments to redeem banknotes and cover the cost of monitoring. The bank monitors the defaulting type A agents.

There are incentive problems due to private information. In addition to the private information of borrowers, portfolio returns are also the private information of the bank. With aggregate uncertainty, the returns depend on the aggregate state and are stochastic in nature. The bank has the incentive to default on redemption unless it is somehow disciplined. The optimal contract must be incentive compatible both for individual borrowers and for the bank. I assume that any type C agent can monitor the bank at the cost of \( \theta > 0 \) units of effort. Monitoring the bank typically involves duplication of efforts and can be very costly. One of the main tasks of this paper is to find a way to solve the two-sided incentive problem while saving the cost of monitoring the bank.

### 3 The banking contract

The bank loan contract takes the following form:

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\(^2\)I assume that if more than one agent post the same optimal contract, only one of them is chosen randomly as the bank. It will become clear later that borrowers always prefer a unique bank as long as it is feasible.

\(^3\)The story can be that the banker must give up the key to his storage so that his endowment becomes publicly observable. This requirement is meant to resolve the conflict of interests of the banker for being a monitor and a project owner at the same time.

\(^4\)Collusion, the collective deviation of any parties related by the contract, is not considered in this paper.
3.1 Terms of the contract

1. Each borrower is entitled to $M$ units of banknotes at $t = 0$. After receiving endowment at $t = 3$, the borrower reports $\tilde{y}$ to the bank. If $\tilde{y} \in \Omega \subseteq [0, 1]$, he will be monitored by the bank; otherwise, monitoring does not occur. Repayment costs $z(\tilde{y})$ units of goods. The borrower is required to sell $t(\tilde{y})$ units of goods in the market and then repay his loan with $z(\tilde{y}) - t(\tilde{y})$ units of goods and $p_A t(\tilde{y})$ units of banknotes, where $p_A$ is the market price of goods for banknotes on island $A$.

2. Each unit of banknote promises an expected value of $R$. After collecting repayments and monitoring, the bank announces $\tilde{s}$ as the aggregate state and devotes $D(\tilde{s})$ units of goods to redeeming banknotes. Each redeemed banknote is entitled to $D(\tilde{s})/m^R$ units of goods, where $m^R$ is the redemption demand. Redemption is served simultaneously. Since the bank monitors the defaulting borrowers with probability one, any hidden goods are bound to be discovered. Part of the hidden goods should have been supplied to the market. The bank must transfer this corresponding amount to those who have traded their banknotes for goods in the market. Redeemers and buyers retain the option of verifying $\tilde{s}$.

3.2 Timing of events

All terms of the contract are public information. Without loss of generality, normalize $M = 1$. All market trades are competitive. According to the contract, the timing of events is summarized as follows:

1. $t = 0$, (i) a bank arises on island $A$; (ii) the bank contracts with $A$s;

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5To avoid the issue of private money over-issue, I assume that the bank can issue notes only at $t = 0$.
6Following Williamson (1986), here I restrict my attention to non-stochastic monitoring.
7This assumption is intended for the simplicity of analysis. An alternative assumption is that redemption is served sequentially. This alternative assumption complicates the analysis considerably in two dimensions. First, now the bank must describe the payments for redemption as a function of an agent’s place in line. Second, the price of goods may vary over time as redemption is served. Although these issues are interesting, they are secondary to the main issue in this paper that prices can reveal the information about the aggregate state of the economy. I thank a referee for pointing out this difference between sequential and simultaneous redemption.
2.  $t = 1$, competitive trades between As and Bs;

3.  $t = 2$, competitive trades between Bs and Cs;

4.  $t = 3$, (i) Cs each decide whether to trade banknotes in the market or redeem them at the bank; (ii) competitive trades between Cs and As; (iii) market closes and the bank collects repayments and monitors according to $\Omega$; (iv) the bank announces $\tilde{s}$ and redeems banknotes; (v) Cs monitor the bank if necessary.

4 The banking equilibrium

By competitive banking, the bank designs the loan contract, $C = (z, t, \Omega, D)$, to maximize the expected utility of a borrower. Since only incentive compatible contracts are considered, in the rest of the paper the reported values and the true values are not distinguished unless otherwise stated.

First consider the type $C$ agents. Let $p_i$ denote the market price of type $i = A, B, C$ goods for banknotes. Recall that $R$ is the expected net returns of a banknote. It follows that

$$R = \int_0^1 [D(s) + T(s)] f(s) ds - \Theta,$$

where $T(s) = \int_0^1 t(sw) g(w) dw$ is the aggregate supply of goods in the market and $\Theta$ is the expected cost of monitoring the bank. Banknotes are valued as long as $R > 0$. Type $C$ agents can either redeem banknotes at the bank or use them to purchase goods in the market. The expected net returns of a banknote are equal to the expected benefit minus the expected cost of monitoring the bank. Let $m_C$ be a type $C$ agent’s holdings of banknotes. Taking $p_C$ and $R$ as given, a representative type $C$ agent chooses $(c^2_C, m_C)$ to maximize his expected utility, $\ln c^2_C + m_C R$, subject to the budget constraint, $p_C c^2_C + m_C = p_C \bar{y}$. Let $(c^2_C, m^*_C)$ denote the optimal choices of a type $C$ agent.

Similarly, let $m_B$ be a type $B$ agent’s holdings of banknotes. Taking $p_B$ and $p_C$ as given, a representative type $B$ agent chooses $(c^1_B, m_B)$ to maximize his expected utility, $c^1_B + \frac{m_B}{p_C}$, subject to the budget constraint, $p_B c^1_B + m_B = p_B \bar{y}$. Again, let $(c^1_B, m^*_B)$ denote the optimal choices of a type $B$ agent.
Now given $c^{1*}_B$, the bank’s problem is

$$V_i = \max_{\mathcal{C}} \left\{ \bar{y} - c^{1*}_B + \varepsilon \int_0^1 [y - z(y)] h(y) \, dy \right\}$$

s.t. \quad \int_0^1 z(y) h(y) \, dy - \int_0^1 T(s) f(s) \, ds - \int_0^1 D(s) f(s) \, ds - \mu \int_\Omega h(y) \, dy = 0 \tag{1}$$

The constraint holds with equality because competition drives the banking profit down to zero. The expected banking profit is equal to the expected aggregate loan repayment in kind, $\int_0^1 z(y) h(y) \, dy - \int_0^1 T(s) f(s) \, ds$, minus the expected cost of redemption $\int_0^1 D(s) f(s) \, ds$ and the expected cost of monitoring borrowers $\mu \int_\Omega h(y) \, dy$. Note that the timing is such that the bank starts collecting repayment after the market closes. Therefore, the bank cannot use repayments of banknotes to buy goods in the market. It can only benefit from the repayments in kind.

It is natural to focus on symmetric equilibria because agents are ex ante identical:

**Definition 1** A banking equilibrium consists of market prices $\{p_A, p_B, p_C\}$, a contract $\mathcal{C}$ and the associated expected net returns of a banknote $R$, the redemption demand $m^R$, and individual agents’ choices $\{c^1_B, m_B, c^2_C, m_C\}$ such that: (i) given prices and $R$, the choices of a representative agent of type $i = B, C$ are optimal; (ii) given $c^1_B$, the bank designs $\mathcal{C}$ to maximize the expected utility of a type $A$ agent; (iii) given $m^R$, type $C$ agents are indifferent between redeeming banknotes and selling them in the market; (iv) all markets clear.

According to condition (iii) in the above, the equilibrium redemption demand $m^R$ must satisfy:

$$\frac{\int_0^1 D(s) f(s) \, ds}{m^R} = \frac{\int_0^1 T(s) f(s) \, ds}{1 - m^R},$$

which implies that $m^R = \frac{\int_0^1 D(s) f(s) \, ds}{\int_0^1 [D(s) + T(s)] f(s) \, ds}$. Then the equilibrium market price on island $A$
is given by:

\[ p_A = \frac{1 - m^R}{T(s)} = \frac{\int_0^1 T(s) f(s) \, ds}{T(s) \int_0^1 [D(s) + T(s)] f(s) \, ds}. \]

Thus,

\[ T(s) = \frac{p_A \int_0^1 [D(s) + T(s)] f(s) \, ds}{\int_0^1 T(s) f(s) \, ds}. \]

Obviously, as long as the function \( T(s) \) is fully revealing, agents are able to pin down the exact aggregate state \( s \) simply by observing \( p_A \). Hence they know the exact amount of goods the bank should spend on redemption, provided that \( D(s) \) is also a fully revealing function.

The bank will be monitored if it posts any \( \tilde{s} \neq s \). Note that this aggregate information is not available to individual agents unless a market exists on island \( A \). Therefore, the cost of monitoring the bank is completely saved as long as borrowers are required to sell in the market. The market price \( p_A \) fully reveals the aggregate state and keeps the bank perfectly disciplined, which proved the following proposition:

**Proposition 1**  The bank is perfectly disciplined and thus \( \Theta = 0 \), provided that (i) \( T(s), D(s) \geq 0 \) for all \( s \); (ii) \( T(s) \neq T(s') \) and \( D(s) \neq D(s') \) for any \( s \neq s' \in [0, 1] \).

Recall that the expected gross returns of banknotes are \( \int_0^1 [D(s) + T(s)] f(s) \, ds = R + \Theta \). Given \( R \), the expected profit of the bank, the right-hand side of the constraint in (1), is the highest with \( \Theta = 0 \) because there is no need to compensate banknote-holders for the expected cost of monitoring the bank. Therefore, it is in the interest of the bank to implement the mechanism in Proposition 1 to discipline itself.

In equilibrium, \( p_B = \frac{1}{y-c^B_H} \) and \( p_C = \frac{1}{y-c^C_H} \). It is straightforward to derive that \( c^{1*}_H = c^{2*}_C = \frac{\bar{y}}{1+R} \). Given \( c^{1*}_H \) and \( \Theta = 0 \), (1) becomes

\[
V_1 = \max_c \left\{ \frac{y R}{1 + R} + \varepsilon \int_0^1 [y - z(y)] h(y) \, dy \right\} \\
\text{s.t.} \quad \int_0^1 z(y) h(y) \, dy - \mu \int_0^1 h(y) \, dy - R = 0.
\]
Proposition 2  The optimal repayment schedule is given by

\[ z^*(y) = \begin{cases} 
  x, & \text{if } y \geq x \\
  y, & \text{if } y < x
\end{cases}, \]

where \( x \in [0, 1] \) is a constant. Correspondingly, the optimal monitoring state is \( \Omega^* = \{ y : 0 \leq y < x \} \).

Proof of Proposition 2 is provided in the Appendix.\(^8\)

By Propositions 1-2, the bank’s problem becomes

\[ V_1 = \max_{x \in [0,1]} \left\{ \frac{yR(x)}{1 + R(x)} + \varepsilon \int_x^1 (y - x) h(y) \, dy \right\} \tag{3} \]

where

\[ R(x) = \int_0^x y h(y) \, dy + x \left[ 1 - H(x) \right] - \mu H(x). \tag{4} \]

The solution to the above pins down \( x \) and hence \( R(x) \). After that, it remains to determine \( t(y) \) and \( D(s) \) so as to completely characterize the banking contract. Notice that the contract must be incentive compatible for type \( A \) agents to sell exactly \( t(y) \) units of goods in the market. That is, the bank must make sure that a borrower does not have the incentive to sell more than \( t(y) \) units and then demand redemption for the extra banknotes.

Given \( D(s) \) and any equilibrium \( \{ p_A, m^R \} \), the expected payoff for a borrower to sell one unit goods and then demand redemption on the banknotes is \( \frac{p_A D(s)}{m^R} \). In the equilibrium, \( p_A = \frac{1 - m^R}{T(s)} \) and \( m^R = \frac{\int_0^1 D(s)f(s)ds}{\int_0^1 [D(s)+T(s)]f(s)ds} \). A borrower will sell exactly \( t(y) \) units of goods as required provided that \( \frac{p_A D(s)}{m^R} \leq 1 \) for any \( s \). That is,

\[ \frac{D(s)}{T(s)} \leq \frac{\int_0^1 D(s) f(s) \, ds}{\int_0^1 T(s) f(s) \, ds}, \quad \forall \, s. \tag{5} \]

\(^8\)This proof follows the proof of Proposition 1 in Williamson (1986), pp166.
Proposition 3  Given $x, t(y) \in [0, z(y)]$ must satisfy (i) $T(s) \neq T(s')$ for any $s \neq s' \in [0, 1]$; (ii) $\int_0^1 T(s) f(s) ds = \frac{R(x)}{1+\phi}$, where $\phi \geq 0$ is a constant.

Proposition 4  $D(s) = \phi T(s)$, for all $s$.

Proofs of Propositions 3-4 are provided in the Appendix.

By Proposition 3-4, the choices of $t(y)$ and $D(s)$ are indeterminate and can take on a continuum of values. However, the amount of goods devoted to redemption must be proportional to the aggregate supply of goods to market $A$. Let $x^*$ be the optimal solution and $V_1^*$ be the optimum of (3). The optimal contract is given by $C^* (x^*) = \{ z^*(y;x^*), t^*(y;x^*), \Omega^* (x^*), D^*(s;x^*) \}$. Define $\Psi_1 (x) = \frac{\gamma R'(x)}{1+R(x)} - \varepsilon [1-H(x)]$ and $\Psi_2 (x) = \frac{R(x)R'(x)}{1+R(x)} + \Psi_1 (x)$.

Assumption 1  $\varepsilon < \overline{y} \left[ 1 - \mu g(0) \int_0^1 f(s) \frac{1}{s} ds \right]$.

Assumption 2  $\Psi_1'' (x) > 0$ and $\Psi_2'' (x) > 0$ for all $x \in [0, 1]$.

Assumptions 1-2 ensure the following proposition:

Proposition 5  A unique banking equilibrium exists and the optimal contract $C^*$ is characterized by $x_1^* \in (0, 1)$.

Proof of Proposition 5 is provided in the Appendix.

The optimal contract is essentially a debt contract as in Williamson (1986). A borrower either makes a fixed level of repayment or defaults and gets monitored by the bank. Here what is not standard about the debt contract is that a specified amount of the repayment must be made in banknotes and the rest in real goods. As a result, borrowers must trade for banknotes in order to repay their loans. The induced market at the repayment stage generates information revealing prices that perfectly discipline the bank. The cost of monitoring the bank is completely saved even in an environment with aggregate uncertainty.
**Inside money and outside money.** Trivially, there will be no incentive problem of a bank if it does not promise redemption. That is, \( D(s) = 0 \) for any \( s \in [0, 1] \). Note that redeemability is not a necessary condition for banknotes to be valued. Indeed, the bank has two ways of making its notes valuable: the redemption channel and the repayment channel. With \( D(s) = 0 \) for any \( s \in [0, 1] \), banknotes can only reflux through the repayment channel. Nonetheless, banknotes are still valued because borrowers are obliged to trade goods for banknotes at the repayment stage.

This further indicates that the optimal contract can also be implemented with outside money. Suppose the bank is prohibited from issuing banknotes (i.e. inside money) while each type \( A \) agent is endowed with \( M \) units of outside money. The bank can write a contract with type \( A \) agents, which is essentially \( C^* \) with \( D(s) = 0 \) for any \( s \in [0, 1] \) and with outside money replacing every role of banknotes in the contract. The outcome will be exactly the same. That is, the cost of monitoring the bank can also be completely saved with the help of outside money.

**Monitoring technology.** In this model, it is assumed that type \( A \) projects can only be verified by type \( A \) agents although type \( C \) agents are able to monitor the bank on island \( A \). This assumption allows me to focus on the main issue being addressed: solving the incentive problem of a bank. Note, however, the conclusions of this paper do not hinge on this assumption.

Without this assumption, type \( C \) agents can monitor as efficiently as type \( A \) agents. Type \( A \) agents’ personal IOUs can be accepted by type \( C \) agents. By assuming a larger population on islands \( B \) and \( C \) than on island \( A \), one can show that this arrangement involves duplication of monitoring efforts by type \( C \) agents. This result corresponds to a well established point in the literature (Diamond [1984] and Williamson [1986]) that direct lending is typically associated with duplication of monitoring costs.

A seemingly promising solution is to have a type \( C \) bank, which collects type \( A \) agents’ IOUs from type \( C \) agents and monitors type \( A \) agents on behalf of all type \( C \) agents.
Duplication of monitoring is eliminated in this dimension. Nevertheless, now the $C$ bank single-handedly handles all the type $A$ goods. With aggregate uncertainty, it has every incentive to misreport to type $C$ agents. Therefore, type $C$ agents must monitor their bank. Duplication of monitoring efforts still exists and the problem of monitoring the monitor prevails. As has been established, banking by type $A$ agents gains access to an efficient device to cope with the problem of monitoring the monitor, which is to let market prices discipline the bank. This cannot be done by type $C$ banking. Thus type $A$ banking strictly dominates type $C$ banking because it completely saves the cost of monitoring the bank and hence improves efficiency.

5 Multiple Banks

Previously it has been established that the cost of monitoring the bank can be completely saved if money is required as part of the loan repayment. In this section, I study scenarios of multiple banks. In particular, I address whether the previous result is robust to the existence of multiple banks and whether the outcome will differ if inside money and outside money are implemented, respectively, in the banking contract.

5.1 Banking with inside money

Now suppose there are $n$ communities on island $A$, where $n \geq 2$ is an integer. Each community has a population of measure $\frac{1}{n}$. The endowment of an agent in community $j = 1, \cdots, n$ is given by $Y_j = S_j W$. The idiosyncratic shock $W$ is defined as before, while the local shock $S_j$ for each community $j$ is i.i.d. from $f(\cdot)$. The endowment of a project in community $j$ can only be verified by someone from community $j$. That is, type $A$ agents cannot monitor projects across communities. The rest of the model is characterized the same as previously.

Naturally, $n$ banks arise on island $A$, one bank for each community. Banks write
contracts with local community members. Contracts take the same form as specified in the previous section. Banks require loans to be repaid with goods and banknotes of their own. Normalize the amount of banknotes issued by a bank to \( M_j = \frac{1}{n} \) for all \( j \). Banknotes of various banks are identifiable and they co-circulate across islands. There will be \( n \) markets on each island. In each market, goods are traded for notes of a particular bank. As before, all terms of the contract are public information. Agents take prices as given and trade competitively in all markets.

Now consider type \( C \) agents’ responses. The expected net returns of one unit of \( j \)-banknote are

\[
R_j = \int_0^1 [D_j(s_j) + T_j(s_j)] f(s_j) ds_j - \Theta_j.
\]

Taking \( \{p^j_C, R_j\}_{j=1}^n \) as given, a representative type \( C \) agent maximizes his expected utility subject to the budget constraint:

\[
\max_{(c^2_C, m^j_C)} \ln c^2_C + \sum_{j=1}^n m^j_C R_j \\
\text{s.t.} \quad c^2_C + \sum_{j=1}^n \frac{m^j_C}{p^j_C} = \bar{y}
\]

where \( m^j_C \) is the agent’s holdings of \( j \)-banknotes.

Taking \( \{p^j_B, p^j_C\}_{j=1}^n \) as given, a representative type \( B \) agent maximizes his expected utility subject to the budget constraint:

\[
\max_{(c^1_B, m^j_B)} c^1_B + \sum_{j=1}^n \frac{m^j_B}{p^j_B} \\
\text{s.t.} \quad c^1_B + \sum_{j=1}^n \frac{m^j_B}{p^j_B} = \bar{y}
\]

where \( m^j_B \) is the agent’s holdings of \( j \)-banknotes. Once again I will focus on symmetric equilibria.

**Definition 2** A symmetric equilibrium with \( n \) banks and inside money consists of prices \( \{p^j_A, p^j_B, p^j_C\}_{j=1}^n \), contracts \( \{C_j\}_{j=1}^n \) and the associated expected net returns on banknotes \( \{R_j\}_{j=1}^n \), the equilibrium redemption demand \( \{m^R_j\}_{j=1}^n \), and individual agents’ choices
\[ \left\{ c^j_B, \{ m^j_B \}^n_{j=1}, c^j_C, \{ m^j_C \}^n_{j=1} \right\} \text{ such that: (i) given prices and returns, the choices of a representative agent of type } i = B, C \text{ are optimal; (ii) given } \{ R_{k \neq j} \}^n_{k=1} \text{ and the best responses } \left\{ c^j_B, \{ m^j_B \}^n_{j=1} \right\}, \text{ bank } j \text{ designs } C_j \text{ to maximize the expected utility of a local borrower; (iii) given } m^j_R, \text{ type } C \text{ agents are indifferent between redeeming } j \text{-banknotes and selling them in market } j; (iv) all markets clear; (v) symmetry of banks: } C_j = C \text{ and } R_j = R \text{ for all } j. \]

Similar to the case with one bank, in the equilibrium \( m^j_R = \frac{\int_{0}^{1} D_j(s_j)f(s_j)ds_j}{\int_{0}^{1} [D_j(s_j) + T_j(s_j)]f(s_j)ds_j} \) and \( p^j_A = \frac{1-m^j_R}{T_j(s_j)}. \) Hence,

\[
T_j(s_j) = \frac{\int_{0}^{1} T_j(s_j)f(s_j)ds_j}{p^j_A \int_{0}^{1} [D_j(s_j) + T_j(s_j)]f(s_j)ds_j}.
\]

Analogous to Proposition 1, one can prove the following proposition:\(^9\)

**Proposition 6** With \( n \) banks and inside money, bank \( j \) is perfectly disciplined by the corresponding market price \( p^j_A \) and thus \( \Theta_j = 0 \) for all \( j \), provided that (i) \( T_j(s_j), D_j(s_j) \geq 0 \) for all \( s_j = 0 \); (ii) \( T_j(s_j) \neq T_j(s^j_k) \) and \( D_j(s_j) \neq D_j(s^j_k) \) for any \( s_j \neq s^j_k \in [0, 1] \).

By Proposition 6, the expected net returns of one unit of \( j \)-banknote are given by \( R_j = \int_{0}^{1} [D_j(s_j) + T_j(s_j)]f(s_j)ds_j \) for all \( j \). Market clearing requires \( m^*_B = m^*_C = \frac{1}{n} \). Let \( q^j_i \) be the amount of goods sold by a type \( i = B, C \) agent in market \( j \). It is straightforward to derive that for all \( j \), \( q^*_B = q^*_C = \frac{\frac{n}{n+\sum_{k=1}^{n} R_k}}{n+\sum_{k=1}^{n} R_k} \). Notice that \( \frac{q^*_B}{q^*_j} = \frac{q^*_C}{q^*_j} = \frac{R_j}{R_k} \) for any \( j, k \). Intuitively, banknotes with higher expected returns have higher purchasing power.

Define \( R_{-j} = \sum_{j \neq k=1}^{n} R_k \). Given \( q^*_B \) and \( R_{-j} \), bank \( j \) seeks to maximize the expected utility of a local borrower:

\[
V_n = \max_{c_j} \left\{ \frac{nR_j}{n + R_j + R_{-j}} + \varepsilon \int_{0}^{1} [y_j - z_j(y_j)]h(y_j)dy_j \right\}
\]

\[
s.t. \quad \frac{1}{n} \int_{0}^{1} z_j(y_j)h(y_j)dy_j - \frac{\mu}{n} \int_{0}^{1} h(y_j)dy_j - \frac{R_j}{n} = 0.
\]

\(^9\)Implicitly, here I only consider inside money that can be redeemed. As mentioned before, when \( D_j(s_j) = 0 \) for all \( s_j \), there is no incentive problem of a bank.
where the first term in the objective is simply \( q_j^* / (\frac{1}{n}) \), which is the amount of type B goods a borrower of bank \( j \) can buy. The constraint is the same as in the one-bank case, except that here each bank has borrowers of measure \( \frac{1}{n} \) and issues banknotes of measure \( \frac{1}{n} \). Similarly to the one bank case, we have the following proposition:

**Proposition 7** With \( n \) banks, the optimal repayment schedule of any bank \( j \) is given by

\[
z_j^*(y) = \begin{cases} 
x_j, & \text{if } y_j \geq x_j \\
y_j, & \text{if } y_j < x_j
\end{cases}
\]

where \( x_j \in [0, 1] \) is a constant. Correspondingly, the optimal monitoring state is \( \Omega_j^* = \{y_j : 0 \leq y_j < x_j\} \).

Proof of Proposition 7 is provided in the Appendix.

Now (6) is simplified to:

\[
V_n = \max_{x_j \in [0,1]} \left\{ \frac{n \bar{y} R_j (x_j)}{n + R_j (x_j) + \bar{R}_{-j}} + \epsilon \int_{x_j}^{1} (y_j - x_j) h(y_j) dy_j \right\}
\]  

(7)

where \( R_j (x_j) = \int_0^{x_j} y_j h(y_j) dy_j + x_j [1 - H(x_j)] - \mu H(x_j) \). Analogous to the one-bank case, the main task of characterizing the rest of the contract is to pin down the optimal \( x_j \). Given \( x_j, t_j \) and \( D_j \) must satisfy the conditions in Propositions 3-4.

Assuming interior solutions, the first-order condition of (7) is:

\[
\frac{n \bar{y} (n + \bar{R}_{-j}) R'_j (x_j)}{[n + R_j (x_j) + \bar{R}_{-j}]^2} - \epsilon [1 - H(x_j)] = 0
\]

(8)

In a symmetric equilibrium, \( x_k = x \) and \( R_k (x_k) = R(x) \) for all \( k \), which implies \( \bar{R}_{-j} = (n - 1) R(x) \). Substituting these into (8), it becomes

\[
\frac{1 + \frac{n-1}{n} R(x)}{[1 + R(x)]^2} \bar{y} R'(x) - \epsilon [1 - H(x)] = 0.
\]

(9)
Let $x^*_n$ be the optimal choice and $V^*_n$ be the optimum of (7).

**Proposition 8** If $\mu < E(y) + \frac{n}{n-1}$, a unique symmetric equilibrium with $n$ banks and inside money exists and the optimal contract $C^*_n$ is characterized by $x^*_n \in (0, 1)$.

**Proposition 9** $V^*_n < V^*_1; x^*_n > x^*_1; R(x^*_n) > R(x^*_1)$.

Proofs of Propositions 8-9 are provided in the Appendix.

Propositions 8-9 demonstrate that banks compete for higher purchasing power of their own banknotes, which results in higher equilibrium returns of banknotes than with a single bank ($R[x^*_n] > R[x^*_1]$). This is because type $B$ and type $C$ agents value banknotes according to their returns. A bank tends to offer higher returns on its banknotes so that its borrowers can purchase more goods. However, the higher the banknote returns, the higher the repayment level required for borrowers. As a result, banks strive for an optimal market share of banknotes at the expense of the welfare of their borrowers ($V^*_n < V^*_1$). This also shows that borrowers prefer to have a unique bank as long as it is feasible.

### 5.2 Banking with outside money

Suppose banks are prohibited from issuing banknotes. Each type $A$ agent is endowed with $M$ units of outside money at the beginning of time.\textsuperscript{10} Outside money is identical. Once again, normalize $M = 1$. Outside money is not redeemable, i.e. $D_j(s_j) = 0$ for all $s_j$.\textsuperscript{11} The rest of the terms of a typical contract $C_j$ are the same as stated before. Banks announce local shocks $\{\tilde{s}_j\}^n_{j=1}$ simultaneously.

\textsuperscript{10}Here money is not actually lent to type $A$ agents by banks. But they still write contracts stipulating that type $A$ agents should make payments to banks at $t = 3$, as if they had borrowed from banks. As is explained previously, in this model money needs the support of banking in order to be valued.

\textsuperscript{11}This comes naturally with the model because type $A$ and type $C$ agents do not meet until $t = 3$. If they could meet upfront, banks might contract with type $C$ agents as well, promising to redeem outside money for goods. In that case, $D_j(s_j)$ could be chosen in a variety of ways as with inside money. However, the result would not be any different as has been shown that the equilibrium outcome only depends on the repayment level $x$. 

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Accordingly, outside money is valued because agents know that type $A$ agents will need it to fulfill their contractual obligations. Since outside money is identical, there will be only one market on every island, where goods are traded for outside money.

**Definition 3** A symmetric equilibrium with $n$ banks and outside money consists of market prices $\{p_A, p_B, p_C\}$, contracts $\{C_j\}_{j=1}^n$ and the associated expected net returns on outside money $\{R_j\}_{j=1}^n$, and individual agents’ choices $\{c^1_B, m_B, c^2_C, m_C\}$ such that: (i) given prices and returns, the choices of a representative agent of type $i = B, C$ are optimal; (ii) given $\{R_k\}_{k \neq j}^n$ and the best responses $\{c^1_B, m_B\}$, bank $j$ designs $C_j$ to maximize the expected utility of a local contracted agent; (iii) all markets clear; (iv) symmetry of banks: $C_j = C$ and $R_j = R$ for all $j = 1, \ldots, n$.

The aggregate supply of goods in market $A$ is given by

$$\sum_j T_j(s_j) = \sum_j \frac{1}{n} \int_0^1 t_j(s_j w) g(w) \, dw.$$ 

As long as $T_j$ is fully revealing of $s_j$ for all $j$, $p_A$ can be used to discipline banks. To see this, suppose $\tilde{s}_k = s_k$ for all $k \neq j$, then $T_j(\tilde{s}_j) + \sum_{k \neq j} T_k(\tilde{s}_k) = T_j(s_j) + \sum_{k \neq j} T_k(s_k) \neq \frac{1}{p_A}$ if and only if $\tilde{s}_j \neq s_j$. Therefore, the price $p_A$ nests the information of all banks. Given that all the other banks choose to announce their local shocks truthfully, it is also optimal for a bank to announce the true local shock it has received. Hence the following proposition:

**Proposition 10** With $n$ banks and outside money, the market price $p_A$ is information revealing if (i) $T_j(s_j) \geq 0$ for all $s_j$; (ii) $T_j(s_j) \neq T_j(s'_j)$ for any $s_j \neq s'_j \in [0, 1]$. Banks are disciplined collectively by $p_A$ and the expected cost of monitoring a bank $\Theta_j = 0$ for all $j$.

Similarly to the case of a single bank, in equilibrium $p_B = \frac{1}{y-c^1_B}$ and $p_C = \frac{1}{y-c^2_C}$. It is straightforward to derive $c^1_B = c^2_C = \frac{-1}{1+\frac{1}{n} \sum_{k=1}^R R_k}$. Note that the value of outside money is determined by the aggregate returns $\frac{1}{n} \sum_{k=1} R_k$. Again define $\mathbb{R}_{-j} = \sum_{j \neq k=1}^R R_k$. Given
$c^*_B$ and $R_{-j}$, the problem of bank $j$ is:

$$V'_n = \max_{C_j} \left\{ \bar{y} - \frac{\bar{y}}{1 + \frac{1}{n} (R_j + R_{-j})} + \varepsilon \int_0^1 [y_j - z_j (y_j)] h (y_j) dy_j \right\}$$

s.t. $\frac{1}{n} \int_0^1 z_j (y_j) h (y_j) dy_j - \frac{\mu}{n} \int_{\Omega_j} h (y_j) dy_j - \frac{R_j}{n} = 0.$

It is straightforward to prove that Proposition 7 also applies here. (The proof is omitted for brevity.) Accordingly, the above problem can be simplified to:

$$V'_n = \max_{x_j \in [0,1]} \left\{ \frac{\bar{y} [R_j (x_j) + R_{-j}]}{n + R_j (x_j) + R_{-j}} + \varepsilon \int_0^1 [y_j - z_j (y_j)] h (y_j) dy_j \right\}, \quad (10)$$

where $R_j (x_j) = \int_0^{x_j} y_j h (y_j) dy_j + x_j [1 - H (x_j)] - \mu H (x_j)$. Assuming interior solutions, the first-order condition is:

$$\frac{n \bar{y} R'_j (x_j)}{[n + R_j (x_j) + R_{-j}]^2} - \varepsilon [1 - H (x_j)] = 0$$

In a symmetric equilibrium, the above becomes

$$\frac{\bar{y} R' (x)}{n [1 + R (x)]^2} - \varepsilon [1 - H (x)] = 0. \quad (11)$$

Let $x^*_n$ be the optimal choice and $V'^*_n$ be the optimum.

**Proposition 11** If $\varepsilon < \frac{\bar{y}}{n} \left[ 1 - \mu g (0) \int_0^1 f (s) \frac{1}{s} ds \right]$, a unique symmetric equilibrium with $n$ banks and outside money exists and the optimal contract $C^*_n$ is characterized by $x^*_n \in (0, 1)$.

**Proposition 12** $V'^*_n < V^*_1$; $x'^*_n < x^*_1 < x^*_n$; $R (x'^*_n) < R (x^*_1) < R (x^*_n)$.

Proofs of Propositions 11-12 are provided in the Appendix.

When private provision of money is permitted, a bank issues its own banknotes to the benefit of its own borrowers. In contrast, here the use of outside money is not exclusive of
contracted agents of any bank. The returns of outside money are given by the aggregate of the returns offered by all banks, \( \frac{1}{n} \sum_{k=1}^{n} R_k \). The support for outside money by any one bank \( R_k \) benefits all type A agents. The equilibrium returns on outside money are inefficiently low as banks make private decisions on the amount of goods to offer for outside money. Intuitively, banks get free rides from one another on supporting outside money, causing a less than optimal level. Moreover, outside money does not change the result that borrowers are worse off with the presence of multiple banks \( (V^*_n < V^*_1) \).

### 5.3 Welfare

Assign equal weights to all agents in the economy. Given the equilibrium level of repayment \( x \in (0, 1) \), the welfare level is given by \( W(x) = U_A(x) + U_B(x) + U_C(x) \).

**Proposition 13** \( W(x^*_n) < W(x^*_1) < W(x^*_1) \).

Proof of Proposition 13 is provided in the Appendix.

**A numerical example.** Set \( f(s) = 2s \), \( g(w) = 1 \), \( \mu = 0.2 \), \( \varepsilon = 0.2 \), \( \bar{y} = 0.8 \). Figure 2 illustrates the equilibrium outcomes of this example. Welfare \( W(x) \) is maximized at \( x = 0.45 \) and the expected utility of a type A agent \( V(x) \) is maximized at \( x = 0.41 \). With multiple banks, a prohibition on private money issue will shift the economy from the region of \( x \in (0.41, 0.45) \) to the region of \( x \in (0, 0.41) \), resulting in a strictly lower level of welfare.

[Insert Figure 2]

In this model, type B and type C agents are essentially lenders. Although they do not directly lend to banks, they are willing to trade goods for money, which is (direct or indirect) liabilities of banks. It is as if type B agents provided credit to banks and then
sold the credit to type $C$ agents. Eventually, type $C$ agents benefit from money like a deserving lender.

Proposition 13 establishes that with multiple banks, it is inefficient to prohibit the issue of private money. While the inside money regime fosters money competition, the outside money regime eliminates competition and, to make things worse, triggers incentive problems of free-riding among banks. The equilibrium returns of money $R$ decline with the imposition of outside money. Lenders (type $B$ and type $C$ agents together) are worse off because their welfare is strictly increasing with $R$. Although the welfare of borrowers (type $A$ agents) may not be decreased, the overall effect on social welfare is negative ($W[x_n^*] < W[x_n^*]$). To summarize, it is inefficient to prohibit banks from issuing banknotes. If it has to be done, then it is a good idea to make outside money differentiable. That is, the policy of prohibition should be accompanied with the imposition of bank-specific outside money. As a result, each bank can work with a unique kind of outside money. Then the outcome would be the same as with inside money.

6 Conclusion

This paper studied money and banking in a model with private information and aggregate uncertainty. The cost of monitoring a bank can be completely saved if the bank requires money as a means of loan repayment. As a result, a market arises at the repayment stage and generates an information-revealing price that perfectly disciplines the bank. Thus the incentive problem of a bank is overcome costlessly. This mechanism can be readily applied to scenarios of multiple banks. If private issue of money is permitted, there will arise as many markets at the repayment stage as the number of banks. In each market, a unique kind of inside money is traded for goods. The market price fully reveals the bank-specific information. If private money issue is prohibited, only one market will arise at the repayment stage, where outside money is traded. The market price disciplines banks collectively.
The model suggests that in the presence of multiple banks, a prohibition on private money issue gives rise to inefficient outcomes. While the inside money regime fosters money competition, the outside money regime eliminates competition and triggers incentive problems of free-riding among banks. As a result, the equilibrium returns of outside money are inefficiently low as banks make private decisions. The overall effect on welfare is negative. The policy of prohibition should be accompanied with the imposition of bank-specific outside money to avoid the negative effect on welfare.
Appendix

Proof of Proposition 2. Consider the borrower reports \( \tilde{y} \notin \Omega \) so that monitoring does not occur. It follows that the repayment level must be the same for all \( y \in \overline{\Omega} \). Otherwise, suppose \( z(y_1) < z(y_2) \) for any \( y_1, y_2 \in \overline{\Omega} \) and \( y_1 \neq y_2 \). Then the borrower will always report \( y_1 \) instead of \( y_2 \). Hence, incentive compatibility requires that \( z(y) = x \) for \( y \in \overline{\Omega} \) and \( z(y) < x \) for \( y \in \Omega \), where \( x \) is a constant. Moreover, repayment schedule must be feasible, that is, \( 0 \leq z(y) \leq y \). Together, we need \( 0 \leq z(y) \leq y < x \).

Now it remains to prove that it is optimal to have \( z(y) = y \) for \( y \in \Omega \). Define \( \Omega_1 = \{ y : 0 \leq y < x \} \) and \( \Omega_2 = \{ y : 0 \leq \tilde{z}(y) < x \} \) for any function \( \tilde{z} : [0, 1] \to [0, y] \). Obviously, \( \Omega_1 \subseteq \Omega_2 \). Substituting the constraint into the objective, (2) becomes

\[
V_1 = \max_{C} \left\{ \frac{yR}{1 + R} + \varepsilon E(y) - \varepsilon R - \varepsilon \mu \int_{\Omega} h(y) \, dy \right\}.
\]

For any given \( R \) that the bank chooses to offer on banknotes, the bank’s problem remains to minimize \( \int_{\Omega} h(y) \, dy \). By \( \Omega_1 \subseteq \Omega_2 \), it follows that \( \int_{\Omega_1} h(y) \, dy \leq \int_{\Omega_2} h(y) \, dy \). Therefore, it is optimal to have \( \Omega = \{ y : 0 \leq y < x \} \) and \( z(y) = y \) for \( y \in \Omega \). ■

Proof of Propositions 3-4. Condition (i) of Proposition 3 is by Proposition 1. Note that \( t(y) \geq 0 \) implies that \( T(s) \geq 0 \) for all \( s \). Define \( \phi = \left[ \int_{0}^{1} D(s) \, f(s) \, ds \right] / \left[ \int_{0}^{1} T(s) \, f(s) \, ds \right] \). Suppose \( \frac{D(s)}{T(s)} \leq \phi \), with strict inequality for some \( s \). It follows that

\[
\phi = \frac{\int_{0}^{1} D(s) \, f(s) \, ds}{\int_{0}^{1} T(s) \, f(s) \, ds} < \frac{\phi \int_{0}^{1} T(s) \, f(s) \, ds}{\int_{0}^{1} T(s) \, f(s) \, ds} = \phi,
\]

which is an obvious contradiction. Therefore, (5) implies that \( \frac{D(s)}{T(s)} = \phi \) for all \( s \), which proved Proposition 4. By condition (i) of Proposition 3, it follows that \( D(s) \neq D(s') \) for any given \( s \neq s' \in [0, 1] \), which guarantees that condition (ii) of Proposition 1 holds. To guarantee the expected returns on banknotes are \( R(x) \) for given \( x \), \( \int_{0}^{1} [D(s) + T(s)] \, f(s) \, ds = R(x) \). Thus condition (i) of Proposition 3 holds by Proposition 4. ■
Proof of Proposition 5. Assuming interior solutions, the first-order condition of (3) is:

\[
\frac{\gamma R'(x)}{[1 + R'(x)]^2} - \varepsilon [1 - H(x)] = 0.
\] (12)

Note that the left-hand side is exactly \(\Psi_1(x)\). Derive \(\Psi_1'(x) = \frac{\gamma R'(x)[1+R(x)]-2[R'(x)]^2}{[1+R(x)]^3} + \varepsilon h(x)\). By \(h(y) = \int_y^1 f(s) g\left(\frac{y}{s}\right) \frac{1}{s} ds\), we have \(h'(y) = -f(y) g'(1) \frac{1}{y} + \int_y^1 f(s) g'\left(\frac{y}{s}\right) \frac{1}{s^2} ds\). It follows that \(h(0) = g(0) \int_0^1 f(s) \frac{1}{s} ds\), \(h(1) = 0\) and \(h'(1) = -f(1) g(1)\). Then it is straightforward to derive from (4) that \(R(0) = 0\), \(R'(0) = 1 - \mu h(0)\), \(R(1) = E(y) - \mu\), \(R'(1) = 0\) and \(R''(1) = \mu f(1) g(1)\). Therefore, \(\Psi_1(0) = \gamma [1 - \mu h(0)] - \varepsilon\), \(\Psi_1(1) = 0\), \(\Psi_1'(1) = \frac{\gamma \mu f(1) g(1)}{1 + E(y) - \mu}\) > 0. By continuity, if \(\Psi_1(0) > 0\), there exist at least one \(x \in (0,1)\) such that \(\Psi_1(x) = 0\) and \(\Psi_1'(x) < 0\). Assumption 1 ensures that \(\Psi_1(0) > 0\) holds. Since \(\Psi_1(0) > 0\), \(\Psi_1(1) = 0\), \(\Psi_1'(1) > 0\) and \(\Psi_1''(x) > 0\) for all \(x \in [0,1]\) by Assumption 2, there exists a unique \(x_1^* \in (0,1)\) that satisfies \(\Psi_1(x) = 0\) and \(\Psi_1'(x) < 0\).

Proof of Proposition 7. The proof is analogous to the proof of Proposition 2. For any given \(R_j\) that bank \(j\) chooses to offer, the problem remains to minimize \(\int_{\Omega_j} h(y_j) dy_j\). Refer to the proof of Proposition 2 for details. ■

Proof of Propositions 8-9. Define the left-hand side of (9) as \(LHS(x)\). Then \(LHS'(x) = \frac{(n-1)(1+R)-2[n+(n-1)R]}{n(1+R)^2} \gamma R'^2 + \frac{n+(n-1)R}{n(1+R)^2} \gamma R'' + \varepsilon h(x)\). Previously in the proof of Proposition 4, it has been derived that \(R(0) = 0, R'(0) = 1 - \mu h(0), R(1) = E(y) - \mu, R'(1) = 0\) and \(R''(1) = \mu f(1) g(1)\). It follows that \(LHS(0) = \gamma [1 - \mu h(0)] - \varepsilon\), \(LHS(1) = 0\), and \(LHS'(1) = \frac{n+(n-1)[E(y) - \mu]}{n(1+|E(y)-\mu|)} \gamma \mu f(1) g(1) > 0\) if \(\mu < E(y) + \frac{n}{n-1}\). By continuity, if \(LHS(0) > 0\), there exist at least one \(x \in (0,1)\) such that \(LHS(x) = 0\) and \(LHS'(x) < 0\). Note that \(LHS(0) = \Psi_1(0)\). Assumption 1 ensures \(LHS(0) > 0\). Therefore, \(x_n^* \in (0,1)\).

Define set \(X = \{x \in (0,1): R(x) > 0\) and \(R(x) > R(x')\) for all \(x' \in [0,x]\}\). The following proves that \(x_n^* \in X\) must be true. Obviously, \(R(x_n^*) > 0\). Now suppose \(R(x_n^*) \leq R(x')\) for some \(x' \in [0,x_n^*)\). (i) If \(R(x^*) = R(x')\), then \(V_n|_{x=x'} - V_n|_{x=x_n^*} > 0\) because
\[ \varepsilon \int_{x}^{1} (y_j - x) h(y_j) dy_j \text{ is strictly decreasing in } x. \] Hence \( x_n^* \) cannot be the optimal choice, which is a contradiction. (ii) Consider \( R(x_n^*) < R(x') \). Since \( R(0) = 0 \) and \( x' < x_n^* \), continuity implies that there must exist some \( x'' \in [0, x') \) such that \( R(x'') = R(x_n^*) \). By the same argument, \( x_n^* \) cannot be the optimal choice, which is another contradiction. Therefore, it must be true that \( R(x_n^*) > R(x') \) for all \( x' \in [0, x_n^*) \) and that \( x_n^* \in X \). Since \( R(x_1) < R(x_2) \) for any \( x_1 < x_2 \in X \), \( x_n^* \) must be unique. This proved Proposition 8. Note that it can be proven analogously for the one-bank case that \( x_1^* \in X \).

From (7),
\[ V_n^* = \frac{\bar{y} R(x_n^*)}{1 + R(x_n^*)} + \varepsilon \int_{x_n}^{1} (y_j - x_n^*) h(y_j) dy_j. \]
The right-hand side is exactly the objective of \( V_1 \). Recall that \( x_1^* \) is the maximizer of \( V_1 \) and it satisfies (12), that is,
\[ \frac{\bar{y} R'(x)}{[1 + R(x)]^2} = \varepsilon \left[ 1 - H(x) \right]. \tag{13} \]
By (9), \( x_n^* \) satisfies
\[ \frac{\bar{y} R'(x)}{[1 + R(x)]^2} = \frac{\varepsilon [1 - H(x)]}{1 + \frac{n-1}{n} R(x)} < \varepsilon \left[ 1 - H(x) \right]. \tag{14} \]
The inequality holds for \( n > 1 \) because \( R(x_n^*) > 0 \). Therefore, \( x_n^* \neq x^* \), which implies \( V_n^* < V_1^* \). By (14), \( \Psi_1(x_n^*) < 0 \) where \( \Psi_1(x) \) is the left-hand side of (12). It has been shown in the proof of Proposition 4 that \( \Psi_1(0) > 0, \Psi_1(1) = 0, \Psi_1'(1) > 0 \) and \( \Psi_1''(x) > 0 \) for all \( x \in [0, 1] \). It follows that \( x_n^* > x_1^* \). Hence, \( R(x_n^*) > R(x_1^*) \) because \( x_1^* \in X \) and \( x_n^* \in X \), which proved Proposition 9. ■

**Proof of Propositions** 11-12. The proof of Proposition 11 is analogous to the proof of Proposition 8. The main steps are to show (i) \( x_n^* \in (0, 1) \) if \( \varepsilon < \frac{2}{n} \left[ 1 - \mu g(0) \int_{0}^{1} f(s) \frac{1}{s} ds \right] \) and (ii) \( x_0^* \in X \) and \( x_n^* \) is unique, where \( X = \{ x \in (0, 1) : R(x) > 0 \text{ and } R(x) > R(x') \text{ for all } x' \in [0, x] \} \). (Details of the proof are omitted for brevity.)
From (10),
\[ V^*_n = \frac{\overline{y} R(x^*_n)}{1 + R(x^*_n)} + \varepsilon \int_{x^*_n}^{1} (y_j - x^*_n) h(y_j) dy_j. \]  
(15)

Note that the right-hand side is exactly the objective of \( V_1 \). Recall that \( x^*_1 \) maximizes \( V_1 \) and it satisfies (12), that is,
\[ \frac{\overline{y} R'(x)}{[1 + R(x)]^2} = \varepsilon [1 - H(x)]. \]  
(16)

By (11), \( x^*_n \) satisfies
\[ \frac{\overline{y} R'(x)}{[1 + R(x)]^2} = n\varepsilon [1 - H(x)] > \varepsilon [1 - H(x)]. \]  
(17)

The inequality holds for all \( n \geq 2 \). Therefore, \( x^*_n \neq x^*_1 \) and hence \( V^*_n < V^*_1 \). By (17), \( \Psi_1(x^*_n) > 0 \) where \( \Psi_1(x) \) is the left-hand side of (12). It has been shown in the proof of Proposition 4 that \( \Psi_1(0) > 0, \Psi_1(1) = 0, \Psi'_1(1) > 0 \) and \( \Psi''_1(x) > 0 \) for all \( x \in [0, 1] \). It follows that \( x^*_n < x^*_1 \) and thus \( R(x^*_n) < R(x^*_1) \) because \( x^*_1 \in X, x^*_n \in X \) and \( R(x_1) < R(x_2) \) for any \( x_1 < x_2 \in X \). By Proposition 9, \( x^*_n < x^*_1 < x^*_n \) and \( R(x^*_n) < R(x^*_1) < R(x^*_n) \), which proved Proposition 12.  

**Proof of Proposition 13.** In the symmetric equilibrium, \( c^1_B = c^2_B = \frac{\overline{y}}{1 + R(x)} \). Hence,
\[ W(x) = \frac{\overline{y} R(x)}{1 + R(x)} + \varepsilon \int_x^1 (y - x) h(y) dy + \frac{\overline{y} R(x)}{1 + R(x)} + \frac{\overline{y} R(x)}{1 + R(x)} + \ln \frac{\overline{y}}{1 + R(x)} + R(x) \]
\[ = \frac{\overline{y} R(x)}{1 + R(x)} + \varepsilon \int_x^1 (y - x) h(y) dy + \overline{y} + \ln \overline{y} - \ln [1 + R(x)] + R(x). \]

Suppose \( W(x) \) is maximized at some \( x \in (0, 1) \). Then it must satisfy \( W'(x) = 0 \) and \( W''(x) < 0 \). Then the first-order condition is given by
\[ \left\{ \frac{\overline{y}}{[1 + R(x)]^2} + \frac{R(x)}{1 + R(x)} \right\} R'(x) - \varepsilon [1 - H(x)] = 0. \]  
(18)

The left-hand side of the above equation is exactly \( \Psi_2(x) \). Analogous to the proof of
Proposition 4, by Assumptions 1-2 it is straightforward to show that there exists a unique \( \hat{x} \in (0, 1) \) such that \( W' (\hat{x}) = 0 \) and \( W'' (\hat{x}) < 0 \). (18) implies that

\[
R' (\hat{x}) = \frac{\varepsilon [1 - H (\hat{x})] [1 + R (\hat{x})]^2}{\bar{y} + R (\hat{x}) [1 + R (\hat{x})]}.
\]

By (9),

\[
R' (x_n^*) = \frac{\varepsilon [1 - H (x_n^*)] [1 + R (x_n^*)]^2}{\bar{y} [1 + \frac{n-1}{n} R (x_n^*)]} > \frac{\varepsilon [1 - H (x_n^*)] [1 + R (x_n^*)]^2}{\bar{y} + R (x_n^*) [1 + R (x_n^*)]}.
\]

The inequality holds because \( R (x_n^*) > 0 \) and \( 0 < \bar{y} \leq 1 \). The above implies \( \Psi_2 (x_n^*) > 0 \).

By Assumption 1-2, \( \Psi_2 (0) > 0 \), \( \Psi_2 (1) = 0 \) and \( \Psi_2'' (x) > 0 \) for all \( x \in [0, 1] \). Therefore, it must be true that \( x_n^* < \hat{x} \). Proposition 12 implies \( x_n^{**} < x_1^* < x_n^* < \hat{x} \). It follows that \( W (x_n^{**}) < W (x_1^*) < W (x_n^*) < W (\hat{x}) \), which proved Proposition 13. ■
References


Figure 1 Potential monetary trades