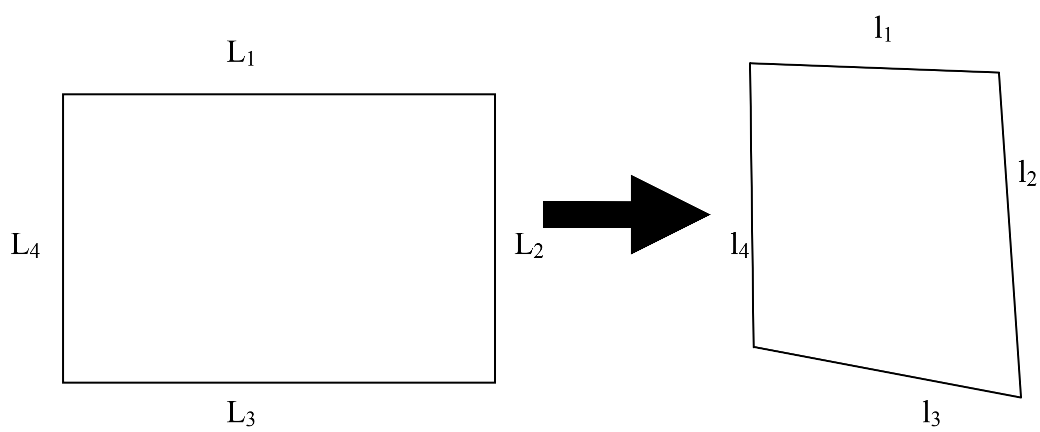
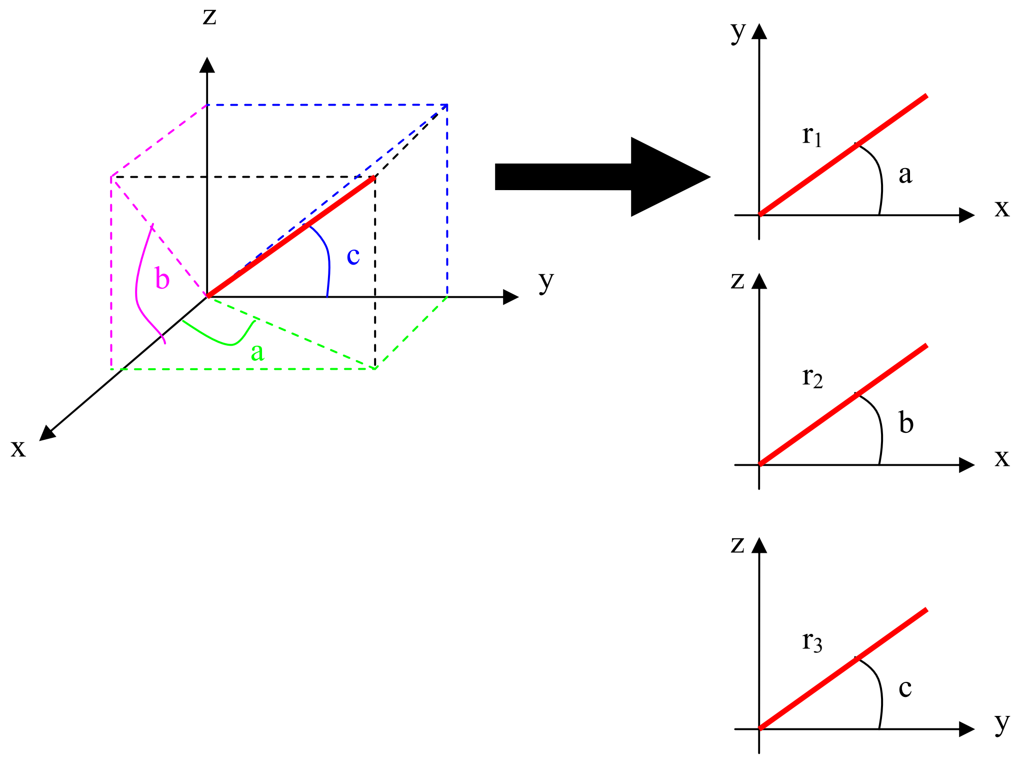
MR
28 July 2007

Consider the general form of the problem: the arbitrary, three-dimensional rotation of a rigid rectangle with sides of known lengths L_i. The observed lengths of the rotated body are l_i. Since all L_i are known, all (X_i,Y_i) are also known by arbitrarily assigning a coordinate basis. Further, in the rotated form, (x_i,y_i) are known (and, therefore, l_i). Thus, the only unknowns are z_i. (NOTE: Upper case letters always refer to the two-dimensional figure, whereas lower case letters refer to the rotated image. Also note that the rotated image may always be arranged such that one rotational component, about either the x or y axis, of one edge is zero.)



Next, consider the arbitrary rotation of any line segment in three dimensions by decomposing the rotations into their two-dimensional components. The line segment created by the rotated object projects a length r_i into each plane and is related to the initial length by some angle of orientation a, b, or c, depending on the plane.



Finally, the true three-dimensional length R of each line segment is known and is related to all r_i by

(1) $$R = \sqrt{r_1^2 + r_2^2 + r_3^2}\,.$$

Thus, simple geometry is required to extract the z-coordinate of each corner pixel in the rotated image. The z-coordinate is dependent on each of the angles of rotation a, b, and c such that they must all be found simultaneously, yielding a system of seven equations (below) and seven unknowns (a, b, c, r_1, r_2, r_3, and z) for each line segment.

(2) $$a = \tan^{-1}\left(\frac{y}{x}\right),$$ (3) $$r_1 = \frac{x}{\cos(a)}$$

(4) $$b = \tan^{-1}\left(\frac{z}{x}\right),$$ (5) $$r_2 = \frac{x}{\cos(b)}$$

(6) $$c = \tan^{-1}\left(\frac{z}{y}\right),$$ (7) $$r_3 = \frac{y}{\cos(c)}$$