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Momentum distribution and correlation function of quasicondensates in elongated traps

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We calculate the spatial correlation function and momentum distribution of a phase-fluctuating, elongated three-dimensional condensate in a trap and in free expansion. We take the inhomogeneous density profile into account via a local-density approximation. We find an almost Lorentzian momentum distribution, in stark contrast with a Heisenberg-limited Thomas-Fermi condensate.

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Low-dimensional, degenerate Bose gases are expected to have significantly different coherence properties than their three-dimensional (3D) counterparts. In one-dimensional (1D) uniform systems, no true condensate can exist at any temperature T because of a large population of low-lying states that destroys the phase coherence (see Ref. [1] and references therein). For a trapped gas, the situation is different—the finite size of the sample naturally introduces a low-momentum cutoff, and at sufficiently low temperature, $T \ll T_\phi$, a phase coherent sample can exist [1]. Above T_ϕ, the degenerate cloud is a so-called *quasicondensate*—the density has the same smooth profile as a true condensate, but the phase fluctuates in space and time. As shown in Ref. [2], this analysis holds also for 3D condensates in elongated traps even if, strictly speaking, radial motion is not frozen. Such 3D, phase-fluctuating condensates have been recently observed experimentally in equilibrium [3] and nonequilibrium [4] samples.

Phase fluctuations of the condensate are caused mainly by long-wavelength (or low-energy) collective excitations [1,2,5]. In elongated traps, the lowest-energy modes are 1D excitations along the long axis of the trap [6]. Furthermore, in the long-wavelength limit, density fluctuations are small and can be neglected for the calculation of the correlation function [1,7]. Then, the single-particle density matrix is, assuming cylindrical symmetry,

$$\langle \hat{\Psi}^\dagger(\rho,z)\hat{\Psi}(\rho,z')\rangle \approx \chi(\rho,Z,s)e^{-(1/2)\Delta\phi^2(Z,s)}. \tag{1}$$

We have introduced $\Delta\phi^2(Z,s)=\langle[\phi(z)-\phi(z')]^2\rangle$, the variance of the phase difference between two points z,z' on the axis of the trap, with mean coordinate $Z=(z+z')/2$ and relative distance $s=z-z'$, and the overlap function $\chi=\sqrt{n_0(\rho,z)n_0(\rho,z')}$, where n_0 is the (quasi)condensate density. The variance $\Delta\phi^2(Z,s)$, the key quantity to characterize the spatial fluctuations of the phase of the condensate, has been calculated in Ref. [2], and an analytical form has been given, which is valid near the center of the trap (i.e., for $Z,s \ll L$, with L the condensate half length). The first goal of this paper is to find an analytical approximation for the variance $\Delta\phi^2(Z,s)$ valid across the whole sample. This is motivated by the fact that experiments with quasicondensates [15,16] are quite sensitive to the inhomogeneity of the sample. In position space, interferometry [10,11] gives access to the spatial correlation function $\mathcal{C}^{(1)}(s)$ (see, e.g., Ref. [12])

$$\mathcal{C}^{(1)}(s)=\int d^3\mathbf{R}\langle \hat{\Psi}^\dagger(\rho,Z+s/2)\hat{\Psi}(\rho,Z-s/2)\rangle. \tag{2}$$

Equivalently, one can measure the axial (i.e., integrated over transverse momenta) momentum distribution $\mathcal{P}(p_z)$, which is the Fourier transform of $\mathcal{C}^{(1)}(s)$ [12,13,14]:

$$\mathcal{P}(p_z)=\frac{1}{2\pi\hbar}\int ds\,\mathcal{C}^{(1)}(s)e^{-ip_z s/\hbar}. \tag{3}$$

A powerful tool to measure $\mathcal{P}(p_z)$ is Bragg spectroscopy with large momentum transfer, as demonstrated in Ref. [14] for a 3D condensate, and recently applied in our group to perform the momentum spectroscopy of a quasicondensate [15]. It is clear that both $\mathcal{C}^{(1)}$ and $\mathcal{P}$ are sensitive to the inhomogeneity of the system. Our second goal is to obtain explicit expressions for these two important quantities.

This paper is organized as follows. First, we summarize the results of Ref. [2], and give an energetic interpretation of T_ϕ. Next, we discuss in detail a local-density approach (LDA) to compute the variance of the phase for any mean position in the trap. This approximation is found to be accurate for $T \geq 8T_\phi$, when applied to a trapped condensate. Using the LDA, we then address the problem of a phase-fluctuating condensate in free expansion. In particular, we point out that at a higher temperature, the phase fluctuations dominate over the mean-field release velocity and govern the shape of the momentum distribution.

We consider N_0 condensed atoms, trapped in a cylindrically symmetric harmonic trap, with an aspect ratio $\lambda=\omega_z/\omega_\perp \ll 1$. If $\mu \gg \{\hbar\omega_\perp,\hbar\omega_z\}$, the condensate is in the 3D Thomas-Fermi (TF) regime [16]. The density has the well-known inverted parabola form: $n_0(\mathbf{r})=n_{0\mathrm{m}}(1-\tilde{\rho}^2-\tilde{z}^2)$, with the peak density $n_{0\mathrm{m}}=\mu/g$ related to the chemical potential μ. From now on, we will use the reduced coordinates $\tilde{\rho}=\rho/R$ and $\tilde{z}=z/L$, with $R^2=2\mu/M\omega_\perp^2$ and $L^2=2\mu/M\omega_z^2$, respectively.

As shown in Ref. [1], the phase fluctuations in trapped gases are mostly associated with thermally excited, low-energy quasiparticles (the quantum fluctuations are negligible). Under these conditions, the variance $\Delta\phi^2(\tilde{Z},\tilde{s})$ is

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$$\Delta\phi^2(\tilde{Z},\tilde{s}) \approx \sum_j \frac{2k_BT}{\hbar\omega_j}|\phi_j(\tilde{Z}+\tilde{s}/2)-\phi_j(\tilde{Z}-\tilde{s}/2)|^2, \quad (4)$$

where the sum extends over the 1D axial excitations, with energy $\hbar\omega_j$ and occupation number $N_j \approx k_BT/\hbar\omega_j$ for $N_j \gg 1$. For a 3D condensate in an elongated trap, the amplitude ϕ_j is proportional to a Jacobi polynomial $P_j^{(1,1)}$, and $\omega_j = \omega_z\sqrt{j(j+3)}/2$ [6] for integer j. The explicit result for the variance is then [2]

$$\Delta\phi^2(\tilde{Z},\tilde{s}) = \frac{T}{T_\phi}f(\tilde{Z},\tilde{s}), \quad (5)$$

with $f(\tilde{Z},\tilde{s}) = \Sigma_j F_j[P_j^{(1,1)}(\tilde{Z}+\tilde{s}/2)-P_j^{(1,1)}(\tilde{Z}-\tilde{s}/2)]^2$ and the coefficients $F_j=(j+2)(j+3/2)/4j(j+1)(j+3)$. Below the characteristic temperature $T_\phi = 15N_0(\hbar\omega_z)^2/32\mu k_B$, the phase profile is almost flat, and the single-particle density matrix (1) is limited by the overlap function χ; therefore the characteristic width of $\mathcal{C}^{(1)}$ (i.e., the coherence length) is of the order of L. On the other hand, if $T \gg T_\phi$ the variance $\Delta\phi^2$ dominates the behavior of $\mathcal{C}^{(1)}$, and the coherence length is substantially smaller than L. Near the center of the trap ($\tilde{Z},\tilde{s} \ll 1$), Petrov *et al.* [2] have derived the simple law $\Delta\phi^2(\tilde{Z},\tilde{s}) \approx (T/T_\phi)|\tilde{s}|$ and introduced the characteristic phase-coherence length $L_\phi = LT_\phi/T$ which depends implicitly on the temperature, on the number of condensed atoms, and on the trapping geometry.

We can understand this expression for L_ϕ from energetic considerations. A random-phase gradient of the condensate wave function, on a length scale L_ϕ, requires an average kinetic energy $E_\phi \sim N_0\hbar^2/ML_\phi^2$. This kinetic energy is supplied by the thermal excitations that drive the fluctuations of the phase [5]. As these excitations are quasiclassical ($N_k \gg 1$), this energy is of order k_BT times the number of relevant modes. In 1D k space, the distribution of the relevant excitations extends over $\sim 1/L_\phi$, and the spacing between modes is $\sim 1/L$ because of the finite size of the system—this gives L/L_ϕ relevant modes. By equating the two expressions for E_ϕ, we recover finally $L_\phi \sim LN_0(\hbar\omega_z)^2/\mu k_BT$.

As indicated earlier, it is important to take the full spatial dependence of $\Delta\phi^2(\tilde{Z},\tilde{s})$ into account for quantitative comparison with experiments. In any case, Eq. (5) can be evaluated numerically. However, we gain physical insight with an analytical approach based on the LDA, also used in Ref. [3] to calculate the evolution of the density in time of flight. This approximation considers that the condensate is locally equivalent to a homogeneous medium, however, with a slowly varying density that depends on the trapping potential. If $T \gg T_\phi$, the coherence length is sufficiently small compared to L, so that the LDA is valid for the calculation of correlation properties.

The first step is to consider a finite cylinder of length $2L$, with radial harmonic trapping and periodic boundary conditions along z (and therefore homogeneous axial density). For this geometry, we find in the TF regime $n_0(\mathbf{r}) = n_{0\mathrm{m}}(1-\tilde{\rho}^2)$ for the condensate wave function. The low-lying excitations are found using standard Bogoliubov theory [5] after averaging over the transverse degrees of freedom [6]. The Bogoliubov spectrum for the excitation frequencies is $\omega_k^B = [\omega_k(\omega_k + Mc_{1d}^2/\hbar)]^{1/2} \approx c_{1D}k$ for small k, with the free particle energy $\hbar\omega_k = \hbar^2k^2/2M$ and the 1D speed of sound $c_{1D} = \sqrt{\mu/2M}$ [17]. The Fourier component for phase fluctuations with wave vector k is

$$\phi_k = \sqrt{\frac{\omega_k^B}{2\omega_k}}\frac{1}{\sqrt{\mathcal{V}}} \approx \sqrt{\frac{Mc_{1D}}{\hbar k}}\frac{1}{\sqrt{\mathcal{V}}}, \quad (6)$$

where the final expression holds for low-lying phonon states ($k \to 0$), and $\mathcal{V} = 2\pi n_{0\mathrm{m}}R^2L$. In a second step, we take into account the trapping potential by the substitution

$$\mu \to \mu - \frac{1}{2}M\omega_z^2z^2. \quad (7)$$

This implies directly the following replacements:

$$\begin{aligned}
&\text{density:} && n_{0\mathrm{m}}(\propto\mu) \to n_{0\mathrm{m}}(1-\tilde{z}^2),\\
&\text{speed of sound:} && c_{1D}(\propto\sqrt{\mu}) \to c_{1D}\sqrt{1-\tilde{z}^2},\\
&\text{radius:} && R(\propto\sqrt{\mu}) \to R\sqrt{1-\tilde{z}^2},\\
&\text{half length:} && L \to L. \quad (8)
\end{aligned}$$

With these substitutions, we recover the 3D TF density profile. We require that the excitation frequency $c_{1D}k$ is not modified as well, which implies replacing k with $k(1-\tilde{z}^2)^{-1/2}$ and using a density of states $\mathcal{N}(k)dk = (L/\pi)(1-\tilde{z}^2)^{-1/2}dk$. For the position-dependent variance of the phase, we find [18]

$$\Delta\phi^2(\tilde{Z},\tilde{s}) \approx \frac{T}{T_\phi}\frac{|\tilde{s}|}{(1-\tilde{Z}^2)^2}. \quad (9)$$

The $\tilde{Z}$-dependent phase-coherence length $L_\phi(1-\tilde{Z}^2)^2$, appearing in Eq. (9), can be substantially smaller near the edges of the trap than in the center. We will see that this reduces the average coherence length below its value at the center of the trap.

We deduce from Eqs. (1), (2), and (9) the correlation function

$$\mathcal{C}^{(1)}_{\mathrm{trap,T}}(\tilde{s}) \approx \frac{15N_0}{8}\int_0^{\sqrt{1-\tilde{s}^2/4}} d\tilde{z}(1-\tilde{z}^2-\tilde{s}^2/4)^2 \times \exp\left(-\frac{T}{2T_\phi}\frac{|\tilde{s}|}{(1-\tilde{z}^2)^2}\right). \quad (10)$$

In deriving Eq. (10), we have used the approximation for the overlap function $\chi(\rho,Z,s) \approx (1-\tilde{\rho}^2-\tilde{z}^2-\tilde{s}^2/4)$, which is valid near the center of the trap. In Fig. 1, we compare the result (10) to the correlation function following from the numerical integration of Eq. (5). In the $T=0$ limit, the correlation function is limited by the overlap χ. Because of the

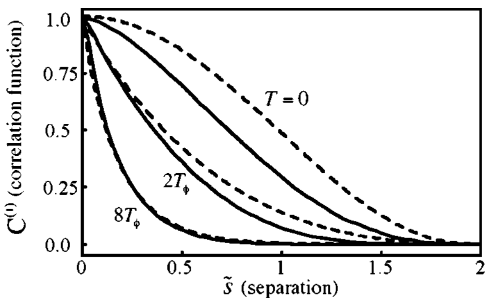

FIG. 1. Spatial correlation function $\mathcal{C}^{(1)}(\tilde{s})$ of a trapped quasicondensate. The solid lines follow from the numerical evaluation of the result (5) derived in Ref. [2], for $T=0$, $T/T_\phi=4$, and $T/T_\phi=8$, in order of decreasing width. The dashed lines follow from the local-density approximation (10). Note that for convenience, $\mathcal{C}^{(1)}(\tilde{s})$ as shown here is normalized to 1 rather than to N_0, different from what is done in the text.

approximate form of χ, our result $\mathcal{C}^{(1)}_{\mathrm{trap,T=0}}\approx N_0[1-(\tilde{s}/2)^2]^{5/2}$ is about 25% too broad, and one should rather use the Gaussian approximation to $\mathcal{C}^{(1)}$, derived in Ref. [13]. As T increases, $\mathcal{C}^{(1)}$ turns to an exponential-like function, and our approximation approaches the numerical calculation. For $T>8T_\phi$, the LDA result is very close to the numerical one (maximum error $\approx 3\%$). For $T\gg T_\phi$, Eq. (10) can be further simplified by keeping the $\tilde{s}$-dependent term only in the exponential. The Fourier transform then gives the momentum distribution

$$\mathcal{P}_{\mathrm{trap,T}}(p_z)\approx\frac{15N_0p_\phi}{32\pi}\int_{-1}^{1}d\tilde{z}\frac{(1-\tilde{z}^2)^4}{(1-\tilde{z}^2)^4p_z^2+p_\phi^2/4},\quad(11)$$

where $p_\phi=\hbar/L_\phi$ is a typical momentum associated with the phase fluctuations. This function is self-similar in p_z/p_ϕ, and approximated to better than 4% by a normalized Lorentzian with a half-width at half maximum (HWHM) of $\Delta p=0.67p_\phi$. This Lorentzian shape of the momentum distribution differs qualitatively from the fully coherent case, where it is almost Gaussian and limited by the Heisenberg principle [14]. The increase of the phase fluctuations with increasing T not only broadens the momentum distribution, but also induces the appearance of "wings," which form the "high-energy tail" of the quasicondensate. To quantify the accuracy of our approximation, we have calculated numerically the Fourier transform of the correlation function. We find empirically that the HWHM is accounted for by the formula $\Delta p^2\approx(2.04\hbar/L)^2+(0.65\hbar/L_\phi)^2$. The first term corresponds to the Heisenberg-limited momentum width, and the second to the phase fluctuations. For $T\geqslant 8T_\phi$, the height and width agree to better than 4% with the Lorentzian approximation. For lower T, the overlap function χ still affects the momentum distribution.

Note finally that the momentum distribution is Lorentzian only in the domain $k\ll R^{-1}$. Outside this region, the 3D nature of the excitations should be taken into account properly. However, this does not affect the quasicondensate peak we are investigating here, but only the much smoother thermal background [13,14].

The results of the above paragraphs are valid for an equilibrium situation. However, coherence measurements involving Bragg scattering [10,14] suffer from two major difficulties in a very elongated trap [15]: mean-field broadening of the resonance [14] and elastic scattering from the recoiling atoms and the condensate towards initially empty modes [19]. Both of these problems can be solved by opening the trap abruptly, and allowing the Bose-Einstein condensates to expand to decrease its density before measurement. In the remainder of this paper, we discuss how expansion modifies the momentum distribution and the correlation function, assuming that the expansion time is chosen to be long enough to neglect the collisions.

For a pure elongated condensate, abruptly released from the trap at $t=0$, the explicit solution was found in Ref. [20]. The condensate density keeps its initial Thomas-Fermi shape, with the coordinates rescaled. The (small) axial momentum from the released mean-field energy is linear in position: $p_z\approx p_{\mathrm{exp}}\tilde{z}$, with $p_{\mathrm{exp}}=(\pi/\sqrt{2})\lambda Mc_S$ for $\tau=\omega_\perp t\gg 1$, and $c_S=\sqrt{\mu/M}$ is the 3D speed of sound. The axial momentum distribution mirrors the (integrated) density distribution:

$$\mathcal{P}_{\mathrm{exp,T=0}}(p_z)=\frac{15}{16p_{\mathrm{exp}}}\left[1-\left(\frac{p_z}{p_{\mathrm{exp}}}\right)^2\right]^2.\quad(12)$$

This expression holds for a pure condensate, at $T=0$, as indicated. For a phase-fluctuating condensate at finite T, it is necessary to consider the time evolution of the fluctuations as well. As shown in Ref. [3], the momentum distribution partially converts into density modulations after time of flight. An explicit solution was derived for the density fluctuations in the axially homogeneous case. Using the continuity equation (after radial averaging), we find for $\tau\gg 1$,

$$\phi_k(z,\tau)\approx\phi_k(z,0)\,\tau^{-(\omega_k^{\mathrm{B}}/\omega_\perp)^2}\cos\left(\frac{\omega_k}{\omega_\perp}\tau\right).\quad(13)$$

If $\omega_z t\ll(\mu/\hbar\omega_z)(T/T_\phi)^2$, then for all $k\lesssim L_\phi^{-1}$, the phase distribution is essentially frozen: $\phi_k(z,\tau)\approx\phi_k(z,0)$. Physically, this condition states that for such a time of flight, the excitations that have significant contributions to the phase fluctuations have not yet been converted into density modulations. This condition is not at all restrictive for typical experimental parameters [3,4,15], and we suppose it is met in the remainder of the paper.

Using the rescaled wave function from Ref. [20], together with Eq. (9), we find the correlation function for the expanding quasicondensate:

$$\mathcal{C}^{(1)}_{\mathrm{exp,T}}(\tilde{s})\approx\frac{15N_0}{16}\int_{-1}^{1}d\tilde{z}(1-\tilde{z}^2)^2\times\exp\left(i\frac{\pi\mu}{\hbar\omega_\perp}\tilde{z}\tilde{s}-\frac{T}{2T_\phi}\frac{|\tilde{s}|}{(1-\tilde{z}^2)^2}\right).\quad(14)$$

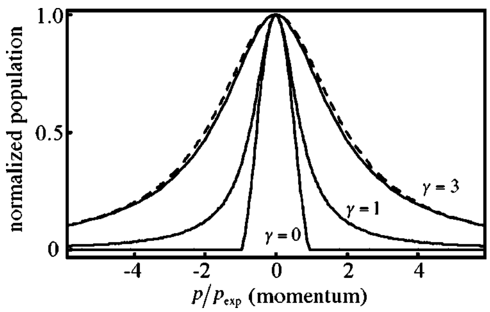

FIG. 2. The momentum distribution in expansion g_γ [Eq. (16)] for $\gamma = p_\phi/p_{\rm exp} = 0,1,3$. As γ increases, g_γ continuously transforms from a quartic profile to a Lorentzian-like profile (see text). For $\gamma = 3$, we find little change from the momentum distribution in the trap (dashed line, a Lorentzian with HWHM 0.67γ, see text). The functions have been rescaled by their maximum values to facilitate comparison.

The phase factor in Eq. (14) accounts for the local expansion momentum introduced above (recall $\pi\mu/\hbar\omega_\perp \gg 1$). The Fourier transform gives the momentum distribution

$$\mathcal{P}_{\rm exp,T}(p_z) \approx \frac{N_0}{p_{\rm exp}} g_{\gamma = p_\phi/p_{\rm exp}}\left(\frac{p_z}{p_{\rm exp}}\right). \tag{15}$$

The function g_γ is given by

$$g_\gamma(x) = \frac{15\gamma}{32\pi}\int_{-1}^{1} d\tilde{z}\frac{(1-\tilde{z}^2)^4}{(1-\tilde{z}^2)^4(x-\tilde{z})^2 + \frac{\gamma^2}{4}}, \tag{16}$$

and the ratio $\gamma = p_\phi/p_{\rm exp}$ controls the component of the momentum distribution which dominates. In the limit $T \to 0$, using $\gamma/(x^2+\gamma^2) \to \pi\delta(x)$ as $\gamma \to 0$, we recover the zero-temperature result (12). On the other hand, if $\gamma \gg 1$, we expect the momentum distribution to be similar to the distribution in the trap (11). Figure 2 shows a numerical calculation of g_γ for various values of γ. We find that already for $p_\phi \gtrsim 2p_{\rm exp}$, the momentum distribution is almost entirely dominated by phase fluctuations and, as in the trapped case, is very well approximated by a normalized Lorentzian with HWHM $= 0.67 p_\phi$. Here, we note the following two points: first, that the Heisenberg width $\sim \hbar/L$ is negligible at any temperature, and second that, for large enough condensates, we can have $p_{\rm exp} \gg p_\phi$ even if the coherence length is smaller than L.

In summary, we have analyzed the measurement of phase fluctuations in elongated Bose condensates. Within a local-density approach, we have been able to take the density profile into account, and derived analytical formulas for the correlation function and the momentum distribution of static and freely expanding quasicondensates. In the regime of interest, the formula compares well to a numerical evaluation based on the results of Ref. [2], which are exact in the long-wavelength limit. In particular, we show how the shape of the momentum distribution tends to a Lorentzian with half-width $\approx 0.67\hbar/L_\phi$ as one goes further in the phase-fluctuating regime. We believe that these results may be helpful to understand quantitatively the experiments involving quasicondensates [3,4,15].

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