

Exponentially localized magnetic fields for single-spin quantum logic gates

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An infinite array of parallel current-carrying wires is known, from the field of neutral particle optics, to produce an exponentially localized magnetic field when the current direction is antiparallel in adjacent wires. We show that a finite array of several tens of superconducting Nb nanowires can produce a peak magnetic field of 10 mT that decays by a factor of 10^4 over a length scale of 500 nm. Such an array is readily manufacturable with current technology, and is compatible with both semiconductor and superconducting quantum computer architectures. A series of such arrays can be used to individually address single single-spin or flux qubits spaced as little as 100 nm apart, and can lead to quantum logic gate times of 5 ns. © 2004 American Institute of Physics.

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I. INTRODUCTION

Among the many and varied proposals for constructing quantum computers, spintronic solid state devices occupy a special place because of the prospects of integration with the existing semiconductor technological infrastructure.¹ At the same time, superconducting devices have taken an early lead in demonstrating the viability of the building blocks of quantum computing (QC) in the solid state, with recent reports of controlled single-qubit operations and entanglement generation.²

In both the spintronics QC proposals, such as quantum dots,^{3–6} donor spins in Si,⁷ electrons on helium,⁸ and the superconducting QC proposals,⁹ the ability to apply highly localized and inhomogeneous magnetic fields would be a definite advantage, if it could be done without excessive technical difficulties. In fact the early proposals suggested manipulating individual spin qubits using such localized magnetic fields, e.g., by a scanning-probe tip or by coupling to an auxiliary ferromagnetic dot,³ but there are significant speed, controllability, and other difficulties associated with such methods. Because of these difficulties, in particular the spintronics requirement to resolve single spins, many alternatives to the use of localized magnetic fields have been proposed in spin-based QC. These alternatives typically avoid the use of magnetic fields altogether: e.g., *g*-factor engineering combined with all-electrical control,¹⁰ optical spin manipulation,¹¹ or encoding into the states of several spins.^{12,13} Other alternatives include gate teleportation, which requires control lable exchange interactions and certain two-spin measurements,¹⁴ and qubits encoded into anti-ferromagnetic spin clusters, in which case the magnetic field needs to be controlled only over the length scale of the cluster diameter.¹⁵ In the context of superconducting qubits it is

also possible to avoid using localized magnetic fields by introducing an appropriate encoding.¹⁶

Here we revisit the possibility of applying highly localized magnetic fields. We show that a scheme inspired by magnetic mirrors for cold neutrons,¹⁷ and more recently cold atoms^{18–22} is capable of generating a magnetic field that decays exponentially fast over a length scale comparable to the spacing between nanofabricated quantum dots, and has strength and switching times that are compatible with QC given available estimates of decoherence times. Our scheme uses arrays of parallel current-carrying wires, that is readily implementable with currently available nanotechnology,^{1,23} and appears well suited for integration with quantum dot nanofabrication methods, as well as with superconducting flux qubits and spin-cluster qubits, where the length scales are larger. Thus we believe that QC with localized magnetic fields deserves a fresh look.

II. EXPONENTIALLY LOCALIZED MAGNETIC FIELD FROM AN INFINITE WIRE ARRAY

In order to have a concrete application in mind we shall from now on refer to semiconductor quantum dot spin-qubits.^{3–6} However, our results are equally applicable to other qubits that are manipulated by localized magnetic fields, such as superconducting flux qubits.⁹ The first requirement for single-spin magnetic resolution is a magnetic field profile that decays exponentially fast over length scales comparable with the interspin spacing. We will now show, in close analogy to results from magnetic mirrors, how to produce such an exponentially localized magnetic field. The basic design is one of an array of parallel current carrying wires, with the current direction alternating from wire to wire: see Fig. 1.

We first consider the idealized case of infinitely long wires. In this case the field can be calculated analytically (see, also, Refs. 17 and 18). Let B_α^N be the magnetic field in

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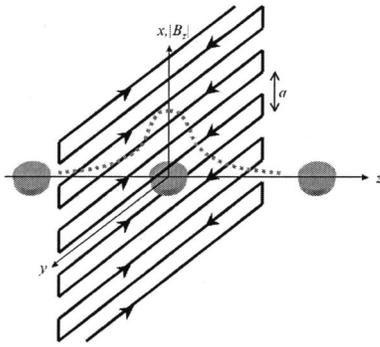


FIG. 1. Schematic depiction of quantum dots (disks) with an array of current-carrying wires. Array period is a , current is I and alternates direction as indicated by arrows. The resulting magnetic field profile is drawn schematically.

the $\alpha = x, z$ direction generated by N infinitely long wires. We add magnetic field contributions from each wire, to get the field components from N wire pairs

$$B_z(x, z) = \frac{\mu_0 I}{2\pi a} \sum_{n=0}^{N-1} (-1)^n \left(\frac{\left(\frac{1}{2} + n\right)/2 - x}{\left[\left(\frac{1}{2} + n\right)/2 - x\right]^2 + z^2} + \frac{\left(\frac{1}{2} + n\right)/2 + x}{\left[\left(\frac{1}{2} + n\right)/2 + x\right]^2 + z^2} \right),$$

$$B_x(x, z) = \frac{\mu_0 I}{2\pi a} \sum_{n=0}^{N-1} (-1)^n \left(\frac{z}{\left[\left(\frac{1}{2} + n\right)/2 - x\right]^2 + z^2} - \frac{z}{\left[\left(\frac{1}{2} + n\right)/2 + x\right]^2 + z^2} \right),$$

where I is the current through each wire and $k = 2\pi/a$ is the reciprocal array constant. This sum can be computed analytically in the limit $N \rightarrow \infty$ using the residue theorem result

$$\sum_{n=-\infty}^{\infty} (-1)^n f(n) = -\pi \sum_{\zeta_k} \frac{1}{\sin \pi \zeta_k} \text{Res}(f, \zeta_k),$$

where ζ_k are the poles of $f(\zeta)$, yielding

$$B_x^\infty(x, y, z) = \frac{\mu_0 I}{a} \frac{2 \sin(kx) \sinh(kz)}{\cos(2kx) + \cosh(2kz)}, \tag{1}$$

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$$= \frac{\mu_0 I}{a} \text{sech}(kz) \xrightarrow{z \gg a} \frac{2\mu_0 I}{a} e^{-k|z|}. \tag{2}$$

The z -component result shows the basic point: An exponentially localized magnetic field can be generated using a wire

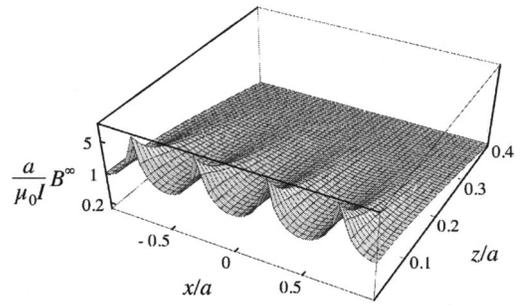


FIG. 2. The magnetic field magnitude from an infinite array of infinitely long wires (“doubly infinite”), on a logarithmic scale. Note that the field is flat between the wires, and decays exponentially in z .

array. The flat top of the sech profile is a useful design feature, since it implies no exponential sensitivity in the range $z \lesssim a$.⁶ The field magnitude

$$B^\infty = |\mathbf{B}^\infty| = \sqrt{2} [\cos(2kx) + \cosh(2kz)]^{-1/2} \tag{3}$$

oscillates with period a in the x -direction: see Fig. 2.

III. MULTIPLE ARRAYS

For the purposes of QC, we should ideally be able to address each spin separately. To this end we propose to center a separate wire array on each quantum dot. Then, as long as the dots are spaced on the order of the lattice constant a , we have exponentially sensitive addressability of each dot. The introduction of multiple arrays is useful in another respect: We can adjust the magnitudes and directions of currents in different arrays so as to exactly cancel the field at all (or only some) other dots except the desired one (or ones). To see this, let $b(z) \equiv B_z^\infty(z)/I = \mu_0/a \text{sech} kz$. The field at position z from K arrays of wires, with the j th array having current I_j and intersecting the z -axis at position z_j (typically the center of one of the dots), is

$$B^{\{K\}}(z) = \sum_{j=-(K-1)/2}^{(K-1)/2} I_j b(z - z_j). \tag{4}$$

Suppose we wish the field to have magnitude c_j at position z_j . Formally, we need to solve

$$B^{\{K\}}(z_j) = c_j, \quad j \in [-(K-1)/2, (K-1)/2]. \tag{5}$$

This is a linear system of K equations in the K unknowns I_j , so it can always be solved in terms of the K positions z_j . E.g., the fields with and without the correction are shown, for $K = 5$, in Fig. 3.

IV. FINITE SIZE EFFECTS

For a finite system ($N < \infty$ wires, finite length and thickness, nonideal shape, etc.), we can only expect the above results to hold to an approximation. Much of the theory of corrections to finite size effects has already been worked out in Refs. 19 and 22, in the context of atomic mirrors. Since for atomic mirrors the primary concern is specular reflection, there the focus was on reducing the variation of the field magnitude in the planes parallel to the wire arrays. For us

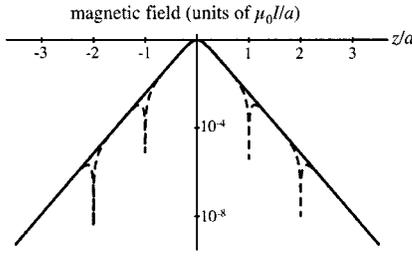


FIG. 3. Magnetic field generated by a single doubly infinite wire array centered at $z=0$ (solid), vs the field generated by $K=5$ such arrays (dashed), with currents chosen to cancel the field at positions $z=\pm 2a, \pm a$.

this criterion is unimportant; instead, our focus is on making the field as localized as possible along the z -axis.

The first important conclusion in the case of a finite number of wires N is that there is a transition from exponential to quadratic (i.e., $1/z^2$) decay.²² The transition takes place at the inflection point (zero second derivative) of $B^N(z)$; however, this is difficult to obtain analytically. To roughly estimate the transition point we compute where the field from a single pair of wires, positioned at the edge of an array of N wires ($y = \pm Na/4$), generates a field of magnitude equal to that from an infinite array:

$$\frac{\mu_0 I}{\pi} \frac{Na/4}{z^2 + (Na/4)^2} = \frac{\mu_0 I}{a} e^{-2\pi z/a}. \quad (6)$$

Since the transition happens for $x/a \ll N$ we neglect x/a in the denominator. The solution is then

$$z_t \approx \frac{\pi a}{20} \ln\left(\frac{\pi N}{4}\right). \quad (7)$$

Numerical calculations show that Eq. (7) overestimates the position of the transition point by about a factor of 5; however, after this correction is made, analytics and numerics agree well across several orders of magnitude of N .

The logarithmic dependence of the transition point on the number of wires might appear to pose a severe scalability constraint on our method. However, this is not the case when we take into account the threshold for fault tolerant quantum error correction.²⁴ For, it follows from the threshold result that we only need to make the ratio of the residual field to the peak field (applied to the desired spin) to the residual field smaller than, say 10^{-4} . The crucial question thus becomes for what N this can be achieved, and this brings us to the idea of “endcaps.”

As observed in Refs. 19, 21, and 22, near the center of the array the magnetic field that would be produced by the semi-infinite array of “missing” wires is the same (to first order in $4/N$) as that of a pair of wires carrying current $I_{\text{cap}} = I/2$ and placed with their centers shifted by $a/4$ from the outermost wires in the array. Thus, to cancel this field, one can simply place two “endcap” wires carrying currents $\pm I/2$ at these positions. In the context of atomic mirrors this is important to improve flatness, and hence specular reflection. In Ref. 22 it was observed that flatness can be further enhanced by using an *odd* number of wires.

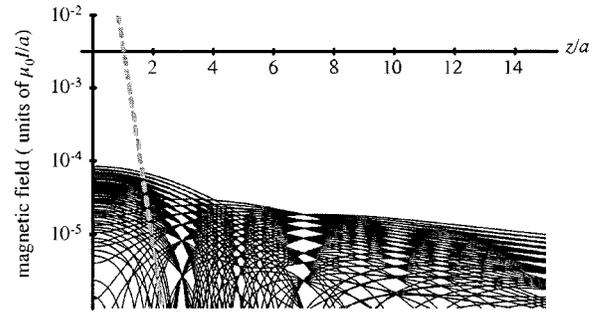


FIG. 4. Deviation $|\mathbf{B}^\infty - \mathbf{B}^{30}|$ of the field of a finite array (solid lines) from the field $|\mathbf{B}^\infty|$ of an infinite array (gray dashed line, shown for comparison). This array has $N=30$ wires (including two endcap wires), such that main wires are at $\pm a/4, \pm 3a/4, \dots, \pm 27a/4$, and endcap wires are at $\pm 7a$, carrying current $I_{\text{cap}} = I/2$. Field deviations are plotted for a range of endcap currents ($\pm 10^{-3} I_{\text{cap}}$) and positions ($\pm a/100 = 2.5$ nm). Note that the residual field never exceeds the threshold of 10^{-4} .

Some of these schemes can be used to improve *field localization*, a criterion not considered originally. For instance, we find that the number of wires can be reduced drastically—from $N \approx 10^4$ to as few as $N = 22$ wires—when endcaps are used to achieve a residual field smaller than 10^{-4} . By contrast, using an odd N is disastrous for localization: For example, residual fields appear at the 3% level for $N = 23$, even including endcap correction. Higher (even) wire number increases the robustness of the cancellation against experimental uncertainty in the current and position of the endcaps: $N = 30$ is required to maintain $B/\mu_0 I \leq 10^{-4}$ for a fractional current variation of $\pm 10^{-3}$ and a positional uncertainty of ± 2.5 nm, as shown in Fig. 4. Finally, we note that the corrections arising from the finite length of the wires and the short, perpendicular connecting wires, can also be compensated for by the use of judiciously placed compensating wires.²²

V. FEASIBILITY AND IMPLEMENTATION CONSIDERATIONS

We now come to estimates of whether the fields and size scales required are feasible in practice. Let us first calculate the magnetic field strength required for single-qubit operations. A spin can be rotated by a relative angle $\phi = g \mu_B B \tau / 2\hbar$ by turning on the field B for a time τ (where $g \approx 1$ is the g -factor, and μ_B is the Bohr magneton). Recent estimates of dephasing times are $50 \mu\text{s}$ for electron spins in GaAs quantum dots (a calculation, assuming spectral diffusion is dominant),²⁵ and a measurement of 60 ms for T_2 of phosphorus donors in Si.²⁶ If we use the more pessimistic of these numbers, and assume a fault tolerance threshold of 10^{-4} for QC,²⁴ we may estimate the desired operation time as $\tau \sim 10^{-4} \times 50 \mu\text{s} = 5$ ns for an angle $\phi = \pi/2$. Thus the desired field strength is $B = 2\hbar \phi / g \mu_B \tau \approx 7$ mT.

To evaluate the feasibility of such a specification, we consider an array with periodicity $a = 250$ nm, wire radius $r = 50$ nm, $N = 32$ wires, and length $L_y = 10 \mu\text{m}$ along the y -direction. These dimensions are compatible with the 100 nm length scales of quantum dots.⁶ In order to reach the desired field strength of 7 mT, $I = a B_z^\infty / \mu_0 \approx 1.4$ mA would be required. However, decoherence due to heating with such

a current could be a major issue. An upper bound estimate for the required temperature T can be given by $T \ll E_Z/k_B$, where $E_Z = g\mu_B B/2$ is the Zeeman splitting of the spins in the applied magnetic field B . In our case, we have constrained B by the gate time τ , so we can write $k_B T \ll \pi\hbar/2\tau$, or $T \ll 2.4$ mK for $\tau = 5$ ns. This is feasible with dilution refrigeration technology if the heat load is on the order of 100 pW,²⁷ comparable to the dissipation of quantum dots.⁶

For normal metal wires, such a heat load restriction is prohibitive. The power P dissipated is $P = j^2 \rho A \ell$, where j is the current density, $\ell = N(2L_y + a)$ is the total wire length, and ρ is the resistivity. Below 10 K, oxygen-free copper can have $\rho \approx 3 \times 10^{-11}$ Ω m.²⁸ At $P \leq 100$ pW, $I \leq 3.6$ μ A, which would give $\tau \geq 2.0$ μ s—nearly a thousand times slower than our original goal. Although this would be acceptably fast if the decoherence time were 60 ms, as in Ref. 26, the high-purity ρ we have used is optimistic for nanofabricated wires, and the heat generated would increase linearly with the number of qubits manipulated.

One can circumvent resistive heating by using superconductors. A wire with radius $r \leq \xi_0$, where ξ_0 is the coherence length, can also avoid heating mechanisms associated with vortex movement through the superconductor. For $r \sim \lambda$ or smaller, where λ is the penetration depth, the critical current density is due to depairing: $j_c = (2/3)^{3/2} B_c / (\mu_0 \lambda)$.²⁹ These constraints are compatible when $\lambda \leq \xi_0$, i.e., mostly type I superconductors. For Nb, $B_{c1} = 0.206$ T, $\lambda = 52$ nm, and $\xi_0 = 39$ nm,³⁰ so $j_c = 1.7 \times 10^{12}$ A/m². Note that high-temperature superconductors typically have higher λ and thus lower j_c . In any case, the critical current density of Nb is more than is required: A gate time of $\tau = 5$ ns would require $j \approx 0.1 j_c$.

One final note about the devices considered is that their counter-wound geometry minimizes magnetic field energy:²⁰ the inductance of a single array discussed above is on the order of 10^{-15} H. This is important for ease of fast and independent switching.

VI. CONCLUSIONS AND EXTRAPOLATIONS

In conclusion, our results indicate that QC with localized magnetic fields deserves renewed consideration. We have shown that a method to produce exponentially decaying magnetic fields using an array of current-carrying wires, known in the cold neutron and atom optics communities, is adaptable to solid state quantum computer implementations. Our estimates indicate that in all respects the method is technologically feasible, provided superconducting wires with sufficiently high critical current density (such as Nb) are used.

Our work is motivated by the quest to perform single-spin or flux-qubit rotations, which is a component of a universal set of quantum logic gates. The geometry shown in Fig. 1 yields a field that is localized in the z -direction; in order to perform arbitrary single-qubit rotations we need to localize the field along another, perpendicular direction. An independent field vector can be produced by a second set of interleaved arrays placed at 45° with respect to the original

arrays, with current flowing along $(\mathbf{z} + \mathbf{y})/\sqrt{2}$. With a judicious array placement and $na = d/\sqrt{2}$, for qubit spacing d and any integer n , the field direction will be along \mathbf{x} at all qubits (this design will need to be optimized similarly to our considerations above—an issue we do not intend to address here). The additional spatial constraints would require only a four-fold increase in a for the same currents and wire sizes. If on the other hand, introducing a second array is undesirable, “software” solutions using recoupling and encoding techniques have been developed to still allow for universal QC.¹³ These techniques would be considerably simplified by the ability to perform single-qubit operations along one direction.

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