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In defense of Countabilism

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Abstract Inspired by Cantor's Theorem (CT), orthodoxy takes infinities to come in different sizes. The orthodox view has had enormous influence in mathematics, philosophy, and science. We will defend the contrary view—Countablism—according to which, necessarily, every infinite collection (set or plurality) is countable. We first argue that the potentialist or modal strategy for treating Russell's Paradox, initially proposed by Parsons (2000) and developed by Linnebo (2010, 2013) and Linnebo and Shapiro (2019), should also be applied to CT, in a way that vindicates Countabilism. Our discussion dovetails with recent independently developed treatments of CT in Meadows (2015), Pruss (2020), and Scambler (2021), aimed at establishing the mathematical viability, and therefore epistemic possibility, of Countabilism. Unlike these authors, our goal isn't to vindicate the mathematical underpinnings of Countabilism. Rather, we aim to argue that, given that Countabilism is mathematically viable, Countabilism should moreover be regarded as true. After clarifying the modal content of Countabilism, we canvas some of Countabilism's many positive implications, including that Countabilism provides the best account of the pervasive independence phenomena in set theory, and that Countabilism has the power to defuse several persistent puzzles and paradoxes found in physics and metaphysics. We conclude that in light of its theoretical and explanatory advantages, Countabilism is more likely true than not.

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1 Introduction

Our goal is to motivate, defend, and apply the following claim:

> *Countabilism*: Necessarily, every infinite collection (set or plurality) is countable.

Typically, discussions of size and infinity are confined to the philosophy of mathematics. However, Countabilism is best understood as a *metaphysical* claim concerning ontology and modality. Even the staunchest nominalist, who denies the existence of all abstract objects and treats the whole of mathematics as a useful fiction, is welcome to endorse Countabilism. Countabilism applies just as much to tables, planets, spacetime points, and propositions, as it applies to mathematical abstracta.

We wish to emphasize this point because the near-universal rejection of Countabilism, primarily due to Cantor's Theorem, has had an enormous influence outside of mathematics—notably, in contemporary metaphysics and the physical sciences. Nonetheless, we will argue, Countabilism is true. We start by presenting Cantor's Theorem (CT) and noting its parallels with Russell's Paradox (RP) (Sect. 2). We then consider a recent attempt to undercut CT, due to Whittle (2015, 2018), and argue that it fails (Sect. 3). We then turn to considering whether, given the structural similarities between CT and RP, certain commonly accepted strategies for responding to RP might also provide the basis for an alternative understanding of the import of CT. We argue that the *potentialist* or *modal* strategy for treating RP, first proposed by Parsons (2000) and developed by Linnebo (2010), Linnebo (2013) and Linnebo and Shapiro (2019), should also be applied to CT in a way that vindicates Countabilism (Sect. 4). Our discussion dovetails with recent independently developed treatments of CT in Meadows (2015), Pruss (2020), and Scambler (2021), aimed at establishing the mathematical viability, and therefore epistemic possibility, of Countabilism.[1] While opening the door to Countabilism, however, these authors don't walk through; as Scambler (2021) registers, "like each of Pruss

[1] Meadows (2015) "expands upon a way in which we might rationally doubt that there are multiple sizes of infinity. [...] elements of contextualist theories of truth and multiverse accounts of set theory are brought together in an effort to make sense of Cantor's troubling theorem" (191). Scambler (2021) develops the modal or 'indefinite extensibility' approach to CT that we favor, "generalizing [Linnebo's] theory **L**, which offers a modal, indefinite extensibility solution to Russell's Paradox, to a theory I call **M** [after Meadows], which offers an analogous 'solution' to Cantor's theorem" (10), such that "in the modal setting Cantor's theorem can be reconciled with the existence of only one infinite cardinality" (2); and which provides "a way of reconciling mathematics after Cantor with the idea there is one size of infinity" (21); we will be helping ourselves to Scambler's formal results down the line. Pruss (2020) also develops a modal approach to CT, arguing that "there is an epistemic possibility that all infinite sets have the same size as the natural numbers" (604).

and Meadows, I do not conclude [...] that all sets are really countable after all" (21).[2]

Unlike these authors, our goal isn't to add to the mathematical underpinnings of Countabilism. Instead, we will be concerned with whether, given that Countabilism can be understood in a mathematically viable way, Countabilism should be regarded as *true*.[3] After clarifying the modal content of Countabilism (Sect. 5), we explore some of the many positive implications that Countabilism has in the philosophy of mathematics (Sect. 6), in physics (Sect. 7), and in metaphysics (Sect. 8). We conclude that in light of its theoretical and explanatory advantages, Countabilism is more likely true than not (Sect. 9).

2 Cantor's theorem and Russell's paradox

Cantor's Theorem generally applies to any finite or infinite set, and states that there is no bijection—no one-to-one correspondence—between a set and its 'power set', containing all subsets of that set. When applied to the countably infinite set of natural numbers N, Cantor's Theorem is commonly taken to establish the existence of 'uncountable' infinities, as follows:

Cantor's Theorem (CT): The cardinality of $P(N)$ is uncountably infinite.

Proof

1. Suppose for reductio that $P(N)$, like N, is countably infinite, so that there exists a bijection $f : N \rightarrow P(N)$.
2. Consider the set $C = \{x \in N : x \notin f(x)\} \subset N$.
3. $C \in P(N)$; hence $C = f(c)$ for some $c \in N$.
4. By construction, $c \in C \leftrightarrow c \notin f(c) = C$: contradiction.

$\therefore$ $P(N)$ is not countably infinite.

As a piece of formal mathematics, CT is unimpeachable. There is, however, prima facie reason to be skeptical of the understanding of infinity that CT is supposed to have vindicated. One major reason is that the reasoning behind CT generates a plethora of puzzles and paradoxes of the Russellian variety. The most salient such paradox is, of course, Russell's Paradox (RP), one version of which follows from an application of CT to the case of the universal set V, or set of all sets, on the assumption that the relevant bijection f is the identity function between V and $P(V)$ (see Crossley (1973)):

[2] Scambler moreover suggests that the moral to draw is broadly anti-objectivist: "I think the results are best understood in the context of a kind of anti-objectivism about the question of whether there are different sizes of infinity: the guiding idea being that, when set-theoretic practice is formalized one way, one will find one verdict, and when it is formalized differently one will find quite a different one, with nothing objective to tell between them" (1099).

[3] Relatedly, we aim to push back against the sort of anti-objectivist line of thought registered in the previous footnote.

Russell's Paradox (RP): There is no set of all sets.[4]

Proof

1. Suppose for reductio that V is the set of all sets.
2. By the power set axiom, $P(V)$ exists.
3. Since $V = P(V)$, the identity function $f : V \to P(V)$ is a bijection.[5]
4. Consider the set $C = \{x \in V : x \notin f(x)\} \subset V$.[6]
5. By construction, $C \in C \leftrightarrow C \notin f(C) = C$: contradiction.

$\therefore$ There is no set of all sets.

As Klement (2010a) further observes, "Cantor's diagonalization method generalizes beyond mappings involving classes or sets" (18) to give rise to paradoxes involving predications, properties, propositions, and descriptive senses, among other categories. Russell himself remarked, in a 1902 letter to Frege, that "from Cantor's proposition that any class contains more subclasses than objects we can elicit constantly new contradictions". These deep structural parallels between CT and RP should give us some pause. As a sociological point, it is surprising that while an enormous amount of philosophical scrutiny has been directed at RP, very little such scrutiny has been directed at CT.[7] No doubt this reticence reflects an apt appreciation for all the mathematical fruits that can be grown in Cantor's paradise—though as we'll discuss down the line, recent mathematical results indicate that these these fruits can be grown in countable soil.[8]

[4] It is more common to encounter RP as the claim that there is no set of all sets not containing themselves, but in the context of the axiom of foundation, the plurality of all sets not containing themselves is the same as the plurality of all sets *simpliciter*.

[5] Here we intend V to be the set of all *pure* sets (sets whose transitive closure only includes sets), so that every $x \in V$ is a set whose elements are all included in V, implying $V \subset P(V)$.

[6] This set will exist by the standard axiom schema of *restricted* comprehension.

[7] We'll discuss certain historical and contemporary exceptions to this rule shortly.

[8] We should note that our position is not that there is anything inherently problematic about diagonalization arguments. After all, several of the most important proofs in logic appeal to some kind of diagonalization procedure, such as Gödel's Incompleteness Theorems and the undecidability of the Halting problem. Relatedly, we are not questioning that CT and RP (and other diagonalization proofs) are perfectly valid formal results. We will only be arguing that CT does not have the *philosophical* import that it's generally taken to have. Moreover, our reasons for thinking this do not generalize to every diagonalization argument, and our general skepticism about the uncountable typically will not apply to these other arguments (since, e.g., neither Gödel's Incompleteness Theorems nor the undecidability of the Halting problem require any uncountable mathematics). See Meadows (2015: 206–7) for further discussion of why skepticism about the philosophical import of CT does not generalize to other diagonalization arguments.

3 Whittle's strategy for blocking Cantor's theorem

A recent exception to the uncritical acceptance of the import of CT is found in Whittle (2015a, b, 2018); however, as we'll now argue, Whittle's strategy for blocking CT is uncompelling, such that if room is to be made for Countabilism in light of CT, a different strategy is required.

To start, Whittle observes that CT shows that infinities come in different sizes only given the following principle:

> *Size* $\rightarrow$ *Function*: For any sets A and B: A is the same size as B only if there is a bijection from A to B.

Size $\rightarrow$ Function is commonly accepted; but why? Whittle argues that the only promising motivation involves an inference to the best explanation to the truth of the contrapositive: why *else* would there fail to exist a bijection, in a given case, if not for the fact that A and B have different sizes? By attending to the structural analogy between CT and RP, Whittle suggests an alternative explanation—namely, the availability of a diagonal function between the domains and co-domains at issue:

> *Diagonal*: For any $f : X \rightarrow P(X)$, there is a $Df \in P(X)$ such that $Df = \{x \in X : x \notin f(x)\}$.

As Whittle sees it: since Diagonal explains the failure of bijection at issue in CT with no reference to the respective sizes of the collections, the principal motivation for Size $\rightarrow$ Function is undercut.

It strikes us that someone might reasonably maintain that Size $\rightarrow$ Function constitutes a better explanation of this failure of bijection than Diagonal, as systematically accommodating the clear truth of Size $\rightarrow$ Function for finite collections. The main problem with Whittle's argument, however, is that there is a seemingly compelling motivation for Size $\rightarrow$ Function which he does not address.

To appreciate this motivation, it's useful to first consider Whittle's treatment of a different motivation for Size $\rightarrow$ Function, according to which the associated biconditional claim is "an analysis" of the same-size relation:

> [Size $\leftrightarrow$ Function] does not tell us what it is for infinite sets to be of the same size. Why? Because the size of a set—whether infinite or finite—is an intrinsic property of that set. It is a property the set has purely in virtue of what it is like; specifically, in virtue of which members it has. Thus, what it is for A and B to be the same size is for them to share a certain sort of intrinsic property. It is not for there to exist a certain sort of function between the sets [...]. (33–34)

McGee (2015: 28) objects to this line of thought, on grounds, first, that size is not intrinsic, but is rather "tied inextricably to size comparisons", and second, that an intrinsic notion of size is problematically obscure. While we are sympathetic to McGee's concern that a conception of size wholly divorced from size comparisons (e.g., bijections) is obscure, we are also sympathetic to Whittle's point that size is intrinsic. Even if (perhaps counterpossibly) there was only a single set in existence, there still would be a fact of the matter about *how many* members that particular set has.[9] Ultimately, what is needed is a compelling argument showing that, even if the size of a collection is intrinsic, facts about size must always be associated with the possibility of forming certain one-to-one correspondences.[10] We now provide such an argument.

To start, the operative notion of (cardinal) size is one ineliminably connected to number: the size of a collection just is the *number* of objects in the collection; correspondingly, two sets will be of the same size only if they each contain the same number of objects. But what constitutes or determines what number—what answer to the 'how many members?' question—is assigned to a given collection? It is that number which is the output from a possible or potential act of counting (or an objective abstraction therefrom—the existence of conscious counters is not at issue here), which in turn involves matching each object in the collection with elements in an initial segment of ordinal (natural) numbers. The first object is assigned 1, the second is assigned 2, and so on.[11] But such an assignment of objects to ordinal numbers *itself* constitutes a one-to-one function from objects in the collection to those ordinal numbers. Putting these thoughts together, two sets A and B have the same size iff they have the same *number* of objects, and two sets have the same number of objects iff there are possible acts of counting out the members of those sets that yield the same number/ordinal, which is the case iff there are possible bijections $f_A : A \rightarrow o_A$ and $f_B : B \rightarrow o_B$, where o_A and o_B are the same ordinal.[12] This account of size (i) explains why facts about size are connected with facts about the possibility of certain kinds of bijections and (ii) is consistent with size being intrinsic. Facts concerning the possibility of counting out a set in a particular way are entirely explained by (and supervene on) the intrinsic features of that very set.

Now, in the case of infinite collections, there are certain complications. In particular, this process of counting may result in assigning different ordinal numbers to the objects in our collection, depending on the order in which we count. However, our account of size applies nonetheless. It is sufficient for two sets to be the same size if there is *some* possible act of counting out the members of the two sets that yields the same number/ordinal. In the infinite case, it is clearly incorrect to require that two sets are the same size only if *every* possible act of counting out the members of the two sets yields the same number/ordinal. Since N can be counted out in different ways, this would imply that N is not the same size as itself! At a bare minimum, the 'same size as' relation should be reflexive.[13]

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[9] Thanks to Øystein Linnebo for discussion here.

[10] Note that this task may be accomplished even if Whittle is right that there is something like an intrinsic conception of size. Compare certain realist accounts of dispositions (as per, e.g., Martin (1996)) according to which these are intrinsic in that they might be had at a 'lonely' world, but which are necessarily such as to produce certain manifestations in certain circumstances.

[11] Indeed, it is plausible that the notion of number itself arises from just such acts of counting—or tallying, as the simplest form of counting—and that the fact that this is so provides the basis for arithmetical relations between numbers being a priori and not subject to empirical disconfirmation; see Wilson (2000).

[12] We think that even a Nominalist should be able to make sense of the *possibility* of there being such bijections. We discuss the relevant modality in more depth in Sect. 5.

[13] One potential concern with this account of size is that it implicitly relies on the axiom of choice. In particular, someone who rejects the axiom of choice can consistently believe that some sets do not have a

From this understanding of size, it's a short hop to Size → Function. Suppose that A and B are the same size. Then, they will have the same number of elements. So, there will be possible numberings of A and B, corresponding to possible bijections $f_A : A \to o_A$ and $f_B : B \to o_B$, where $o_A = o_B$. From here, we can simply compose f_A with the inverse of f_B, yielding the desired bijection at issue in Size → Function, namely $f_B^{-1} \circ f_A : A \to B$.

4 Post-Russellian strategies for blocking Cantor's theorem

We turn now to a different approach to blocking CT and its presumed import. Given the deep structural parallels between CT and RP, it is worth exploring whether any popular strategies offered in response to RP can also be used to respond to CT. We will find that, while each of these strategies *can* be generalized in a way that blocks CT, the most promising strategy, which moreover vindicates Countabilism, is the *modal* response to RP.[14]

Recall that RP is directed against the following account of when some collection forms a set:

> *Naive Comprehension*: Necessarily, for any things, there is a set of exactly those things.

Footnote 13 continued
well-ordering. Consequently, such sets would not have *any* bijection from themselves to any ordinal number. Given our conception of size, this would in turn imply that the 'same size as' relation is not reflexive, since such sets would not have the same 'number' of elements as themselves (in fact, such sets could not be assigned any 'number' at all in the absence of such a well-ordering). We have three responses to this concern. First, as a dialectical matter, Whittle (2015) himself is perfectly content with the axiom of choice. Second, pursuing this objection to our account of size is mathematically revisionary, whereas Whittle's initial case against Size → Function was meant to be mathematically neutral. At the very least, endorsing a conception of size that is mathematically revisionary implies a significant cost. Lastly, the kind of denial of the axiom of choice that is needed for this objection to work is fairly radical. Our account of size only relies on the *possibility* of there being an appropriate bijection to an ordinal number. In order to pursue this kind of objection, one would have to maintain that there is a set such that it's *impossible* for there to be a well-ordering of that set. A pluralist approach to set theory, according to which there are possible set-theoretic universes where the axiom of choice holds and possible set-theoretic universes where the axiom of choice does not hold, could still allow for this modal principle. Given the fact that **ZFC** is consistent so long as **ZF** is consistent, rejecting the mere possibility of there being suitable well-orderings strikes us as a fairly radical view. We will further discuss the modality at issue in our conception of size in Sect. 5.

[14] We won't be addressing *every* strategy for blocking RP. One notable strategy that we won't discuss (beyond the following remarks) proceeds via dialetheism, accepting Naive Comprehension and the contradiction it generates, and altering the logic along paraconsistent lines (as in Priest (1995)). This approach could presumably be extended to CT, and moreover in a way that would fail to vindicate Countabilism (since moving towards a paraconsistent logic does not prevent there being, say, uncountably many spacetime points). It remains, in our view, that a dialethic approach to CT would be unsatisfactory, in facing the same main problem as the approach applied to RP: namely, the move to dialetheism and paraconsistent logic is too theoretically costly in light of the availability of viable and consistent approaches to the set-theoretic paradoxes.

Naive Comprehension seems initially intuitive: after all, what could possibly *stop* a given plurality of things from forming a set? RP shows, however, that Naive Comprehension must be false. The reason why RP is called a ‘paradox’ in the first place is because it contradicts Naive Comprehension, which can seem self-evidently true prior to encountering RP.

4.1 Limitation of size

The limitation of size strategy was introduced by von Neumann (1925) and further developed by Aczel (1988). According to this approach, “some things form a set unless there are too many of them. ...The reason Russell’s paradox doesn’t lead to ruin is that the sets that don’t contain themselves are more numerous than any set” (McGee 2015: 23).

The main objection to the limitation of size strategy is that the operative notion of ‘too big’ is amorphous, and existing suggestions for a precisification of the notion are problematic. For example, Linnebo (2010) has argued that there is no non-arbitrary way to motivate one particular size as being the ‘threshold’ size at which a collection is too large to form a set. With respect to RP, a common suggestion is that what it is to be ‘too big’ is to be equinumerous with the plurality of all ordinals *oo* that actually exist. Linnebo (2010) objects to this suggestion, as follows:

> Consider the question why there are not more ordinals than *oo*. For instance, why cannot the plurality *oo* form a set, which would then be an additional ordinal, larger than any member of *oo*? According to the view under discussion, the explanation is that *oo* are too many to form a set, where being too many is defined as being as many as *oo*. So the proposed explanation moves in a tiny circle. The threshold cardinality is what it is because of the cardinality of the plurality of all ordinals; but the cardinality of this plurality is what it is because of the threshold. I conclude that the response fails to make any substantial progress, and that the proposed threshold remains arbitrary. (153–154)

If we were to apply this strategy to CT, then perhaps we could say that the boundary between those pluralities that do form sets and those that do not corresponds to the boundary between the countable and the uncountable. This immediately raises all sorts of questions about how mathematics could be done without recourse to uncountable sets, but it would serve to trivially block CT. As it happens, it would do so in a way that would fail to vindicate Countabilism. For Countabilism is not merely a claim about sets, but rather a claim about any plurality. And on this approach, if there are pluralities of mathematical objects that are ‘too big’ to form sets, then such pluralities would not be countable, contra Countabilism.

It remains, in our view, that a limitation of size approach to either RP or CT is unsatisfactory. In particular: even if the boundary between the countable and the uncountable is non-arbitrary, it remains unclear why this boundary should track whether or not some objects are able to form a set. The deeper problem with the limitation of size approach to RP, and hence to CT, is that the size of a plurality just seems totally irrelevant to whether the plurality should be able to form a set. As long as the plurality of objects is perfectly definite and precise, it should be able to form a set. So, limitation of size does manage to avoid RP, but it doesn't seem like it has much motivation otherwise; and similarly for its application to CT. By way of contrast, as we will discuss down the line, the modal approach to naive comprehension is independently attractive wholly apart from RP.

4.2 Nominalism

A different approach to RP is to deny the existence of sets altogether. According to this Nominalist perspective, what the set-theoretic paradoxes show is that there is something inherently problematic about our notion of *set*.

Rejecting the existence of sets is a principled way to reject Naive Comprehension, and hence RP. Such a rejection would also serve to block CT, which is the main reason to believe in uncountable collections. However, by itself, Nominalism doesn't entail Countabilism, because it is consistent with Nominalism that there might be uncountably many *concrete* objects, such as uncountably many spacetime points.

It remains, in our view, that a Nominalist approach to either RP or CT, at least on its own, is unsatisfactory. For a start, Russellian paradoxes can also be run on non-set-theoretic objects, such as properties and propositions; so to work in full generality, this nominalistic approach would have to deny the existence of every kind of object associated with Russellian-like paradoxes. More importantly, unless the Nominalist is prepared to be highly revisionary of mathematical practice, they have to find *some* way to interpret mathematical practice, including set theory, in an ontologically neutral way. The Nominalist should therefore have *some* story about how the working mathematician should think about sets in a non-paradoxical way, even if reality happens to contain no such things.[15] The modal approach to set theory that we will be defending provides a way of thinking about sets that even a Nominalist can endorse.

4.3 Predicativism

The Predicativist approach to RP seeks to limit which pluralities of objects can form sets in a way that is distinct from the limitation of size approach. As Feferman (2005) notes, Russell's initial characterization of the predicative/impredicative distinction was one simply registering whether or not a condition (propositional function) defined a set or class:

> To begin with, the terms *predicative* and *non-predicative* (later, *impredicative*) were introduced by Russell (1906) in his struggles dating from 1901 to carry out the logicist program in the face of the set-theoretical paradoxes. Russell called a propositional function $\phi(x)$ predicative if it defines a class, i.e., if the

[15] We don't intend to be arguing that Nominalism is false by making this point. In fact, one of us has defended Nominalism on independent grounds (see Builes (forthcomingb)). Rather, our point is simply that even a Nominalist should have a way of making sense of the practice of set theory.

> class $\{x : \phi(x)\}$ exists, and non-predicative otherwise. Thus, for example, the propositional function $x \notin x$ figuring in Russell's paradox is impredicative. (590)

Feferman goes on to observe that the usefulness of this distinction for purposes of resurrecting logicism from RP requires identifying some principled criterion of impredicativity—a task usefully taken up by Poincaré:

> Poincaré came up with two distinct diagnoses of the source of the paradoxes [...]. The first was that there is in each case a vicious circle in the purported definition. [...] Poincaré's second diagnosis is distinct in its emphasis, namely that the source of each paradox lies in the assumption of the 'actual' or 'completed' infinite. (591)

Poincaré's second source of impredicativity is closely related to the modal strategy we will next consider down the line, so in this subsection we focus on Poincaré's first source of impredicativity, as reflecting vicious circularity in a purported definition or condition.[16] While there have been a number of precisifications of this notion of vicious circularity (see Gödel (1944); Goldfarb (1988); Feferman (2005); Horsten (2019)), the guiding thought behind Predicativism is the following:

> *Predicativism*: Every set must be definable without quantification over a class to which it belongs.

Predicativism is a principled position in the philosophy of mathematics, endorsed by logicians and philosophers such as Poincaré (1906), Russell (1908), Weyl (1918), and Feferman (2005), that explains why Naive Comprehension is problematic: the set of all sets V is explicitly defined by reference to a class (the class of all sets) in which it belongs (since V is itself supposed to be a set), violating the non-circularity condition.

Can a rejection of impredicativity be used to block CT? As CT is usually formulated, the theorem does involve impredicative resources. Indeed, Poincaré (1912) took this understanding of impredicativity to apply to CT, in an important early historical critique of that theorem and its import.[17] Hence, as Goldfarb (1988) notes, "Poincaré [uses] the vicious circle principle to bar from membership in a set anything that in some sense presupposes that set. In this form, the principle can also

[16] We subsume the strategy of response to RP that involves appealing to a Ramified theory of types (see, e.g., Russell (1908)) under the present strategy, since Ramified type theory assumes a sort of predicativism/no circularity principle.

[17] Poincaré's critique targeted a version of CT aiming to show that arbitrary descriptions in a fixed language (corresponding to a countable set ordered by alphabetized sentences consisting of a finite number of words) could not be put in one-to-one correspondence with points of space; he characterizes Cantor's purported result as "an illusion", since "to classify these sentences and the corresponding points according to the letters which form the sentences [...] is to construct a classification which is not predicative" (61). See also Weyl's (1918: 26–7) critique of Cantor's theorem, which highlights that the result reflects the 'purely mathematical' creation of an impredicative condition. It is worth noting that these early critiques of Cantor's theorem were less radical than Brouwer's intuitionism: an entirely classical conception of natural numbers is retained, and classical logic is retained for reasoning about (predicatively acceptable) sets of natural numbers. Thanks to a referee here.

be used to block the Cantor and Russell paradoxes" (72). More recently, Klement (2010a) observes:

> [Cantor's] reasoning is validated within most forms of set theory and is difficult to counter. However, it is not completely incontrovertible. In particular, the supposition that [*C*] corresponds to a well-defined subclass of [*N*] might be open to doubt, since it is defined in terms of a function whose [range] is [*N*'s] powerclass, and perhaps there is a vicious circle in this if [*C*] is to be included in that very range. (18)

Moreover, impredicative reasoning is arguably essential for Cantor's proof, following a result proved by Heck (1996), showing the consistency of an entirely predicative system in second-order logic that both satisfies Frege's 'Basic Law V' and blocks CT.[18] As it happens, while Predicativism might block RP and CT, it (again) doesn't fully secure Countabilism on its own. Even if Predicativism is correct about sets (and other kinds of mathematical objects), it might still be that there are violations of Countabilism in the realm of concrete objects.

It remains, in our view, that a Predicativist approach to RP and CT is unsatisfactory. Our main concern is that Predicativism is revisionary of mathematical practice.[19] Moreover, as Gödel (1944) influentially argued, the philosophical justification for Predicativism seems to be based on a kind of constructivism in the philosophy of mathematics. It does seem to be viciously circular to suppose that the *construction* of a mathematical object can presuppose the existence of some totality to which it belongs. That totality of objects would only be there 'after' we have finished constructing each of the objects in that totality! However, if there is a totality of objects that exist independently of our constructions, then there doesn't seem to be any problem with the existence of objects that can only be described by reference to a collection to which they belong. This broadly 'constructivist' spirit behind Predicativism also goes against mathematical practice—again, unlike the modal strategy we will next consider.

4.4 The modal strategy

The 'modal' response to RP was initially suggested by Parsons (2000) and recently developed by Linnebo (2010, 2013) and Linnebo and Shapiro (2019). As Linnebo and Shapiro highlight, this strategy is clearly related to Poincaré's identification of a second source of impredicativity, associated with the problematic assumption of

[18] See Uzquiano (2015) for a more recent discussion on the relationship between CT and Predicativism. In particular, Uzquiano discusses a version of Bernays' theorem, a result related to Cantor's Theorem about classes, that can be proven with only predicative class comprehension. However, as Uzquiano notes, "while the proof of Bernays' theorem does not require impredicative class comprehension, the link with the Cantorian lemma does presuppose it" (9), where the Cantorian lemma is the claim that every class has more subclasses than members.

[19] See Feferman (2005) and Crosilla (2017) for discussion of which parts of mathematics can and cannot be justified on predicative grounds.

'completed' infinities.[20] The modal strategy will be our preferred strategy for resisting the usual anti-Countabilist import of CT.

4.4.1 The modal strategy as applied to RP

Let's get the strategy on the table. Recall Naive Comprehension:

> *Naive Comprehension*: Necessarily, for any things, there is a set of exactly those things.

As Linnebo (2013) argues, we should replace Naive Comprehension with a corresponding modal principle:

> *Modal Naive Comprehension*: Necessarily, for any things, it is possible that there be a set of exactly those things.

Modal Naive Comprehension retains the intuitive plausibility of Naive Comprehension. As noted, the initial attraction of Naive Comprehension reflects its being unclear what could prevent a given plurality from forming a set. Modal Naive Comprehension, like Naive Comprehension, does not draw any problematic distinctions between those pluralities that can form sets and those pluralities that can't form sets.

By endorsing Modal Naive Comprehension, we can get an illuminating perspective on RP. Recall that RP shows that, necessarily, the plurality of all sets does not *actually* form a set. This leaves open, however, whether that plurality *could* form a set. As Menzel (2019) put it:

> In brief: the axioms of set theory are implicitly modal [...]. Thus, the [set-theoretic] hierarchy is indefinitely extensible: necessarily, no matter how 'high' the hierarchy happens to be, it could be higher still. Set theory is thus the study, not of the sets there actually are—that's irrelevant—but the study of the various set theoretic universes there could be.

Here, Menzel describes *height potentialism*. Any set-theoretic structure V is ultimately generated by two sorts of processes: the powerset operation, and the length of the ordinals in V. This is witnessed by the standard cumulative hierarchy of sets, which starts with the empty set and is generated by repeated applications of the powerset operation. The process continues for as long as there are ordinals:

$$V_0 = \emptyset$$
$$V_{\alpha+1} = P(V_\alpha)$$
$$V_\beta = \bigcup V_\alpha, \text{ for all } \alpha < \beta$$
$$V = \bigcup V_\alpha, \text{ for all ordinals } \alpha \text{ in } V$$

According to height potentialists, set-theoretic structures might differ with respect to their 'height', reflecting the length of ordinals in the structure. Within a set-

[20] Another (if approximate) precursor of the modal strategy, applied by Dummett (1993) to both the set-theoretic and semantic paradoxes, involves the claim that the notion of *set* is 'indefinitely extensible', where the notion of extensibility is not explicitly modal.

theoretic structure V, there will be some pluralities of sets (e.g. the plurality of all ordinals *oo* and the plurality of all sets *ss*) that do not in fact form sets. However, there could always be another set-theoretic structure V^*, for which the original V is merely an initial segment, such that every plurality of sets in V *does* form a set in V^*. Similarly, the pluralities of sets that do not form a set in V^* will go on to form sets in V^{**}, and so on. On this picture, *there is no canonical set-theoretic structure*. Rather, there is a series of possible set-theoretic structures that grow ever taller. This picture elegantly avoids the awkward question of exactly how high the ordinals go: no matter how high they go, they could always go higher. In response to RP, the modal strategy maintains that *within* any given set-theoretic structure certain pluralities will not form sets, but every such plurality *could* form a set in some larger set-theoretic structure.

4.4.2 The modal strategy as applied to CT

An entirely analogous move can be made in response to CT: while *within* a given set-theoretic structure certain sets will not stand in one-to-one correspondence to N (because the structure lacks the relevant bijections), every such set is such that it *could* stand in one-to-one correspondence to N in some larger set-theoretic structure (because that larger structure *would* contain the relevant bijection). We correspondingly maintain:

> *Modal Countability*: Necessarily, for every infinite collection (set or plurality), there could be a bijection from the members of the collection to N.

Modal Countability secures Countabilism conditional on the following claim:

> *Size* $\leftrightarrow$ *Possible Function*: For any two collections (sets or pluralities): the two collections are the same size iff it is possible for there to be a bijection from one to the other.

There are a number of reasons why this modal principle is preferable to Size $\leftrightarrow$ Function (the commonly accepted biconditional of the previously discussed Size $\rightarrow$ Function). First, the modalized version should be acceptable to Nominalists. Whether two pluralities are the same size should not crucially depend on the actual existence of any abstracta.[21] Second, the modalized version encodes the account of size that we previously defended: size is constitutively connected to *possible* or *potential* acts of counting, which themselves constitute the relevant bijections. Third, anyone who acknowledges that it should not be a *contingent* matter whether two pluralities have the same size should want to endorse Modal Countability. If the same-size-as relation is to supervene on the intrinsic properties of the pluralities in question, we should adopt the modalized version.

An important consequence of Modal Countability is that the ‘powerset’ of any infinite set is indefinitely extensible. More precisely, necessarily, for any infinite set X and any plurality *xx* of sets which are subsets of X, it is possible for there to be a

[21] We address the potential worry that the non-existence of abstracta is ‘necessary’ in Sect. 5.1.

distinct subset $Y \subset X$, which is not among the *xx*. In other words, for any infinite set *X*, there is no 'maximal' powerset of *X*. Here is an informal proof of this result:

> For reductio, suppose there is some set-theoretic structure *V* that contains *X* and its maximal powerset *P*(*X*). By Cantor's Theorem, *V* will not include a bijection $f : X \rightarrow P(X)$. By Modal Countability, there could be some other set-theoretic structure V^* that contains *X*, *P*(*X*), and a bijection $f : X \rightarrow P(X)$.[22] Because Cantor's Theorem holds in V^*, there will be some subset $Y \subset X$ in V^* that is not contained in *P*(*X*). So, *P*(*X*) is not maximal.

The fact that the powerset of any infinite set is indefinitely extensible has an interesting, systematic, and elegant consequence. Recall that any set-theoretic universe is ultimately generated by two sorts of processes: the powerset operation and the length of the ordinals. Proponents of height potentialism maintain that the length of the ordinals is indefinitely extensible: necessarily, for any ordinals, there could always be more. The modal approach to CT simply extends this line of thought to the powerset operation: necessarily, for any subsets of an infinite set, there could always be more. This is *width potentialism*. For any set-theoretic structure, there is both a taller one and a wider one.

4.4.3 The mathematical viability of the modal strategy

Is there a mathematically precise way of understanding both height and width potentialism? There is. And can this mathematically precise way of understanding height and width potentialism recover all of ordinary mathematics? It can. Since our primary goal is to focus on the *philosophical* case for the truth of Countabilism, here we'll only briefly summarize these mathematical results.

Linnebo (2010, 2013) has developed a modal set theory **L** that embodies height potentialism. Linnebo's modal set theory is formulated in terms of first-order modal logic with both singular and plural variables, so as to be able to formulate Modal Naive Comprehension as an explicit axiom in his theory:

> *Modal Naive Comprehension*: $\Box\forall xx\Diamond\exists y\forall x(x \prec xx \leftrightarrow x \in y)$.

Linnebo's theory also includes several standard set-theoretic axioms, such as the axiom of foundation and extensionality, together with modalized versions of certain axioms, such as a modal version of the axiom of extensionality (in which sets are individuated by their members across worlds) as well as a modal version of the axiom schema of replacement. Intuitively speaking, one can think of the 'possible worlds' in Linnebo's theory as stages in the cumulative hierarchy of the form V_α, where V_β is accessible to V_α just in case $\beta \geq \alpha$. The informal interpretation of $\Diamond\phi$ corresponds to 'ϕ can be made true by adding enough sets', where $\Diamond\phi$ is true at V_α just in case ϕ is true at some higher stage in the cumulative hierarchy. The informal interpretation of $\Box\phi$ corresponds to 'ϕ will remain true no matter what sets are

[22] This follows from the fact that there is a possible set-theoretic structure witnessing the countability of both *X* and *P*(*X*).

introduced', where $\Box\phi$ is true at V_α just in case ϕ is true at all higher stages in the cumulative hierarchy.

The reason why Linnebo's theory is not a form of width potentialism is because he includes an axiom to the effect that the powerset of any given set is 'completeable'. In other words, contrary to width potentialism, Linnebo's theory has as an axiom that for any arbitrary set z, it is possible to form *all possible* subsets of that set:

Completability of Subsets: $\Box\forall z\Diamond\exists xx\Box\forall y(y \prec xx \leftrightarrow y \subseteq z)$.

For our purposes, the most important fact about **L** is that it can reproduce all of ordinary mathematics, because **L** can interpret the full theory of **ZFC** using the following translation:

Definition 1 ($\Diamond$-translation) Where $\Diamond$ is any modal operator, the $\Diamond$-*translantion of a formula* ϕ in the first-order language of set theory is the formula $\phi^\Diamond$, which results from ϕ by replacing every universal quantifier $\forall x$ with $\Box\forall x$, and every existential quantifier $\exists x$ with $\Diamond\exists x$.

Theorem 1 (Linnebo Interpretation Theorem) The theory **L** interprets **ZFC** under the $\Diamond$-translation.[23]

In recent work, Scambler (2021) has expanded on Linnebo's framework to develop a modal set theory **M** (which he names after Meadows (2015) that drops the axiom of Completability of Subsets and axiomatizes a version of set-theoretic potentialism consistent with both width and height potentialism, where any set-theoretic universe has possible 'vertical' and 'horizontal' expansions.

The central mathematical technique used to create 'horizontal' expansions of any set-theoretic universe is Cohen's (1966) method of *forcing*, which was used initially to prove the independence of the axiom of choice and the continuum hypothesis. With forcing, one typically starts with a transitive model M of **ZFC** and a 'forcing notion' $\mathbb{P} \in M$, which is a partially ordered set. Supposing that there exists an 'M-generic filter' $G \subseteq \mathbb{P}$ (meaning that G is a filter that contains members from every dense subset of $\mathbb{P}$ in M), then one can build a forcing extension $M[G]$, where (i) $M \subset M[G]$, (ii) $G \in M[G]$ (but $G \notin M$), (iii) $M[G]$ is also a model of **ZFC**, and (iv) M and $M[G]$ contain the same ordinals (meaning that $M[G]$ is a *horizontal* rather than vertical expansion of M). By choosing an appropriate forcing notion $\mathbb{P}$, one is able to precisely control the set-theoretic truths found in $M[G]$, which is why the the method of forcing is able to produce such a wide range of different models of **ZFC** for the purposes of establishing independence results.

While the mathematical technique of forcing is uncontroversial, there are certain philosophical controversies about how it should be interpreted. According to a 'multiverse' conception of set theory, defended by Hamkins (2012), for any set-

[23] Here we are following Scambler (2021) presentation of **L**, since Linnebo does not build in the axiom of choice in his presentation of the theory. Scambler includes the axiom of plural choice as part of the background plural logic, which, together with Modal Naive Comprehension, ensures the possible existence of a choice set for any set of disjoint non-empty sets.

theoretic universe V, one can always construct a larger set-theoretic structure $V[G] \supset V$ by the method of forcing. The multiverse approach to forcing is especially promising with respect to Countabilism, because from the multiverse perspective one can prove that, for any set-theoretic structure V and any cardinal $\kappa \in V$, one can consider a forcing extension $V[G]$ that 'collapses' that cardinal to a countable size by including a bijection $f : N \rightarrow \kappa$ inside $V[G]$. In other words, arbitrary uncountable cardinals can always be seen to be countable in a larger set-theoretic universe. However, on a conception of set theory where there is a single canonical set-theoretic structure V that includes *all* sets, it doesn't make sense for there to be a 'larger' structure than V itself. On this single universe view, while there can be no forcing extensions of V as a whole, there can still be forcing extensions of models of set theory *inside* V, where $M \subset M[G] \subset V$.

Scambler's modal logic **M** is an instance of the multiverse approach to set theory, which allows vertical extensions of any universe (like **L**) and horizontal extensions of any universe using a non-trivial forcing notion $\mathbb{P}$ in that universe. One can think of the possible worlds of **M** as stages V_α in the cumulative hierachy, but this time one can extend upwards (to V_β for $\beta \geq \alpha$) or outwards (to some $V_\alpha[G]$, where G is a V_α-generic filter for a non-trivial forcing notion $\mathbb{P} \in V_\alpha$). Because of these different processes of expansion, **M** has two basic modal operators: $\langle v \rangle$ and $\langle h \rangle$. $\langle v \rangle$ corresponds to a 'vertical' sense of possibility, where $\langle v \rangle \phi$ is true at a world V_α just in case ϕ holds at a possibility that is vertically accessible from V_α, and $\langle h \rangle$ corresponds to a 'horizontal' sense of possibility, where $\langle h \rangle \phi$ is true at a world V_α just in case ϕ holds at a possibility that is horizontally accessible from V_α. **M** also has a generic modal operator $\Diamond$, where $\Diamond \phi$ is true just in case ϕ holds in some expanded set-theoretic universe that is reachable by arbitrary combinations of vertical and horizontal expansion techniques (two basic axiom schemas of **M** state that '$\langle v \rangle \phi \rightarrow \Diamond \phi$' and '$\langle h \rangle \phi \rightarrow \Diamond \phi$' are true for arbitrary ϕ).

In addition to giving up on Completeability of Subsets, perhaps the central difference between **M** and **L** is the addition of an axiom governing the horizontal expansion process, corresponding to the technique of forcing. The axiom roughly states that for any set-theoretic universe V and any forcing notion $\mathbb{P}$, it is always (horizontally) possible for there to be a V-generic filter on $\mathbb{P}$. If we let $D(x, xx)$ abbreviate the claim that x is a partial order, and the xx are the dense subsets in x, and if we let $FMeets(y, xx)$ abbreviate the claim that y is a filter that intersects all the xx, then the axiom can be written as follows:

Horizontal Extendability: $D(x, xx) \rightarrow \langle h \rangle \exists y[FMeets(y, xx)]$

We won't be presenting every axiom of **M** here (for a full presentation of **M**, see Scambler (2021)). For our purposes, what is crucial is that from the modal set theory **M** one is able to prove the following results:

Theorem 2 The theory **M** interprets **ZFC** under the $\langle v \rangle$-translation.

Theorem 3 The theory **M** proves $\Box \forall xx \Diamond \exists y \forall x(x \prec xx \leftrightarrow x \in y)$.

Letting $f : A \twoheadrightarrow B$ abbreviate the claim that f is a *surjective* function from A to B, then we also have:

Theorem 4 The theory **M** proves $\Box\forall xx([\exists y(y \prec xx) \rightarrow \Diamond\exists f[f : \mathbb{N} \twoheadrightarrow xx]$.

Theorem 2 establishes that **M** can recover all of standard mathematics. **Theorem 3** establishes that **M** is a mathematical framework that entails Modal Naive Comprehension. Finally, **Theorem 4** establishes that **M** entails Modal Countability. These formal results show that the modal approach to RP and CT, embodied by Modal Naive Comprehension and Modal Countability, can be understood in a mathematically precise way that is able to recover all of ordinary mathematics.

5 Clarifying the content of Countabilism

Here we clarify the modal content of Countabilism and its relation to the neighbouring thesis of Finitism.

5.1 The modality at issue in the modal strategy

How are we to understand the kind of modality associated with Modal Naive Comprehension, Modal Countability, Size ↔ Possible Function, and Countabilism? As it happens, there are various options which would serve the Countabilist purposes. Here we discuss what we take to be the three most plausible options.

We start with the most natural interpretation—namely, one in terms of metaphysical possibility. On this sort of approach, just as different metaphysically possible worlds can differ with respect to the physical objects they contain, possible worlds can also differ with respect to the mathematical objects they contain (e.g. different metaphysically possible worlds might contain different set-theoretic universes). By understanding the modal operators in this way, one can derive a number of general metaphysical consequences that go far beyond the interests of the working mathematician. For example, given Modal Countability and Size ↔ Possible Function, no metaphysically possible world can contain more than countably many *concrete* objects (such as uncountably many spacetime points), since, necessarily, every concrete plurality of objects can be put into one-to-one correspondence with N. As Countabilists, we embrace these implications for concrete reality, as we discuss further in Sect. 7.

Perhaps the main concern with this metaphysical approach is the common assumption that the existence or non-existence of abstract mathematical objects is a matter of metaphysical necessity. We have three responses to this concern. First, a number of philosophers have argued against this common assumption on the grounds that no ontological facts, mathematical or otherwise, are metaphysically necessary—the empty world is a genuine possibility, for example (see, e.g., Balaguer (1995), Rosen (2006), and Clarke-Doane (2019)). Second, there are some conceptions of mathematical objects which *require* rejecting the necessitist assumption. For example, on an Aristotelian conception of abstracta, mathematical objects are dependent on the existence of contingent physical objects (see, e.g., Maddy (1990a, b). Lastly, some philosophers interpret mathematical discourse about objects like sets and bijections in a way that is *neutral* on the existence of abstract objects. On these sorts of approaches, the necessary existence or non-existence of abstracta is irrelevant to the truth-conditions (or correctness-conditions) concerning claims about sets or bijections. For example, according to Hellman's (1989, 1996) modal structuralism, claims that are seemingly about mathematical abstracta, such as numbers or sets, should instead be interpreted as claims about possible concrete structures that are models of the relevant arithmetical or set-theoretic axioms. On this kind of approach, the central claim of set-theoretic potentialism is roughly the following: necessarily, for any plurality of physical objects that instantiate a set-theoretic structure, it is possible for there to be some other plurality of physical objects that instantiate a 'larger' set-theoretic structure (either larger in height or width). In addition, various fictionalist approaches to mathematics have been developed that are also neutral on the (non-)existence of abstract mathematical objects. For example, while maintaining that there might be no fact of the matter about whether abstract objects exist, Balaguer (1998, 2021) and Yablo (1998, 2001, 2005) each develop fictionalist approaches to the philosophy of mathematics according to which there is nonetheless an objective sense in which certain mathematical claims (e.g. 'there are infinitely many primes') are 'correct' whereas other mathematical claims (e.g. 'there are finitely many primes') are 'incorrect'. If these ontologically neutral interpretations of mathematics can be made to work, then the Countabilist's central claims about the metaphysical possibility of the existence of certain kinds of bijections can be understood as claims about the possibility of certain kinds of ontologically neutral facts obtaining, which are compatible with the necessary (non-)existence of abstracta.[24]

A second way to interpret the relevant notion of possibility is in terms of what is ideally conceivable (or otherwise rationally intelligible; here we focus on conceivability), such that to say that it is 'possible' for there to be a bijection from a certain plurality of objects to N is simply to say that it is ideally conceivable for there to be a bjiection from that plurality of objects to N.[25] In light of Kripke's (1972/80) influential arguments against the connection between conceivability and possibility (such that, e.g., it is conceivable yet metaphysically impossible for water not to be H_2O), many philosophers will regard this interpretation as importantly distinct from the first interpretation in terms of metaphysical possibility.[26] The main

[24] See Dorr (2008, 2010) and Azzouni (2004) for related ontologically neutral approaches to mathematics. Just as these approaches often claim that they can vindicate the 'nominalistic content' of scientific theories that quantify over mathematical objects, such approaches can use similar means to extract the 'nominalistic content' of Countabilism, as we will further discuss in Sect. 7.

[25] See Chalmers (2002) for discussion of the relevant notion of idealized conceivability.

[26] There are also a number of historical challenges to the connection between conceivability and metaphysical possibility. For example, in her *Essays Upon the Relation of Cause and Effect* (1824), Mary Shepherd maintains that Hume errs in supposing that whatever he can conceive (e.g., that a new existent could occur without a cause, or that some similar cause might produce a different effect) is genuinely possible: "Mr. Hume makes also a great mistake in supposing because we can conceive in the fancy the existence of objects contrary to our experience, that therefore they may really exist in nature; for it by no means follows that things which are incongruous in nature, may not be contemplated by the imagination, and received as possible until reason shows the contrary" (83). For a prominent attempt to reforge an indirect link between conceivability and metaphysical possibility in terms of 'epistemic two-dimensionalism', see Chalmers (1996, 2002, 2006). For criticisms of Chalmers's proposal, see Wilson

advantage of this second proposal is that it avoids the main criticism of the first proposal. Although the (non-)existence of mathematical objects might be metaphysically necessary, the conceivability approach makes no claims about the actual or (metaphysically) possible existence of mathematical objects. This second approach is also closely related to certain broadly logical or conceptual modal notions which have been defended by a variety of philosophers. For example, Russell, in his (1919), endorses a notion of 'logical possibility' according to which facts about ontology are never logically necessary, and Field (1993) defends a notion of 'conceptual possibility', according to which it is conceptually contingent which mathematical objects there are.

Perhaps the main objection to this second approach is that it might not fit well with our overall methodological strategy down the line. Instead of directly arguing that certain kinds of bijections are conceivable or otherwise rationally intelligible, our methodology will be broadly abductive.[27] We think that the theoretical and explanatory advantages of Countabilism in mathematics, science, and philosophy jointly make Countabilism more plausible than its negation. However, one might naturally object that this kind of 'cumulative case' in favor of Countabilism is too weak.[28] Because of the reasoning in Sect. 4.4.2, this second interpretation of Countabilism is committed to the inconceivability (or unintelligibility) of a 'maximal' powerset of N, because the supposition that there could be a bijection from the maximal powerset of N to the natural numbers leads to a contradiction, via Cantor's theorem.

In response, we think that a plausible case can be made that, at least for non-ideal agents like us, abductive reasoning has an important role to play in establishing what is ultimately conceivable or intelligible. Consider, for example, the **ZFC** axioms of set theory. Either these axioms are consistent or inconsistent, where presumably only one of these options would be ideally conceivable to an ideal reasoner. However, given that we are not ideal reasoners, our belief that the **ZFC** axioms are consistent is not justified by some conclusive *a priori* proof, but rather by the overall longterm success that **ZFC** set theory has enjoyed as a unified foundation for mathematics.[29] Examples like this are easily multiplied. For example, among theoretical computer scientists, it is widely believed that $\mathrm{P} \neq \mathrm{NP}$ on broadly abductive grounds. However, to an ideal reasoner, presumably only one of $\mathrm{P} = \mathrm{NP}$ or $\mathrm{P} \neq \mathrm{NP}$ is ultimately coherent or ideally conceivable. We think the

Footnote 26 continued
(2006) and Melnyk (2008). For an alternative approach to implementing epistemic two-dimensionalism which appeals to abduction rather than conceiving, see Biggs and Wilson (2017, 2019).

[27] However, in Sect. 6.3, we argue that width potentialism can be motivated on the grounds that there cannot be brute necessities. Insofar as one thinks that brute necessities are inconceivable, then this would motivate width potentialism as understood in terms of conceivability.

[28] Thanks to a referee for raising this concern, and more generally for encouraging us to expand on what understanding(s) of modality is supposed to be at issue in Countabilism.

[29] As Feferman (2005) observes, "the success of axiomatic set theory—as developed by Zermelo, Skolem and Fraenkel—[succeeded] in allaying fears about the paradoxes. Though not demonstrably consistent, intensive development of the subject without running into any difficulties gave comfort and confidence to its practitioners and gradually won the support of mathematicians at large" (11).

situation is similar with respect to Countabilism. The disagreement between the Countabilist and the anti-Countabilist, at least on this second interpretation, is also a disagreement about what is ultimately coherent or conceivable. The anti-Countabilist maintains that there is a conceivable plurality of objects (e.g. the maximal powerset of *N*) such that it's *inconceivable* that there exists a bijection from those objects to *N*, whereas the Countabilist disagrees. Just as in the case of the consistency of **ZFC** or the question of whether P = NP, we think that non-ideal beings like us are capable of making progress on this question on broadly abductive grounds.[30]

A third and final interpretation of the modality at issue takes it to be one that is distinctive to mathematics. In his formulation of set-theoretic potentialism, Linnebo (2010, 2013) endorses an unanalyzable kind of mathematical or set-theoretic modality, and Scambler (2021) is inclined to understand the modal operators as involving a 'dynamic' notion of mathematical possibility:

> I should say a word about the interpretation of the modal operators. One might be concerned that the results are inherently insignificant, because they make use of a non-trivial sense of mathematical possibility, and no such sense exists. I am not persuaded by negative thoughts of this kind. Clearly set-theorists have been driven to use modal operators in discussing their subject matter, and probably they have something sensible in mind. I myself favor (as has probably been clear throughout) a dynamic reading of the modal operators, wherein they concern possibilities for action on the part of some (unspecified and idealized) individual. This way of thinking of mathematics has been alive since the time of Euclid, and I believe it has just as much application to modern set theory as it did to geometry. Such ideas are also being developed today, in one form or another, by Kit Fine [3] and Martin Pleitz [16]. (22)

We agree with Linnebo and Scambler that there is no in-principle problem with an appeal to a distinctive and perhaps unanalyzable kind of mathematical modality, so we will consider this to be another viable approach for the interpretation of the modal operators found in Countabilism.

We now turn to a question for both the second and third approaches. In discussing the first interpretation in terms of metaphysical modality, we noted the implication that Countabilism has important consequences for what is metaphysically possible, and hence for what is *actual*. If it is metaphysically necessary that everything is countable, then clearly the actual world cannot contain an uncountable plurality of objects. Does the same implication hold for the second and third interpretations? We think it does. Regarding the second interpretation, if it is *ideally inconceivable* (unintelligible) for there to be a genuine uncountable plurality of objects, then on the usual understanding (embraced by proponents of conceivability as a guide to metaphysical possibility, including Hume and Chalmers), it is not metaphysically possible for there be a genuine uncountable plurality of objects

[30] One might also naturally wonder about the utility of abductive reasoning with respect to metaphysical modality. For a defense of the epistemology of abduction with regards to metaphysical modality, see Biggs (2011), Williamson (2013: 423–424), and Biggs and Wilson (2017).

either, since such inconceivability or unintelligibility would reflect the incoherence of the notion at issue, and it is a minimum requirement on a metaphysically possible scenario or world that it should be coherent. Regarding the third interpretation, given the fact that it is *mathematically* impossible for there to be a genuinely uncountable plurality of objects, does it follow that such a plurality is *metaphysically* impossible? We think that a plausible constraint on our understanding of mathematical possibility is that this inference should be satisfied. After all, the realm of mathematical possibility should at least be able to account for the kinds of structures present in the *actual* world. Moreover, the requirement that mathematics should be able to account for the structures present in the actual world should hold robustly: mathematics should be able to account for the structures present in the actual world *regardless* of which metaphysically possible world turns out to be actual. We therefore conclude that all three interpretations rule out the actual and metaphysically possible existence of an uncountable plurality of (concrete or abstract) objects.

In sum, we think that there are three different viable interpretational options for the modal operators in Countabilism, each of which implies that uncountable infinities are metaphysically impossible (and hence not actual). We therefore conclude that skepticism about Countabilism on the grounds that there is no (non-trivial) sense to be made of the modal operators in the formulation of Countabilism is unwarranted.[31]

All this being said, there remain important questions of detail about how the associated modality interacts with specific mathematical implementations of Countabilism, and before any specific mathematical implementation of Countabilism is to be defensible, these details need to be addressed. For example, Scambler's modal set theory **M** makes non-trivial assumptions about the correct principles of plural modal logic that govern the relevant modality, and one's views about the plausiblity of these logical principles might be sensitive to how one chooses to interpret the relevant modal operators.[32] We cannot hope to settle these delicate questions about the correct principles of plural modal logic here, so we will leave the defense of these important details for future work.[33]

[31] As a referee points out, one could perhaps introduce a stipulated notion of 'possibility' that makes Countabilism come out as trivially true, but we don't think any of the three interpretations that we have discussed makes Countabilism come out as trivial. All three interpretations draw on modal notions whose coherence and importance have been defended on independent grounds by several philosophers. One could of course adopt a more radically skeptical stance that questions the intelligibility or objectivity of these modal notions (see, e.g., Clarke-Doane (2019, 2021), but arguing against this kind of view is beyond the scope of the current paper. At the end of the day, we think that the best reason for believing that Countabilism is both true and non-trivial is by looking at its many downstream consequences in mathematics, science, and philosophy. After all, Countabilism cannot be trivial or unintelligible if it has important non-trivial consequences about intelligible matters.

[32] For relevant work on the appropriate modal logic of plurals, see Williamson (2010), Uzquiano (2011), and Linnebo (2016).

[33] Another complicating factor about **M** is that it utilizes three different modal notions, a notion of 'vertical' possibility, a notion of 'horizontal' possibility, and a 'combined' notion of possibility that results from arbitrary iterations of vertical and horizontal possibility. The different interpretations of modality that we have so far been discussing are best viewed as interpretations of the more generalized

5.2 Comparison with finitism

How does Countabilism contrast with *Finitism*, according to which, necessarily, any collection (set or plurality) is finite? Strictly speaking, Finitism is compatible with our formulation of Countabilism. Indeed, the truth of Finitism would render Countabilism immediately true, since in that case there would be no infinite sets or pluralities whatsoever. The Countabilist's main claim is that one cannot move beyond the countably infinite, which the Finitist would of course accept. Finitism should therefore be seen as a strictly stronger claim than Countabilism.

Nonetheless, we think there are good reasons not to go so far as Finitism. Perhaps the main reason is that Finitism is far more revisionary of both mathematics and physics. As Klement (2010b: 30) observes, the supposition that only finite pluralities can form sets would "cripple" mathematics. Relatedly, there is no correlate to the method of forcing that would enable Finitism to account for infinite mathematics. With respect to physics, Finitism is also inconsistent with basic and empirically open physical possibilities, such as a universe that never ends, either temporally or spatially. In addition, while we find it difficult to form a positive conception of an uncountable plurality of objects, it seems to us that countable pluralities of objects are easily conceivable: consider, e.g., an unending sequence of dominoes. Lastly, our reasons for being sympathetic to Countabilism, which we will further discuss below, do not generalize to Finitism.

6 Countabilism: the philosophy of mathematics

We finally turn to the case(s) in favor of Countabilism. As noted above, one important advantage of Countabilism is that, when understood in terms of both height and width potentialism, it offers a unified response to both RP and CT, reflecting the deep structural parallels between both mathematical results. We now canvass several other advantages of Countabilism in the philosophy of mathematics.

6.1 Countabilism and the intuition of limitlessness

Countabilism nicely corresponds to an intuitive understanding of infinity as limitless, such that it cannot itself be surpassed in number, and hence can only have one 'size'. Such an understanding of the infinite as *limitless* or *inexhaustible* has been registered by a number of different mathematicians and philosophers,

Footnote 33 continued
combined notion of modality, whereas Scambler's vertical and horizontal notions of possiblity should be seen as restrictions on this modality from which one can 'generate' the more general combined notion of modality. We also leave the important question of whether the relevant interpretation of the combined modal notion in **M** can be entirely 'factored' into restricted vertical and horizontal modalities for future work. Thanks to a referee for raising this concern.

including Gauss, Cauchy, Poincaré, and Weyl,[34] and it is precisified by the modal indefinite extensibility approach to set theory.

These intuitions of infinity's being limitless or inexhaustible are sometimes expressed in terms of a distinction between 'actual' and 'potential' infinity, as when, e.g., Linnebo and Shapiro (2019) say, "Beginning with Aristotle, and until the nineteenth century, the vast majority of major philosophers and mathematicians rejected the notion of the actual infinite. They argued that the only sensible notion is that of potential infinity" (160). It is important to be clear, however, that the underlying intuition here is not that nothing could be infinite: that would be to endorse Finitism. We do not find it 'intuitive' that there are only finitely many natural numbers! Less radically, the intuition of limitlessness is one that simply maintains than an infinite collection is one that, since itself inexhaustible, cannot be surpassed in number.

An appeal to the limitlessness intuition may not persuade philosophers inclined towards present Cantorian orthodoxy. However, given that the primary reason for endorsing the highly counterintuitive claim that there are different sizes of infinity is now absent (since the import of CT and, more generally, contemporary mathematical results, can be interpreted in a Countabilist setting), we maintain that it makes sense for the *default* position to be the intuitive one according to which there is only one size of infinity.

6.2 Countabilism and independence

An important advantage of Countabilism is that, as we'll now argue, it provides the best explanation for the pervasive independence results in contemporary set theory. The most famous example of independence is the Continuum Hypothesis (CH), which states that there is no set whose cardinality is strictly between that of N and $P(N)$. Following a result from Levy and Solovay (1967), the truth of CH is not decidable within **ZFC**, together with any additional large cardinal axioms. CH, however, is only the tip of the iceberg. Since the advent of Cohen's forcing technique, countless other natural set-theoretic statements have been proven to be independent of **ZFC**.

Broadly speaking, we believe, like Hamkins (2012), that the pervasive independence phenomena in set theory support anti-realist approaches according to which set-theoretic statements like CH lack a determinate truth-value. After all, if the history of set theory had turned out differently, and there was as little independence in set theory as there is in arithmetic, then this would have provided strong support for the realist view; such relative lack of independence would indicate that our set-theoretic axioms successfully pin down a robust conception of the true set-theoretic universe, yielding answers to any naturally formulated set-theoretic questions. If this sort of history would have provided confirmation of the realist view, it follows that the opposite history, where independence is pervasive,

[34] As Poincaré (1912) puts it, '[W]hen we speak of an infinite collection, we understand a collection to which we can add new elements unceasingly [...]' (47), and as Weyl (1918) puts it, 'Inexhaustibility is essential to the infinite' (23).

provides confirmation for the anti-realist view.[35] After all, anti-realist views which entail that CH has no determinate truth-value make the (so far correct) empirical prediction that we will not find an answer to CH and other natural set-theoretic statements.

All this being said, the strongest support here for Countabilism consists in the specific *kind* of anti-realism that it delivers. In what follows, we will consider two specific advantages that a modal potentialist version of Countabilism has over other kinds of anti-realism.

6.2.1 Independence with a univocal concept of set

There are many different accounts of why statements like CH lack a truth value; due to considerations of space we focus on what we see as our main competitor—namely, an account of the sort endorsed by Balaguer (1995), Field (1998), and Hamkins (2012), on which the independence results reflect there being multiple set-theoretic structures that are *equally good* candidates for being the total set-theoretic universe, with CH being true in some such structures and false in others.

Such an account, if feasible, would explain why CH lacks a truth-value in the same way that semantic supervaluationists (e.g., Fine (1975) and Lewis (1982) think that vague sentences lack a truth-value. In the latter case, for example, if 'Bob is bald' lacks a truth value, this reflects that there are multiple equally good ways of precisifying 'bald' in such a way as to draw a determinate boundary between the bald things and the not-bald things, with Bob counting as bald on some precisifications and not-bald on others. As Martin (2001) compellingly argues, however, such a supervaluationist strategy faces a serious problem. A simplified version of his argument is as follows. Suppose that there are two set-theoretic structures V and V^* that are each equally good candidates for being the complete set-theoretic universe. Then we may compare them level by level and see which one 'leaves out' sets that the other has. Clearly, it will have to be the case that $V_0 = \emptyset = V^*_0$. As our inductive step, suppose that $V_\alpha = V^*_\alpha$. Then, if V is to be a candidate for the complete set-theoretic universe, then it must be that $V_{\alpha+1} \supset V^*_{\alpha+1}$, for if V failed to contain sets that were included in V^*, then it would not be a live candidate for the *complete* set-theoretic universe. For similar reasons, it must be the case that $V_{\alpha+1} \subset V^*_{\alpha+1}$, if V^* is to be a live candidate for the complete set-theoretic universe. So, $V_{\alpha+1} = V^*_{\alpha+1}$. The limit stage of the induction is trivial, since for any limit ordinal β, if $V_\alpha = V^*_\alpha$ for all $\alpha < \beta$, it follows by definition that $V_\beta = V^*_\beta$. By similar reasoning, each must contain the ordinals that the other contains if they are both to be candidates for the complete universe of sets, so $V = V^*$. The upshot is that it is impossible for there to be two different but equally good candidates for the entire set-theoretic universe.

In response to Martin's argument, Hamkins (2012) suggests that there are distinct concepts of *set*. For example, V might be a complete universe of sets_1 and V^* might

[35] This follows from a basic theorem of confirmation theory, according to which if some evidence E confirms H over ¬H, then the evidence ¬E confirms ¬H over H.

be a complete universe of sets_2. In order to compare these universes level by level, one must illegitimately assume (according to Hamkins) that there is a unique background concept of 'set' that both these universes answer to. We find this response unsatisfactory, however. To start, there is plausibly a unique concept of *set* that is at the heart of the iterative conception of sets—sets are simply well-founded and extensional objects that are fully determined by their members.[36] Given any plurality whatsoever of (well-founded) sets, there could be a set with exactly that plurality as its members, as per Modal Naive Comprehension. Hamkin's skepticism leads him to be skeptical that there is a unique concept of *well-foundedness*, or even a unique concept of *natural number*. We agree that *if* our notion of *well-foundedness* (or *natural number*) is inherently vague, then so too will be the concept of *set* that is at the heart of the iterative conception. However, we find such a skepticism to be implausibly radical.

We thus seem to face a dilemma: either we can have the explanatory benefits of anti-realism at the cost of implausibly multiplying our concept of *set*, or we can avoid implausibly multiplying our concept of *set* at the cost of taking on board the realist assumption that there are innumerable set-theoretic claims whose truth lies beyond our reach.

Luckily, with Countabilism and its width-potentialist underpinning in hand, we can have our cake and eat it too. We needn't multiply concepts in order to explain why CH lacks a truth-value. We can endorse Martin's conclusion that there can't be *multiple* equally good candidates for being the complete set-theoretic universe, while at the same time denying that there is a *unique* candidate for being the complete set-theoretic universe (whose status determines the truth value of CH), by maintaining that there are *zero* candidates for being the complete set-theoretic universe! This follows directly from height and width potentialism. For any set-theoretic structure V, there could always be a strictly 'better' structure V^* that is either taller or wider than V. So, for any V, V (determinately) fails to be a candidate for the complete set-theoretic universe. Moreover, this explanation of the indeterminacy of CH crucially relies on width-potentialism. If set-theoretic structures only disagreed with respect to how high the ordinals can extend, such structures would agree on CH, which is a claim that is settled merely by $V_{\omega+2}$. However, if set-theoretic structures differ in their width, then CH will not have an answer. For example, as Hamkins (2015) notes, for any set-theoretic structure V, one could always go to a forcing extension $V[G]$ of V which disagrees with V on the truth-value of CH.

[36] We do not have any qualms with there being other conceptions of set besides the iterative conception, such as those that deny the axiom of foundation. We also don't want to defend the claim that the concept of *set* at the heart of the iterative conception is in any way metaphysically superior than other conceptions of set. Our main interest lies in questions about the possible sizes of infinity, which we think has ramifications far outside mathematics. We are only focusing on the iterative conception of set here because such a conception provides an ideal setting for reasoning about possible sizes of infinity. If there are additional non-well founded sets that lie outside the cumulative hierachy, then in any case those sets will not influence any standard mathematics (see, e.g., Kunen (1980)). For an extended defense of the iterative conception of set and its advantages over other conceptions, see Incurvati (2020).

6.2.2 Predicting the right amount of independence

Different versions of anti-realism differ with respect to which kinds of mathematical truths have determinate truth-values. For example, does every statement in the language of arithmetic have a determinate truth-value? Some radical versions of anti-realism say 'no'. For example, one might hold that mathematical truth is fully captured by weaker axiomatic systems such as Robinson Arithmetic (Q), Primitive Recursive Arithmetic (PRA), or Peano Arithmetic (PA) (see, e.g., Koellner (2013)).

In a set-theoretic context, it is useful to measure the strength of different anti-realist positions by focusing on fragments of the set-theoretic universe of the form $(H_\kappa, \in)$, where H_κ is the collection of all sets that are hereditarily of cardinality less than κ. In other words, H_κ is the collection of all sets X such that X, the elements of X, the elements of the elements of X, etc., all have cardinality less than κ. It turns out that claiming that every statement about $(\mathbb{N}, +, \times)$ has a determinate truth-value is equivalent to claiming that every statement about $(H_{\aleph_0}, \in)$ has a determinate truth value.[37] More simply, claiming that arithmetic is determinate is equivalent to claiming that *finite* set theory is determinate. Since **ZFC** provides a more or less complete theory of $(H_{\aleph_0}, \in)$ (and hence of $(\mathbb{N}, +, \times)$), it is very common to hold that claims about finite set theory have determinate truth values.[38]

The next step up is *countable* set theory. Do claims about the hereditarily countable sets, $(H_{\aleph_1}, \in)$ have determinate truth values? Of late, there seems to be a growing consensus among set theorists that the answer is 'yes'. Adding the axiom of 'Projective Determinacy' (PD) to **ZFC** seems to provide as robust a theory for $(H_{\aleph_1}, \in)$ as **ZFC** provides for $(H_{\aleph_0}, \in)$. As Woodin (2001a) remarks, "The only known examples of unsolvable problems about $[(H_{\aleph_1}, \in)]$ are analogous to the known examples of unsolvable problems in number theory: Gödel sentences and consistency statements" (575). With respect to the status of PD, Koellner (2013) notes that the case for the axiom "has gained wide acceptance by the set-theorists (in particular, inner model theorists and descriptive set theorists) who know the

[37] This follows from the fact that both theories are mutually interpretable (in the model-theoretic sense).

[38] Of course, by Gödel's theorem, there can be no truly complete axiomatization of $(H_{\aleph_0}, \in)$, but **ZFC** seems to determine the truth values of 'natural' mathematical statements about $(H_{\aleph_0}, \in)$. There are also less orthodox views on which the axioms of **ZFC** are simply *false* of their intended subject matter, which is (at least for our purposes) the iterative conception of set (see footnote 33). Although entering into the details of the controversies behind the particular axioms of **ZFC** is beyond the scope of the present paper, a few remarks are in order. With respect to the axiom of infinity, it seems to us that the only principled view which denies the possibilty of infinite sets is some version of Finitism, which we (pessimistically) discuss in Sect. 5.2. With respect to the axiom of choice, we are inclined to agree with Pollard (1988) that the axiom of choice for *plurals* (as opposed to sets) should be uncontroversial. With respect to plurals, the axiom of choice simply states that, for any disjoint non-empty sets there are some things which comprise exactly one element from each. Together with Modal Naive Comprehension (which we think is the most plausible substitute for Naive Comprehension), this plural version of the axiom of choice implies the standard axiom of choice for sets (this is how the modal set theory **M** derives the axiom of choice). With respect to the axiom schema of replacement, see Incurvati (2020: 90–100) for three different arguments that the iterative conception of set implies the axiom schema of replacement. For wider discussion of these controversies about **ZFC**, see Clarke-Doane (2013, 2020: Ch. 2).

details of the constructions and theorems involved in the case that has been made for PD" (25).

However, our mathematical theories seem to give out when moving beyond countable set theory. There is no corresponding consensus about the truths concerning $(H_{\aleph_2}, \in)$, which is where the case for CH needs to be decided.[39] As a result, Countabilism exactly predicts the current state of affairs regarding set-theoretic truth. It predicts that truths about (hereditarily) *countable* sets should be within reach, but truths about the uncountable should lapse into indeterminacy. We see this as a strong point in favor of Countabilism.

6.3 Explainability

It is one thing to argue that Countabilism is true; it is another to be able to explain *why* it is true. In this section, we offer an explanation of the truth of width potentialism, and the associated understanding of Countabilism, that is closely connected to the explanation of the truth of height potentialism.

To start, there is a straightforward explanation of why height potentialism is true. Height potentialism is the claim that, necessarily, for any ordinals, there could always be more. In order to deny height potentialism, one has to maintain that there could be some ordinals, *oo*, such that it is *impossible* for the *oo* to form a set, constituting a greater ordinal. However, if this is to be impossible, then there must be an *explanation* for why it is impossible. Otherwise, one would be saddled with an inexplicable brute necessity, which many philosophers have wanted to avoid.[40] The explanation for the truth of height potentialism is simply that there is no reason why any given plurality of ordinals *couldn't* go on to form a set. In other words, the *absence* of an explanation for why the ordinals should stop at any particular place constitutes an explanation for why they can always be extended further.

We think a similar explanation applies to width potentialism. The height potentialist is skeptical of the idea of a 'maximal' collection of all ordinals on the grounds that it would give rise to inexplicable brute necessities (where are the ordinals supposed to stop?). Similarly, it is natural for the width potentialist to be skeptical of the idea of a 'maximal' collection of all the subsets of an infinite set, such as N, on the grounds that such a maximal powerset would raise all sorts of questions, most particularly the cardinality of $P(N)$, for which there don't seem to be any non-arbitrary answers. As discussed in Sect. 6.2, all of the axioms of **ZFC** (supplemented with large cardinal axioms) do very little to constrain the cardinality of $P(N)$. In fact, the *only* constraint imposed by **ZFC** on the cardinality of $P(N)$ is that the cardinality of $P(N)$ is an uncountable cardinal with uncountable cofinality. Insofar as our repeated efforts to find a non-arbitrary answer to the cardinality of

[39] For some speculative research programs which seek to provide answers for CH, see Woodin (2001b, 2017) and Incurvati (2017). Given the line we are pushing here, should such research programs come to fruition, that would count as evidence against the truth of Countabilism.

[40] The rejection of brute necessities has a long tradition, ranging back to Hume's (1739-40) rejection of necessary causal connections, and recently endorsed by, e.g., Lewis (1986), Dorr (2004), and Goswick (2018). See Van Cleve (2018) for an overview.

$P(N)$ fail, this gives us more evidence that there is no explanation for why $P(N)$ should have one cardinality as opposed to another. This fact can then be used to explain *why* width potentialism is true. The *absence* of any explanation for why the maximal powerset of N should 'stop' at one cardinality rather than another constitutes an explanation for why there can be no maximal powerset. If there were a maximal powerset, it would give rise to the same kinds of inexplicable brute necessities as a maximal collection of all the ordinals.

It is still not obvious, however, how one gets an explanation of Countabilism from an explanation of *width potentialism*. One way one can bridge the divide is by appealing to the set-theoretic technique of forcing. The most mathematically developed way we have for 'horizontally' expanding the set-theoretic universe (by 'adding' new subsets of infinite sets) is by forcing, and as we saw in Sect. 4.4.3, forcing vindicates Countabilism insofar as uncountable cardinalities can always be seen to be countable in appropriate forcing extensions. A second way to bridge the divide from width potentialism to Countabilism starts from Cantor's own account of how the ordinals are generated (1883, Sections 1, 11). Starting from ω, Cantor thought that countable ordinals were generated by a successor operation, as well as by taking limits of ordinal sequences:

$$0, 1, 2, ..., \omega, \omega + 1, ..., \omega + \omega = \omega * 2, ..., \omega * 3, ..., \omega^2, ..., \omega^\omega, ...$$

No matter how often we apply these two operations, however, we will only ever get countable ordinals.[41] As Fletcher (2007) notes:

> It seems clear that by applying the two generating principles we can form a long (but always countable) succession of countable ordinal numbers...It seems that we would need to generate uncountably many countable ordinals to justify the leap to ω_1; but the possibility of doing this is itself in question. How can we pull ourselves up by our countable bootstraps into the realm of the uncountable? (539)

We simply cannot pull ourselves into the realm of the uncountable by our countable bootstraps. No matter how many countable ordinals there are, we can always construct a larger countable ordinal by Cantor's two operations. It is impossible to ever *reach* an uncountable ordinal: the countable ordinals are indefinitely extensible. No matter how many countable ordinals there are, there could always be more. On this sort of picture, $\aleph_1$ is akin to Cantor's notion of The Absolute Infinite, which is an 'inconsistent multiplicity'. If one is willing to accept the indefinite extensibility of ordinals in general (as per height potentialism), it is natural to accept the indefinite extensibility of countable ordinals by similar reasoning. The only reason to think that one *could* ever reach an uncountable ordinal is by appealing to maximal powersets. For suppose there were an infinite set X that had a 'maximal' powerset, $P(X)$. Then, by CT, $P(X)$ would be genuinely uncountable. By the Axiom of Choice, $P(X)$ could be well-ordered as an uncountable ordinal α. Once we have pulled ourselves into the realm of the

[41] This follows from the claim that $\aleph_1$ is a regular cardinal.

uncountable ordinals by appeal to maximal powersets, then as a matter of logic there will be some ordinal $\omega_1 \in \alpha$ that is the least uncountable ordinal. ω_1 will then contain the totality of all possible countable ordinals, contradicting the indefinite extensibility of the countable ordinals. However, once the appeal to maximal powersets is blocked (via width potentialism), there is no reason to think that one can go beyond the realm of the countable.[42]

To sum up: there is an explanation of why width potentialism, and hence the associated understanding of Countabilism, is true; and moreover, this explanation is intelligibly connected to the explanation of why height potentialism is true.

7 Countabilism: physics

Orthodox physics uses mathematical objects that are 'uncountable' as a means of modeling certain physical phenomena, such as the use of continuous manifolds made up of zero-dimensional points to model the geometric structure of spacetime. According to Countabilism, these mathematical objects are not well-suited to accurately represent the physical world. This is because, according to Countabilism, there are no canonical examples of 'uncountable' objects. Just as there is no canonical powerset of N, understood as containing *all possible* subsets of N, there is no canonical set of all real numbers R. Necessarily, for any set of real numbers, there could always be a larger set of real numbers. Because of this, there will be no canonical examples of mathematical objects that are defined in terms of the real numbers, such as arbitrary n-dimensional manifolds. Intuitively, such objects will always be inherently 'incomplete' since they can be further supplemented with extra points or real numbers. However, in the case of physical spacetime, it would be very strange to think that there are 'missing' spacetime points! Even if one granted that there could be these kinds of 'holes' in spacetime, any precise account of *which* holes there are in spacetime would be wholly arbitrary. Because Countabilism implies that there are no genuinely uncountable mathematical objects that represent the structure of the physical world, it is therefore to some extent revisionary of orthodox physics. On the face of it, this might be thought to count against Countabilism. However, there are several reasons to see this consequence as a feature, not a bug.[43]

First, as a historical and methodological matter, the supposition that spacetime is actually composed of uncountably many extensionless, zero-dimensional points is comparatively recent and poorly motivated. As Hellman and Shapiro (2018) make clear, the orthodox approach prior to the nineteenth century reflected the

[42] The mathematical fact that **ZFC** *without* the powerset axiom is consistent with the statement 'every set is countable' supports this fact (the model $(H_{\aleph_1}, \in)$ of hereditarily *countable* sets satisfies all of the **ZFC** axioms, excluding the powerset axiom).

[43] We should emphasize that Countabilism does not say that the pure mathematics that these scientific theories are based on (e.g. differential geometry) needs to be revised. It only says that the structure of the physical world is not adequately captured by the 'uncountable' mathematical objects studied in these mathematical theories.

Aristotelian view that a true continuum cannot be composed of zero-dimensional points. It was only through the mathematical work of Bolzano, Cauchy, Dedekind, Cantor, and others that the present approach became orthodoxy. As we have seen in the case of CT, however, the interpretation of the relevant mathematical work motivating this shift is fraught with mistaken philosophical assumptions, involving the reification of the reals as a complete uncountable totality, along with the rest of the Cantorian hierarchy. The right thing to do, it seems to us, is to face up to and correct this methodological wrong turn. Moreover, rejecting the orthodox point-based characterization of the continuum does not necessarily mean rejecting continuous spacetime in favor of discrete spacetime.[44] Hellman and Shapiro (2018) mathematically develop a number of different *regions-based*, or 'gunky', characterizations of the real line which formalize the traditional Aristotelian view of the continuum. From a Countabilist perspective, Hellman and Shapiro's accounts of 'Aristotelian and Predicative Continua' in Ch. 3 are especially relevant, since their accounts of Aristotelian and Predicative Continua do not resort to uncountable infinities. They also generalize their regions-based accounts of the continuum to multi-dimensional Euclidean and non-Euclidean spaces. When discussing further mathematical applications of their regions-based theories in measure theory and differential geometry, they conclude:

> In short, the proponent of a regions-based theory can legitimately 'have it both ways', adhering throughout to an ontology of regions, yet theorizing with points, point-sets, and point-set-sets, as needed for applications. There is nothing in a regions framework that implies that defined superstructural items may not be genuinely useful or even 'required', say in physical applications of mathematics. It is one thing to say that the point-concept may be unavoidable in certain situations (by us humans?), and quite another to say that, 'therefore reality must be constituted of points'. It is not even accurate to say that 'quantification over points is indispensable' (say in physics); the full-fledged reduction already achieved of points to pluralities of sequences of regions simply belies that. (159–60)

Second, as an epistemic matter, it is clear that the assumption that the world is actually composed of an uncountable totality of zero-dimensional points goes far beyond what any possible empirical data could support. After all, our measurement instruments can only ever operate with a finite level of precision. It seems, then, that no matter what observations we make, there will always be an empirically adequate theory that does not assume that spacetime is continuous. Indeed, in recent work, Miller (forthcoming) argues that, notwithstanding that many fundamental physical theories characterize their observables with real number precision, there are no present grounds for thinking that such characterization is tracking worldly structure:

[44] While we focus here on gunky characterizations of spacetime, on which every spacetime region has a proper part, versions of *Existence Monism*, according to which the only concrete objecte object is the whole of spacetime, are also compatible with Countabilism. For more on Existence Monism, see Horgan and Potrc (2008), Cornell (2016), and Builes (2021).

> When we ascribe real number precision to physical quantities we are using an exceptionally rich structure. Fundamental physical facts might come structured so richly, but for all we know, they do not. For this reason, it strikes me as well worth considering the possibility that much of the structure that we employ when we ascribe real numbers to quantities is in fact surplus structure. (16)

Third, as a metaphysical matter, the supposition that spacetime is composed of an uncountable infinity of zero-dimensional points leads to several notorious puzzles and paradoxes. Perhaps the most famous measure-theoretic paradox is the Banach-Tarski paradox, whereby any spacetime region with a fixed radius will have five subregions that can be translated and rotated to obtain two spacetime regions, each with the same size as the original—a result clearly incompatible with our usual understanding of physical space. More generally, such spaces have *non-measurable* regions, with no well-defined size; again, as Forrest (2004) plausibly maintains, this cannot be true of any physical space. Relatedly, Segal (2017) presents a puzzle that, conditional on some plausible metaphysical assumptions, suggests that "it's not possible for something that is made of points to have any size at all" (358). A continuous spacetime made of points also gives rise to classic puzzles of contact, whereby physical objects that occupy closed regions (e.g. point particles) can never come into contact, but must rather always be a finite distance apart (see, e.g., Kline and Matheson (1987)). And as Builes and Teitel (2020) argue, pointy physical spaces lead to other puzzles concerning how one might reduce rates of change quantities (e.g. the acceleration of a material body or the gradient of an electric field) to other fundamental physical quantities. These and related puzzles and paradoxes dissolve upon the rejection of uncountable collections, as per Countabilism.

Fourth, as a purely scientific matter, there are reasons to be skeptical of the existence of such a point-based continuum. Arntzenius (2002) has argued, for example, that the formalism of both Quantum Mechanics (QM) and Quantum Field Theory (QFT) suggests that spacetime is pointless. In the case of QM, from the fact that QM uses *separable* Hilbert spaces (spaces with a countable basis) that lack continuous position eigenstates, Arntzenius concludes that it is more natural to interpret wave functions as living on a *pointless* space, of the kind studied by Caratheodory (1963) and Skyrms (1993). Similarly, in the case of QFT, Arntzenius (2002) writes:

> In quantum field theory there are no well-defined field operators associated with points in spacetime. Rather than field operators defined at points, there are 'smeared' field operators associated with weighted regions [...] the procedure whereby such smeared field operators are defined does not presuppose the existence of spacetime points. (1455)

Moreover, certain prominent approaches to Quantum Gravity, such as Loop Quantum Gravity and Causal Set Theory, suggest abandoning continuous spacetime altogether in favor of a discrete approach (see Rovelli (2001) for relevant discussion). Indeed, it is still an open question whether anything analogous to spacetime exists at the fundamental level (see Huggett and Wüthrich (2013) and Baron (forthcoming)).

In sum, we believe that there are already methodological, epistemological, metaphysical, and scientific reasons to be skeptical of the present orthodoxy of a continuum-based physics formulated in terms of an uncountable totality of zero-dimensional points. In its place, the Countabilist has a number of options. First, they could reformulate orthodox theories using a regions-based conception of the continuum as developed by Hellman and Shapiro (2018).[45] Second, the Countabilist (along with everyone else) could point out that our best physical theories cannot be strictly speaking true anyway. After all, our current continuum-based physical theories are inconsistent with each other (since Quantum Field theory is incompatible with General Relativity), so they should only be regarded as empirically adequate within certain regimes. Consequently, even if the Countabilist ran into trouble trying to reformulate these theories in a way that avoided 'uncountable' physical structures, the fact that Countabilism might be in tension with theories that we already know to be false on independent grounds should not be seen as much of a cost. Third, although it is still very unclear what the final 'theory of everything' might look like, the Countabilist can already point to 'discrete' approaches to Quantum Gravity that avoid the kinds of uncountable mathematical structures present in contemporary theories.

An interesting upshot of these physical consequences is that Countabilism can also help guide our future physical theorizing. Insofar as we are led to a final 'theory of everything' that essentially models the structure of the physical world in terms of 'uncountable' mathematical objects, we should be doubtful that such a theory can accurately reflect the structure of reality. We consider the fact that Countabilism has these concrete, empirical consequences for the physical structure of the world as a theoretical virtue of the view.

8 Countabilism: metaphysics

Since Countabilism is a claim about ontology and modality, it unsurprisingly has many substantive and arguably positive consequences in ontology and modal metaphysics. We briefly mention some highlights here.

First, Countabilism alters our understanding of the space of possibility. For example, Countabilism entails that modal space is far smaller than current orthodoxy supposes. Not only is there no possible world with uncountably many spacetime points, there are also no possible worlds with $\aleph_{17}$ or $\aleph_{\omega+17}$ entities either. More generally, Countabilism falsifies certain natural principles that have been thought to delineate the space of possible worlds, including:

> *Recombination*: For each cardinal number κ, there is a larger cardinal λ such that it is possible that there are exactly λ concrete objects.

45 See also Arntzenius and Hawthorne (2005) for a discussion of how a gunky conception of spacetime should be supplemented with an account of the variation of physical quantities across regions.

Given Countabilism, the case of $\kappa = \aleph_0$ is a counterexample to Recombination. That's a good thing, since as Uzquiano (2015) argues, Recombination leads to paradox.

Second, Countabilism has consequences for mereology. Specifically, Countabilism provides the basis for an argument against a version of *Universalism*, according to which "every plurality of objects has a fusion, and, in particular, the plurality consisting of all things has a fusion" (Korman 2011). For if there are countably many concrete entities, then the possible fusions of those concrete entities are indefinitely extensible (just as the possible subsets of N are indefinitely extensible): necessarily, no matter how many fusions there are of those concrete entities, there could always be more. Universalism aims to secure the maximal number of fusions there could be, but according to Countabilism, such a feat is impossible, just as it is impossible to secure the maximal number of sets or ordinals. If one believes that (i) Universalism is true iff it is necessarily true and (ii) there could be countably many concrete objects, then it follows from Countabilism that Universalism is false. It is worth noting, however, that there is a natural substitute to Universalism, namely *Modal Universalism*, which is the thesis that, for any things, it is possible that they form a fusion.[46] The retreat from Universalism to Modal Universalism precisely mirrors the retreat from Naive Comprehension to Modal Naive Comprehension.

Third, the modal approach to set theory that Countabilism is predicated on makes trouble for 'Plenitudinous Platonism' (PP), defended by Balaguer (1995, 1998) and Linsky and Zalta (1995), Linsky and Zalta (2006), which is the thesis that 'all the mathematical objects that logically possibly *could* exist actually *do* exist' (1998: 6).[47] If PP is true, then all possible well-founded pure sets actually exist, which we may formalize as follows:

Set Plenitude: $\exists xx(\forall x(x \prec xx \rightarrow Set(x)) \wedge \neg\Diamond\exists y(Set(y) \wedge \forall x(x \prec xx \rightarrow y \neq x))$.

Informally, *Set Plenitude* says that Plato's heaven cannot contain more well-founded pure sets than it already contains. However, the modal approach to the set-theoretic paradoxes claims that any plurality of things can form a set:

Modal Naive Comprehension: $\Box\forall xx\Diamond\exists y\forall x(x \in y \leftrightarrow x \prec xx)$.

Modal Naive Comprehension implies that Set Plenitude is false. For suppose Modal Naive Comprehension is true, and consider the plurality *ss* of all well-founded pure sets that actually exist (quantifying unrestrictedly). By Modal Naive Comprehension, it is possible for the *ss* to form a set, which would be a well-founded pure set not contained within *ss*. So, Modal Naive Comprehension entails the possibility of a well-founded pure set that doesn't actually exist, contrary to Set Plenitude. In effect, set-theoretic potentialism implies that talk of 'all possible sets' (or more generally,

[46] Thanks to a referee for this suggestion.

[47] The following argument is generalized and defended at greater length in Builes (forthcominga).

talk of ‘all possible mathematical objects’) is illegitimate. The modal space associated with the modal operators involved in set-theoretic potentialism is indefinitelty extensible, or *open-ended*.[48]

Fourth, the supposition that there are uncountable infinities generates seemingly irresolvable questions. Consider the following two questions: (i) How many angels could dance on the point of a needle? (see Hawthorne and Uzquiano (2011)), and (ii) How many isolated island universes is it possible for the world to contain? (see Rayo (2020)). On an orthodox approach to infinity, it’s hard to know what to say. Countabilism, however, provides a straightforward answer: countably many!

Lastly, the modal approach to set theory that we have defended here naturally generalizes to other kinds of metaphysical categories that generate Russellan paradoxes. For example, in the case of propositions, the Russell-Myhill paradox poses a problem for certain fine-grained ‘structured’ theories of propositions, but Yu (2017) has developed a modal approach to propositions that responds to the Russell-Myhill paradox in much the same way that modal approaches to sets respond to RP.

Here we are only scratching the surface; but even so it should be clear that a great deal of orthodox metaphysics looks very different from the perspective of Countabilism.

9 Conclusion

There’s much to be said in favor of Countabilism, understood along modal width-potentialist lines:

- Countabilism entails no loss of mathematical power: for mathematical purposes, Cantor’s paradise remains intact.
- Countabilism is part of a systematic and unified modal response to both RP and CT, properly reflecting their deep structural parallels.
- Countabilism is compatible with a wide range of approaches to modality.
- Countabilism is pre-theoretically intuitive: it affirms the commonsense claim that there is no way to surpass an infinite collection in number.
- Countabilism provides the best explanation for the independence phenomenon in set theory, by explaining this phenomenon while affirming a univocal concept of *set* and predicting just the right amount of independence.
- Countabilism can be motivated and explained from first principles in much the same way that the indefinite extensibility of the ordinals has been motivated and explained.
- Countabilism defuses a wide range of puzzles and paradoxes in both physics and metaphysics.

[48] One way to resist this argument (and the previous argument about mereology) is by rejecting the legitimacy of absolutely unrestricted quantification, which is the intended reading for the quantifiers in both Countabilism and Modal Naive Comprehension. For defenses of the legitimacy of absolutely unrestricted quantification, see Lewis (1991: 68), Sider (2001: xx-xxiv), van Inwagen (2002), and Williamson (2003, 2013). See Clarke-Doane (2019) and Rayo (2020) for independent reasons to think that metaphysical possibility (or ‘absolute’ possibility) might be open-ended.

On the other side, there is little to be said against Countabilism. Perhaps the main potential concern is that Countabilism is in some ways revisionary of orthodox physics, at least with respect to the common view that spacetime is composed of an uncountable totality of zero-dimensional points. But this orthodoxy in physics, like the post-Cantorian orthodoxy in mathematics, is based in a methodologically problematic understanding of the import of CT as motivating the reification of the real numbers as a complete uncountable totality, along with the rest of Cantor's hierarchy; and again, there are epistemic, metaphysical, and scientific reasons to depart from the present orthodoxy in physics, as is increasingly endorsed by both physicists and philosophers.

We conclude that the many substantive advantages, and lack of undercutting disadvantages, jointly tip the scales: Countabilism is not just epistemically possible and mathematically viable, but is most likely true.[49]

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