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Could Experience Disconfirm the Propositions of Arithmetic?

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I regard the whole of arithmetic as a necessary, or at least a natural, consequence of the simplest arithmetical act, that of counting...
Richard Dedekind[1]

Albert Casullo[2] has argued that the propositions of arithmetic could be experientially disconfirmed, with the help of an invented scenario wherein experiences involving our standard counting procedures, as applied to collections of objects, seem to indicate that $2+2\neq4$. Our best response to this scenario would be, Casullo suggests, to accept the results of our standard counting procedures as correct, and give up our standard arithmetical theory.[3] This suggestion, interestingly enough, is not as bizarre as it initially appears. But indeed a problem lies in the assumption, common to Casullo's scenario and to his suggested resolution, that our arithmetical theory might possibly be independent of our standard counting procedures. Here I show that this assumption is incoherent,

1 Richard Dedekind, 'Continuity and Irrational Numbers,' in *Essays on the Theory of Numbers*, trans. Wooster Woodruff Beman (New York: Dover 1963), 4; originally published in 1888.

2 Albert Casullo, 'Necessity, Certainty, and the A Priori,' *Canadian Journal of Philosophy* **18** (1988) 43-66. Future references to this article will be in the text.

3 Here and throughout I take our standard arithmetical theory to be 2nd-order arithmetic, as axiomatized by Dedekind or Peano; details of this theory will be discussed in a later section.

whether the independence at issue is supposed to make room for the genuine possibility that $2+2\neq4$, or the merely epistemic possibility that we could rationally believe that $2+2\neq4$: given our standard counting procedures, then (on pain of irrationality) our arithmetical theory follows. I conclude that the propositions of arithmetic would not be disconfirmed (or refuted) in Casullo's scenario, or in any scenario that depends similarly upon the world seeming to go (or going) wrong.[4]

I Casullo's Disconfirming Scenario

Casullo's larger agenda in providing a 'disconfirming' scenario (I'll leave off the scare quotes in what follows) is to investigate whether knowledge of mathematical propositions might plausibly be fit into an inductive empiricist framework.[5] In an inductive empiricist framework, 'at least some mathematical propositions can be individually confirmed or disconfirmed and, furthermore, those mathematical propositions which are epistemically basic are confirmed on the basis of experience and inductive generalization' (43). Epistemically basic mathematical propositions are those involving small numbers (small enough, at any rate, to be able to confirm experientially, in the way to be described shortly). Such propositions act as the 'confirmation base' for the mathematical theory in question. Given this base, the inductivist can allow that other mathematical propositions, and indeed the extension of the confirmation base to a full-fledged axiomatic theory, may be justified non-inductively (via, for example, proof-theoretic methods, or abductive or holistic considera-

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4 This conclusion goes only part of the way towards establishing that experience could not disconfirm the propositions of arithmetic. In Casullo's scenario, the participants are justified in believing first, that they have wits enough about them to count, and second, that the objects being counted are stable throughout the counting process. Scenarios in which both assumptions are rejected depart too far from the circumstances of our own experience to provide any illuminating grip on the question of disconfirmation. And as I'll show later, scenarios in which (just) the second assumption is rejected are susceptible to the arguments presented here against Casullo's scenario. But scenarios in which (just) our mathematical wits are called into question, such as those considered by Philip Kitcher in *The Nature of Mathematical Knowledge* (New York: Oxford University Press 1985), are not susceptible to these arguments and may, for all I say here, represent live possibilities.

5 The arguments and results of this paper transcend this particular agenda, however, applying to any account of arithmetical propositions on which these could be disconfirmed (or refuted) by experience, under the general conditions of Casullo's scenario.

tions). But it is the inductive confirmation of the epistemically basic propositions in the confirmation base that gets the theory off the ground: 'the inductivist simply claims that in mathematics, as in science, justification originates in inductive generalization from experience' (45).

One oft-cited argument against the thesis that mathematical propositions, like scientific propositions, are confirmed via inductive generalization from instances, relies on the premise that unlike the propositions of science, mathematical propositions are not *dis*confirmable by experience.[6] The argument (which Casullo calls the Irrefutability Argument) proceeds as follows:

(P1) No experiential evidence can disconfirm mathematical propositions.

(P2) If experiential evidence cannot disconfirm mathematical propositions, then it cannot confirm such propositions.

∴ Experiential evidence cannot confirm mathematical propositions.

Casullo's disconfirming scenario is intended to block the anti-inductivist conclusion by undermining (P1). In this, he follows John Stuart Mill, who provided a disconfirming scenario with a similar intention in *A System of Logic*.[7] Casullo's scenario is, however, an extension of Mill's, and (as we'll soon see) one that evades the reasons usually given for rejecting Mill's scenario as showing that experiential evidence can disconfirm mathematical propositions.

To set the stage for his scenario Casullo describes the process by which one of the epistemically basic propositions, that are to serve as the

6 Prominent among those who have rejected an inductivist approach to mathematics on these grounds are empiricists such as A.J. Ayer, *Language, Truth, and Logic* (New York: Dover 1952), Ch. 4, excerpted and reprinted as 'The a priori' in Paul Benacerraf and Hilary Putnam, eds., *Philosophy of Mathematics* (Englewood Cliffs, NJ: Prentice-Hall 1964) and Carl Hempel, 'On the Nature of Mathematical Truth,' *The American Mathematical Monthly* **52** (1945) 543-56, also reprinted in *Philosophy of Mathematics* who go on to argue that mathematical propositions are analytic, and so known a priori. See also Karl Britton, in 'The Nature of Arithmetic: A Reconsideration of Mill's Views,' *Meeting of the Aristotelian Society* **6** (1947) and Hartry Field, *Science Without Numbers* (Princeton: Princeton University Press 1980), Ch. 1.

7 John Stuart Mill, *A System of Logic* (New York: Harper 1867). Mill's disconfirming scenario is discussed in chapter vi of Book II, and chapter xxiv of Book III.

confirmation basis for a theory of arithmetic, would be established empirically:

> One counts the number of objects in each of two distinct groups of objects, combines the objects into a single group, recounts the objects, and then notes the resulting numerical relations between the component groups and the combined group of objects. (44)

During this process, it is assumed that participants in the scenario are justified in believing that certain background conditions are met. These are the Stability and Correct Counting conditions:

> *Stability condition*: Neither the operations of counting and combining, nor the interactions of the objects to be counted, produce any changes in the relevant features of the objects being counted.
>
> *Correct Counting condition*: No miscounting (say, by repeating a number or missing an object) takes place.

It is primarily in making explicit that these conditions are justifiably believed to hold that Casullo's scenario extends Mill's.[8]

The scenario then proceeds as follows: the participants perform two countings to two, combine the objects, and recount — and the result is a counting to five. They repeat the process over and over, each time with the same result. Each time, they take pains to establish that the Stability and Correct Counting conditions are met, and in each case find that they are. Furthermore, they discredit all hypotheses forwarded in attempts to show that one or both of the conditions are not being met. Under these circumstances, 'the inductivist maintains that... the proposition [that 2+2=4] would be disconfirmed by experiential evidence and, hence, that premise [P1] is false' (49). This leaves the way clear for arithmetic to be an inductive experimental science, where it is 'an open empirical question whether whenever one performs two countings to two, combines the objects, and recounts, the result will be a counting to four' (46).[9]

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8 Britton ('The Nature of Arithmetic,' 2-6) provides convincing textual evidence that Mill 'half-acknowledges' that these 'two general conditions' are assumed to be in place in his scenario, but it takes some doing.

9 Casullo does not argue explicitly for the inverse of (P2) — that if experiential evidence can disconfirm mathematical propositions, then it can confirm such propositions. Presumably his remarks here are intended to leave the inverse of (P2) open as a live possibility (in particular, for the inductive empiricist). Establishing this inverse would be a different project, and one that would have to respond to a priorist accounts that try to show that mathematical propositions are known a priori, in spite

Casullo sees participants in the disconfirming scenario as having two possible responses:

(A) Keep standard arithmetical theory along with standard counting procedures, but maintain that either the Stability or Correct Counting conditions failed to hold, in spite of all evidence to the contrary.[10]

(B) Accept the results of the standard counting procedures as correct and reject standard arithmetical theory.[11]

In the past, (A) has been the usual response to Millian scenarios in which a series of countings seems to disconfirm an elementary arithmetical proposition. Ayer, for example, says

> It might easily happen, for example, that when I came to count what I had taken to be five pairs of objects, I found that they amounted only to nine.... But ... one would not say that the mathematical proposition "2x5=10" had been confuted. One would say that I was wrong in supposing that there were five pairs of objects to start with, or that one of the objects had coalesced, or that I had counted wrongly.... The one explanation which would in no circumstances be adopted is that ten is not always the product of two and five.[12]

Hempel says of a scenario in which a collection of 2 microbes and a collection of 3 microbes, when counted together, give the result 6:

> Would we consider this as an empirical disconfirmation of the ... proposition [that 2+3=5]? Clearly not; rather, we would assume we had made a mistake in counting

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of being potentially disconfirmable by experience. Cf. Donna Summerfield, 'Modest A Priori Knowledge,' *Philosophy and Phenomenological Research* **51** (1991) 39-66.

10 Strictly speaking, Casullo presents a version of (A) where one maintains only that the Correct Counting condition failed to hold, in spite of all evidence to the contrary. However, there seems to be no reason why one couldn't instead maintain that it was the Stability condition that had failed to hold.

11 Here Casullo is assuming an account of individuation of arithmetical theories according to which disconfirmation of even a single proposition of the theory disconfirms the theory as a whole. This is certainly true on an inductive empiricist account, given the foundational nature of the epistemically basic propositions at issue. But more generally, disconfirmation of the sort of elementary arithmetical propositions at issue here would likely render a sufficiently large tear in the fabric of standard arithmetical theory so as to render the theory disconfirmed as a whole, however one took that fabric to be woven.

12 Ayer, 'The a priori,' 318

> or that one of the microbes had split in two between the first and second count. But under no circumstances could the phenomenon just described invalidate the arithmetical proposition in question....[13]

And Douglas Gasking remarks (upping the ante by emphasizing the pervasive nature of the experiences at issue):

> If I counted out 7 matches, and then 5 more, and then on counting the whole lot, got 11, this would not have the slightest tendency to make anyone withdraw the proposition that 7+5=12 and say it was untrue. And even if this constantly happened, both to me and to everyone else, and not only with matches, but with books, umbrellas, and every sort of object — surely even this would not make us withdraw the proposition.... There are plenty of alternative explanations to choose from.... We might try a psychological hypothesis.... Or we might prefer a physical hypothesis.... The one thing we should *never* say, whatever happened, would be that the proposition that 7+5=12 had been experimentally disproved.[14]

These responses, without further argument, fail to appropriately address the possibility raised by Casullo's scenario. To see this, we need to get clear regarding what this possibility is supposed to be. It is part of Casullo's scenario that the participants are *justified in believing* that the Stability and Correct Counting conditions are being met. This leaves it open whether, in the scenario, the conditions are, in fact, being met, or whether they are not, in fact, being met, but the participants just aren't able to figure this out. These two cases correspond to two types of possibility that might be at issue in Casullo's scenario. Suppose first that the conditions are, in fact, being met. Since these conditions apparently exhaust the relevant ways in which things could go wrong, the possibility at issue in this case is what I'll call the 'genuine' possibility that the participant's true justified beliefs in the conditions holding, coupled with their experience, could serve to (not only disconfirm, but) *refute* the proposition that 2+2=4. (This strikes many as immediately incoherent; but see the next section.) Now suppose that one or the other of the conditions isn't, in fact, being met. The possibility at issue in this case is what I'll call the 'epistemic' possibility that (although in fact one or the other of the conditions *doesn't* hold) the participant's justified beliefs in both conditions holding, coupled with their experience, could serve to *disconfirm* the proposition that 2+2=4.

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13 Hempel, 'On the Nature of Mathematical Truth,' 378-9

14 Douglas Gasking, 'Mathematics and the World,' reprinted in *Logic and Language*, Antony Flew, ed. (New York: Anchor 1965), 430-1

Of course, if the participants were irrational or otherwise sufficiently cognitively impaired, either possibility would be (too) easy to defend; which is just to say that questions of what it is rational to believe are only interesting relative to a tacit background assumption that certain cognitive capacities — here, those humans generally possess — are in place. (Such a tacit assumption obviously comes into play, not only in assessing what participants could rationally believe about elementary arithmetical propositions, but also in giving content to the claim that the participants justifiably believe the Stability and Correct Counting conditions to hold.) To be sure, deciding just what cognitive capacities humans generally possess is no small matter. For present purposes, however, it will be enough to assume that the participants are as rational and inventive as *we* would need to be in order for us to say (of some sufficiently but not especially sophisticated group of *us*) that we were indeed justified in believing the Stability and Correct Counting conditions to hold.

Given this tacit background assumption, there is a sense in which each of the aforementioned possibilities is epistemic (since the scenario presupposes that the participants have certain cognitive capacities, which mediate their experience of the scenario) and there is a sense in which each of the above possibilities is genuine (since there will presumably be a fact of the matter whether it is compatible with these capacities that, under the assumptions of the scenario, the participants rationally believe that $2+2\neq4$).[15] Even so, for convenience I'll continue to refer to the first sort of possibility (where the conditions hold in fact, and '$2+2=4$' is *refuted*) as 'genuine,' and the second sort (where one or the other of the conditions doesn't hold in fact, and '$2+2=4$' is *disconfirmed*) as 'epistemic.' Now, although Casullo does not explicitly distinguish between these possibilities, his restricting the participants to having justified belief in (rather than knowledge of) the conditions' holding strongly suggests that it is the epistemic possibility that he takes to be at issue (and in any case, his scenario presumably stands a much better chance of being coherent on this reading). My arguments against the coherence of his scenario will be directed accordingly. Along the way I'll show, however, how these arguments may be adapted for use against a scenario in which the conditions hold in fact, and that nonetheless the participants find (because it is the case) that $2+2\neq4$, should anyone care to put such a scenario forth.

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15 Note that it would be enough for Casullo to establish that participants in the scenario *could*, consistent with the assumptions of the scenario, take their experience as disconfirming the proposition that $2+2=4$; to block (P1) of the Irrefutability argument, he need not argue that they *should*, or even that they *would*, do this.

It is easy to see that the above variations on response (A) fail to appropriately address the epistemic possibility at issue in Casullo's scenario. For in each response it is asserted that any seeming disconfirmation would be resisted by appeal to some 'alternative explanation' in terms of psychological or physical goings-on. But Casullo has built into his scenario that all such 'alternative explanations' of the apparent disconfirmation have been investigated, and ruled out. And there seems nothing incoherent about *this part* of his scenario.[16]

The question remains whether the best response to Casullo's scenario would be to take the given arithmetical proposition (and, by extension, standard arithmetical theory) to have been, as Gasking puts it, 'experimentally disproved' (as per [B]). Casullo says, of claims that the explanation of the apparent disconfirmation must be a matter of miscounting (as per [A]), that 'If this claim is not to be true by stipulation and, hence, question-begging, then the a priorist must provide some plausible hypothesis regarding the source of the mistaken counting' (51). This is incorrect, for it might be that the a priorist could bring some non-question-begging argument to bear which would support choosing (A) over (B), that none the less didn't involve any such hypothesis. (That will, in fact, be the approach in this paper.) Arguments previously given along these lines, however, have been either absent or so weak as to not definitively establish that (A) would be the best response to Casullo's scenario.

Ayer, for example, rejects (B) on grounds that 'we cannot abandon [the propositions of mathematics] without contradicting ourselves, without sinning against the rules which govern the use of language, and so making our utterances self-stultifying,'[17] but even he seems unconvinced by this vague, largely pragmatic justification: 'In rejecting [an inductive empiricist] theory, we are obliged to be somewhat dogmatic' (ibid., 318). Hempel justifies his choice of response (A) by appeal to the standard arithmetical relations being part of the definition of the number terms that result from counting: 'under no circumstances could [counting results] invalidate the arithmetical proposition [that 3+2=5]; for the latter ... merely states that any set consisting of 3+2 objects may also be said to consist of 5 objects. And this is so because the symbols "3+2" and "5" denote the same number: they are synonymous by virtue of the fact that the symbols "2," "3," "5," and "+" are *defined* (or tacitly understood) in such a way that the above identity holds as a consequence of the meaning

16 The above responses would also, for the same reason, fail to appropriately address the genuine possibility that might be at issue in Casullo's scenario.

17 Ayer, 'The *a priori*,' 319

attached to the concepts involved in it.'[18] But Hempel's appeal to definition is indecisive against response (B), since the inductivist challenge can be reframed as the question whether, as a result of untoward experience, we might be compelled to revise those portions of our definitions of number symbols having to do with arithmetical relations.[19]

Gasking offers a Wittgensteinian justification for choosing (A) over (B), claiming that it is a fact about our *use* of mathematical (and in particular, arithmetical) propositions that these are incorrigible (that is, not disconfirmable). But there are two problems with this approach. First, Casullo can deny that Gasking's 'language game' response shows that participants would be justified in choosing (A) over (B), as follows.[20] Consider the following two propositions: (1) it is a fact about our use of mathematical propositions that they are incorrigible; (2) it is a fact about our use of mathematical propositions that it is only under the direst epistemic circumstances that we are even to consider relinquishing the simplest such propositions. Presumably, Gasking would agree that we can't know a priori which of these correctly characterizes our practices. But then even if (1), not (2), is correct, then (not being in such dire epistemic circumstances ourselves), we have no reason for believing that

18 Hempel, 'On the Nature of Mathematical Truth,' 379

19 Admittedly, the inductive empiricist could do more to make intelligible how the definitions of number terms might be amenable to experience. Kitcher, in 'Arithmetic for the Millian,' *Philosophical Studies* **37** (1980), 219, attempts this on Mill's behalf: 'I suggest that we read [Mill] as offering an epistemological thesis about definitions: to be justified in accepting the definitions on which arithmetic rests we must have empirical evidence that those definitions are applicable.... Mill would allow that certain sentences of our language are true in virtue of the connotations of the expressions they contain, and that we can defend our assertion of these sentences by citing our understanding of the language. However ... our defense is adequate only so long as our right to use our language is not called into question. In particular, if experience gives us evidence that certain concepts are not well-adapted to the description of reality our assertion of sentences involving those concepts is no longer justified....' Kitcher supports his reading of Mill, in part, by reference to Mill's discussion of the term 'acid.' It was part of the original definition of this term that an acid had the property of containing oxygen, so that, at one point in time, the assertion of a sentence like 'All acids contain oxygen' could have been defended simply on grounds of understanding the terms involved. After the discovery of hydrochloric acid, consisting only of hydrogen and chlorine, both definition and defense were undermined. On this understanding, disconfirming scenarios can be seen as attempts to show that arithmetical relations stand to the definition of number terms as the property of containing oxygen stood to the original definition of 'acid.'

20 Thanks to Mark Richard for this suggestion.

(1), rather than (2), obtains, and so no reason for thinking that participants in the scenario would be justified in preferring (A) over (B). Second, on Gasking's account the incorrigibility in question would just as well attach to propositions in an alternative mathematical system in which, for example, 3x4=24: 'This latter proposition, if it were part of our mathematical system, would be incorrigible, exactly as "3x4=12" is to us now.'[21] Thus even if Gasking's 'language game' approach does provide some reason for choosing (A) over (B), it is unlikely to be popular. For it carries with it all the disadvantages that attach to accepting Casullo's disconfirming scenario (namely, admitting that we might have believed arithmetical propositions different from those in our standard arithmetical theory) without providing the justificatory basis for accepting non-standard propositions that Casullo's scenario provides (namely, that in the process of disconfirming the standard arithmetical propositions, certain non-standard arithmetical propositions are inductively *confirmed*).

It appears that the usual arguments for choosing (A) over (B) in response to disconfirming scenarios are inconclusive, at best. Moreover, Casullo takes accepting (A) to involve either accepting inexplicable failures in our counting procedures to yield results compatible with our arithmetical theory, or (as per Ayer, Hempel, and Gasking) introducing ad hoc explanations of why the Stability or Correct Counting conditions failed. (Since, in Casullo's scenario, all hypotheses that might have undermined the participant's justified beliefs in the conditions' holding have been investigated, and ruled out, any further explanation of why one or the other condition failed seems bound to be ad hoc.) Given the inconclusive nature of previous arguments in favor of (A), Casullo's suggestion that participants in his scenario should choose (B) over (A) is not unreasonable. If choosing between (A) and (B) ultimately came down to choosing between being ad hoc and being revisionary, we might be inclined to take the latter course.

II Two Aspects of Number

Although, by the above lights, the suggestion that we might dispense with standard arithmetical theory in the face of experience is not unreasonable, there is a strong *prima facie* intuition that disconfirming scenarios such as Mill's and Casullo's are incoherent (which accounts for the tendency for response [A], however dogmatic or problematic, to be preferred over response [B]). Casullo's scenario can be made more

21 Gasking, 'Mathematics and the World,' 442

plausible by noting that in this scenario (as well as in response [B]) Casullo is tacitly separating two aspects of number. The first aspect of number has to do with a number's being the cardinality of a set, as illustrated by the sentence 'There are three objects on that table.' In the scenario this aspect of number is left intact, insofar as the participants are assumed to be in possession of standard counting procedures, that enable them to perform 'countings to two,' 'countings to five,' and so on.[22] The second aspect of number concerns the relations of numbers to each other, as in 'Three plus two is five.' It is these relations (corresponding to addition and multiplication) that the propositions of arithmetic are partly about, and it is propositions predicating such relations that Casullo thinks are 'open empirical questions' insofar as they could be disconfirmed, in spite of standard counting procedures (determinations of cardinality) seeming to operate as usual. Casullo's disconfirming scenario, then, can be understood as asserting that the relational, 'arithmetical,' aspect of number is epistemically independent of the intrinsic, 'cardinal,' aspect. This assertion amounts to the thesis that the following two assumptions could be rationally and concurrently held (for example, by participants in the scenario):

(A1) We are in possession of standard counting procedures that result in correct determinations of cardinality, under conditions where (as we justifiably believe) the Stability and Correct Counting conditions hold (i.e., the cardinality aspect of number is justifiably believed to remain the same).

(A2) Relations (corresponding to addition and multiplication) holding between the results given by standard counting procedures are different from those given by standard arithmetical theory (i.e., the arithmetical aspect of number is justifiably believed to vary).

Casullo's scenario drives a wedge between two aspects of number that we don't ordinarily consider separately. This explains the *prima facie*

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22 Of course (as per the epistemic possibility) if the participants are wrong about either the Correct Counting or Stability conditions holding, it might be the case that a given 'counting to two' failed to accurately reflect the number of objects being counted (although such inaccuracy would have to be persistent, systematic, and undetectable by participants in the scenario, who are justified in believing the conditions to hold). As it turns out, the accuracy of the counting results is irrelevant to the question of whether Casullo's scenario is coherent. For the moment, it is enough to note that Casullo's scenario is not designed to call counting results into question.

implausibility of the scenario, as well as the resistance to allowing that participants might appropriately respond to the scenario (as per [B]) by accepting the results of standard counting procedures as correct and rejecting standard arithmetical theory. More importantly, given that there *are* two aspects of number, it is not obvious that participants in Casullo's scenario couldn't rationally find these to be independent of each other, as required.

III Keeping our Counting Procedures and Rejecting our Arithmetic

Determining whether Casullo's scenario represents a coherent possibility thus comes down to the question of whether the arithmetical aspect of number could possibly be epistemically independent of the cardinal aspect. (I'll sometimes call this 'the independence question.') We can start by investigating the presuppositions of (A1); that is, by investigating what is involved in possessing standard counting procedures, under conditions where the Stability and Correct Counting conditions are justifiably believed to hold. H.N. Casteneda has provided the following formulation of the principles involved in counting:

(C1) To count the objects of a collection or aggregate K is to establish a one-to-one correspondence between the objects of K and a set N of numbers (or numerals) such that:

(C2) N includes 1;

(C3) There is at most one number in N whose immediate successor is not in N;

(C4) The number of objects in K is the number mentioned in (C3) if it exists; otherwise the number of objects in K is infinite;

(C5) The one-to-one correspondence may be carried out (i) by actually attaching one number (or numeral) to each object; or (ii) by forming partial non-overlapping correspondences of type (i), as when we count by twos or fives or hundreds; or (iii) by specifying a rule for actually attaching numerals to as many objects as we please.[23]

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23 Hector Neri Casteneda, 'Arithmetic and Reality,' *The Australasian Journal of Philosophy* **37**, 2 (1959), 103. It is uncontroversial that something like these principles is

Since the participants in the scenario are in possession of standard counting procedures, Casteneda's formulation gives us some indication of the relevant concepts we can take the participants to possess. To start, we can note that, according to this formulation, the participants will possess the concept of a one-to-one correspondence: as per (C1), to count just *is* to establish such a correspondence. Although this notion can be defined in mathematical terms,[24] there is nothing especially technical about the notion of a one-to-one correspondence. The idea is simply to match up all the objects in one collection with all the objects in another collection — something any kid who knows how to set a table can do. As a separate issue, one might think that this formulation of counting already spells deep trouble for Casullo's scenario. For Casteneda's use of the terms 'number' and 'successor' would seem to presuppose the axioms (P1)-(P5) of 2^{nd}-order Peano Arithmetic (PA^2)[25] (henceforth, the

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involved in standard counting procedures. To see how Casteneda's principles could be applied to a concrete example (involving say, counting a collection of apples on a table), do the following:

(1) Take the set N of numbers in (C1) to be drawn from the ordered sequence of natural numbers (1, 2, 3, ...).

(2) Establish a 1-1 correspondence between the apples and the set N by pointing once and only once (mentally or physically) to each apple on the table, each time attaching (mentally or physically) the next number in the sequence of natural numbers (starting with 1). Note that here we would be carrying out (C5), version (i).

(3) When done, investigate N, and apply (C4). If (C2) has been satisfied (this will happen when there is at least one apple on the table), then either there will be a last number in N (this is just what it means to say that (C3) is satisfied), and this last number N will be the number of apples on the table, or the number of apples on the table is infinite. If (C2) was not satisfied, then there were no apples on the table.

24 In mathematical terms, a one-to-one correspondence (a.k.a. a 'bijection') is a relation R:S->T that is 1-1 (distinct elements of S are R-related to distinct elements of T) and onto (for every element in T, there is some element in S that is R-related to it).

25 The axioms of 2^{nd}-order Peano Arithmetic (PA^2) (in whole numbers, for purposes of counting) are:

(PA1) 1 is a number.

(PA2) Every number n has an immediate successor, s(n), which is also a number.

(PA3) If two successors s(m) and s(n) are the same number, then m and n are the same number.

(PA4) 1 is not the successor of any number.

(PA5) Every property P is possessed by all numbers if 1 has P and if, when n has P, s(n) has P. (mathematical induction)

'Peano axioms') and as it is usually presented (for example, by Peano) PA^2 contains both addition and multiplication. If the ability to count presupposes even a tacit acceptance of the Peano axioms, and acceptance of these axioms entails acceptance of addition and multiplication, we might conclude that (justified belief in) the arithmetical aspect of number is not independent of (justified belief in) the cardinal aspect, in the way Casullo's scenario requires.

This conclusion would be too hastily drawn, however. To begin with, note that the train of thought leading to the above conclusion depends upon a thesis to the effect that attention to features of a formal theory (such as whether and how certain operations are therein defined) can provide some indication of the level of cognitive competence associated with understanding various components of the theory.[26] While the thesis has some intuitive appeal (after all, it wouldn't be surprising if our theories mirrored our understanding), the thesis would need further motivation before it could support accepting the above conclusion. More importantly, even if such motivation were given, an approach based on this thesis *doesn't* eventuate in a determinate answer to the independence question. It's worth noting the reasons why this is so, both in order to see the overly hasty response for what it is, and in order to point us in the direction of an approach that does eventuate in a determinate answer.[27]

Suppose, then, that if the definitions for addition and multiplication were seen to follow, in some appropriate sense of 'follow,' from the

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Peano's original formulation invoked 9 axioms, but 4 of these dealt with identity; these latter, being considered part of the assumed underlying logic of the theory, have been dropped from subsequent formulations. See Giuseppe Peano, 'The Principles of Arithmetic, Presented by a new Method' (1889) reprinted in Jean van Heijenoort, ed., *From Frege to Godel: A Source Book in Mathematical Logic, 1879-1931* (Cambridge: Harvard University Press 1967) 83-97.

Peano's 1889 axioms are directly based on Dedekind's 1888 definition of a *simply infinite system* (article 71) and theorem of complete (that is, mathematical) induction (article 80) in 'The Nature and Meaning of Numbers,' *Essays in the Theory of Numbers* 44-115). While Dedekind should be credited as such, I will follow current (inertial) usage in discussing the Peano (rather than the Dedekind, or Dedekind-Peano) axioms.

26 Thanks to Harold Hodes for making clear that this thesis underlies the present discussion, and for his generous assistance in investigating the question of definability of arithmetical operations in PA^2 in what follows.

27 Those who are uninterested in how arithmetical operations may be defined in formalizations of PA^2 can skip ahead to the paragraph starting 'We might wonder, however, if another line of inquiry...' without undue loss of continuity.

Peano axioms, then this would be (at least prima facie) evidence for taking the arithmetical aspect of number *not* to be epistemically independent of the cardinal aspect. A natural place to look for such a sense would be by way of considering how addition and multiplication are defined within formal languages for PA^2. Within a formal language, a definition of a given operation may be either *explicit* or *implicit*. In an explicit definition, the symbol representing the operation being defined on the left-hand side of the definition does not appear on the right-hand side. Now, one might think that the relevant contrast to explicit definitions would be inductive or recursive definitions, where the symbol representing the operation being defined *does* appear on the right-hand side (in the definition's inductive clause or clauses). We need to be careful, however, since in certain languages (with the right sort of 2^{nd}-order or higher-order variables, or with 1^{st}-order variables whose range includes the right sets) some inductive definitions can (for reasons to be discussed shortly) be presented as explicit definitions. Thus the relevant contrast is between definitions that can be presented explicitly, and definitions (including some inductive definitions of operations) that can't be so presented. That said, I'll use the terms 'explicit' and 'implicit' to mark out this contrast.[28] Now, if addition and multiplication were explicitly definable in PA^2, this would plausibly provide prima facie indication that the arithmetical aspect is *not* independent of the cardinal aspect of number. On the other hand, if addition and multiplication were only implicitly definable, this would plausibly provide a contrary prima facie indication.

Unfortunately, the literature contains some formalizations of PA^2 in which addition and multiplication are explicitly definable, and others in which they are not. Consider, for example, Peano and Dedekind's formalizations.[29] Both Peano and Dedekind define addition and multiplication inductively. But in Peano's formalization, his formal language doesn't have the right sort of variables — his variables range only over

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28 This usage of 'implicit definition' applies to interpreted languages, e.g., a language about the natural numbers as interpreted in a standard model. In this way it is weaker than current standard usage in model theory, which requires that *every* model for the definiens (the sentences doing the implicit defining) uniquely determines the interpretation of the definiendum (the constant[s] being implicitly defined). See C.C. Chang and H. Jerome Keisler, *Model Theory*, 3rd ed. (The Netherlands: North-Holland 1990), 90.

29 In these formalizations, the first-order variables range only over natural numbers, and talk of first-order and second-order variables should be understood accordingly.

natural numbers and 1-place properties of the natural numbers; hence his inductive definitions cannot be transformed into explicit definitions, and so are effectively additional axioms for his theory, which introduce two new primitive constants.[30] In Dedekind's formalization, however, his formal language has variables that permit the transformation of his inductive definitions into explicit definitions; hence his definitions of addition and multiplication do not introduce any new primitive constants (or axioms defining these constants).

As indicated, this distinction between the definitions depends on differences between the languages underlying the formalizations of PA^2 at issue. It is a fairly familiar fact that 1^{st}-order PA (PA^1), where the quantifiers $\forall$ and $\exists$ range only over natural numbers, fails to pick out the intended model of the natural numbers, which fact motivates the move to PA^2, where the underlying language contains 2^{nd}-order quantifiers ranging over properties (or functions) of the natural numbers.[31] But it is less often observed that having made this move, there is a choice as regards *which* properties or functions the 2^{nd}-order quantifiers may range

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30 Peano inductively defines addition by introducing the constant '+' and defining this constant as follows:

Take the base clause to be: $n + 1 = s(n)$

Take the induction clause to be: $n + s(m) = s(n+m)$

As an example, consider 2 + 2, i.e., $s(1) + s(1)$. Apply the induction clause once to get

$$s(1) + s(1) = s(s(1) + 1)$$

Apply the base clause to get

$$s(s(1) + 1) = s(s(s(1))) = 4.$$

Peano's definition of addition does not explicitly call attention to the fact that he is introducing an (inductively defined) primitive symbol, and so effectively introducing two new axioms (corresponding to the base and induction clauses above), apparently due to his expressing the successor function in the original axioms as 'x+1.' (Two additional axioms are also needed to inductively define multiplication in Peano's system.) It is for this reason that later expositions of PA do not express the successor function using the '+' symbol. As van Heijenoort ('From Frege to Godel,' 83-4) remarks in his introduction to Peano's article, 'From the outset, Peano uses the notation $x + 1$ for the successor function. He then introduces addition (section 1) and multiplication (section 4) as "definitions".... Peano does not explicitly claim that these definitions are eliminable, but, just as he does for ordinary definitions ... he puts them under the heading "Definition," although they do not satisfy his own statement on that score' (93), namely, that the right side of a definitional equation is 'an aggregate of signs having a known meaning.'

31 This move allows the 1^{st}-order axiom schema for mathematical induction (For P a property: $(P1 \,\&\, (Pn \rightarrow Pn{+}1)) \rightarrow \forall x Px$) to be replaced by the 2^{nd}-order closure of the schema $(\forall P\, ((P1 \,\&\, (Pn \rightarrow Pn{+}1)) \rightarrow \forall x Px))$, and it is this replacement which rules out the unintended models.

over — a choice that bears on whether addition and multiplication can be explicitly defined. These quantifiers must at least range over the monadic (1-place) properties of natural numbers appearing in the mathematical induction axiom (P5). Now, if the underlying language allows *only* quantification over monadic properties, addition and multiplication can only be implicitly defined.[32] But if the underlying language allows the 2^{nd}-order quantifiers to range over dyadic properties or monadic functions of natural numbers, inductive definitions of these operations can be converted into explicit definitions.[33] Now, in Peano's system, the 2^{nd}- order quantifiers range only over monadic properties, while in Dedekind's system, the 2^{nd}-order quantifiers may range over monadic functions. Hence Peano's inductive definitions of addition and multiplication are implicit, while Dedekind's are (presentable as) explicit.[34]

Now, if there were a single standard for choosing between these formalizations (or if different standards agreed), perhaps attention to issues of definability would help us determine whether the arithmetical aspect was epistemically independent of the cardinal aspect of number: if Peano's was preferable, then (prima facie) yes; if Dedekind's, then (prima facie) no. It seems, though, that the differences embedded in these formalizations reflect two attractive conceptions of theoretical elegance. According to the first conception, an elegant theory is as *weak* as it is able to be, and still get the job done (we don't need Goliath to pick up a brick). In the case of arithmetic, the job to be done is the formal representation of the sequence of natural numbers, along with the operations of addition and multiplication on this sequence. On the second conception, an

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32 This follows from Buchi's theorem (Dirk Siefkes, *Buchi's Monadic Second-Order Successor Arithmetic* [Berlin: Springer-Verlag 1970]) that the monadic second-order theory of zero and successor is a decidable theory; if addition and multiplication could be explicitly defined in this theory, it would include PA^1, which then would be decidable. But PA^1 isn't decidable (by Church's theorem), so (by modus tollens) addition and multiplication cannot be explicitly defined in this theory.

33 If the 2^{nd}-order quantifiers range over dyadic properties, we can explicitly define addition as follows: $x+y=z \leftrightarrow \exists R^2 (R^2 1s(x) \,\&\, \forall u \forall v \forall v'(R^2uv \,\&\, R^2uv' \rightarrow v=v') \,\&\, \forall u,v(R^2uv \leftrightarrow R^2 s(u)s(v) \,\&\, R^2yz))$. If the 2^{nd}-order quantifiers range over monadic functions, we can explicitly define addition as follows: $x+y=z \leftrightarrow \exists f\,(f(1)=s(x) \,\&\, \forall u f(s(u))=s(f(u)) \,\&\, f(y)=z)$. We can also provide explicit definitions in a way that utilizes the universal, rather than the existential, prefix.

34 In fact, Dedekind didn't transform his inductive definitions into explicit definitions, although the technique was semi-available at the time of his writing 'The Nature and Meaning of Numbers.' Frege used something like this technique in defining the ancestral of a sequence in his 1879 *Begriffsschrift* (reprinted in *From Frege to Godel*, 1-82), at the end of section 26.

elegant theory is one which is able to get the job done with the least amount of primitive ideology — where the amount of primitive ideology is reflected in the number of primitive symbols and / or axioms. Here we might be willing to accept a certain amount of theoretical overkill, in order to arrive at a *conceptually simpler* framework. Now, by the standard of the first conception, Peano's formalization is more elegant, for it does the job — in particular, expresses a categorical theory of arithmetic — without the additional functional machinery contained in Dedekind's formalization. But by the standard of the second conception, Dedekind's formalization is more elegant, since Peano's formalization contains two more primitive constants (and four more axioms) than Dedekind's. Absent any means of choosing between these two conceptions of theoretical elegance, this line of inquiry fails to move us any closer to an answer to the independence question.

We might wonder, however, if another line of inquiry, in the vicinity of attention to formalizations of PA^2, could bring us closer to an answer. After all, whichever sort of formalization we favor, it remains to be explained *why* addition and multiplication are, however defined, *universally* taken to be part of PA^2. Why is it part of the 'job' to be accomplished by any arithmetical theory that it incorporate these operations? Attention to this prior question, rather than to the question of how addition and multiplication are definable in a particular formalization of PA^2, might be more likely to result in insight into the nature of the relation between the cardinal and arithmetical aspects of number.

Consider Dedekind's discussion of his reasons for regarding 'the whole of arithmetic as a necessary, or at least natural, consequence of the simplest arithmetic act, that of counting':

> I regard ... counting itself as nothing else than the successive creation of the infinite series of positive integers in which each individual is defined by the one immediately preceding; the simplest act is the passing from an already-formed individual to the consecutive new one to be formed.... Addition is the combination of any arbitrary repetitions of the above-mentioned simplest act into a single act; from it in a similar way arises multiplication.[35]

It is not hard to see that the successor function formally represents Dedekind's 'simplest act.' Each positive integer corresponds to a series of 'repetitions' of this act — that is, an act of counting. Addition and multiplication in turn correspond to ('arise from') combining arbitrary such repetitions into single acts. Since the 'single act' resulting from any

35 Dedekind, 'Continuity and Irrational Numbers,' 4

such combination corresponds to repetitions of acts of counting, Dedekind seems to be suggesting that addition and multiplication are in some sense *reducible* to acts of counting (and ultimately to repeated applications of the successor function). Hempel motivates the introduction of the operations of addition similarly. After laying out the first five Peano axioms, he goes on to say 'As the next step, we can set up a definition of addition which expresses in a precise form the idea that the addition of any natural number to some given number may be considered as a repeated addition of 1; the latter operation is readily expressible by means of the successor relation.'[36] Like Dedekind, Hempel seems to be implying that addition is 'nothing over and above' repeated applications of the successor function. Such motivating considerations provide at least some evidence that anyone who accepted the original Peano axioms would necessarily, or at least naturally, be led to accept definitions (whether explicit or implicit, no matter) of addition and multiplication.[37]

Still, it seems that Casullo could resist this verdict, on grounds that even if we *can* (as per Dedekind) see addition as 'the combination of any arbitrary repetitions' of some 'simplest act' associated with counting by 1's, or even if we *can* (as per Hempel) see addition as expressing 'the idea that the addition of any natural number to some given number may be considered as a repeated addition of 1,' we are not conceptually *obliged* to do so. Casullo might maintain that we *could* be able to perform whatever simplest act allows us to count, without automatically being in possession of the 'idea' that not only counting, but *addition*, can be built up out of this simplest act. Moreover, he might maintain that we could understand addition (and even be able to do sums) *without* understanding it as a matter of multiple iterations of the successor function. After all, Hempel himself notes that possession of this idea is associated with a 'next step' (beyond whatever steps are codified in the Peano axioms) in the formal development of arithmetic. This 'next step' effectively amounts to combining two (or more) countings. Casullo could agree that, as it happens, the epistemically basic propositions that we now take to be confirmed do conform to the idea that addition may be understood in terms of combinations of repeated applications of the successor function. But, he might argue, if things were different — if, for

36 Hempel, 'On the Nature of Mathematical Truth,' 382

37 This understanding of addition and multiplication plausibly explains why Peano did not take his 'Definitions' of addition and multiplication to be introducing new primitive constants (and hence, to be new axioms for his theory).

example, we were in the conditions described by his scenario — it would *not* be appropriate to understand addition in this way. And this possibility shows (he might say) that counting and addition could be epistemically independent of one another.

It seems to me that there is something to the observation that there is a difference between the level of cognitive competence associated with repeated applications of the successor function (as in counting), and the level associated with *combinations* of repeated applications of the successor function. The seed of truth in Casullo's disconfirming scenario is precisely that there is such a difference. To say that there is a difference in the levels of cognitive competence associated with counting and adding is not, however, to say that the results of adding *float free* of the results of counting. It is in maintaining the latter that Casullo's scenario goes wrong.

IV Keeping our Tallying Procedures and Rejecting our Arithmetic

To see where the problem lies, we need to move closer to the ground regarding what is actually going on in the act of counting. After all, it is clear that we can count — and most of us do — without having any knowledge of the 2nd-order Peano axioms. Since we do this, there must be some other way, besides via these axioms, in which the terms 'number' and 'successor,' appearing in Casteneda's formulation of counting, could get their meanings fixed. And there is: namely, tallying procedures, as the simplest form of counting.[38] In tallying, one makes a mark for every object in the collection being tallied; when the tally is done, the cardinality of the collection is represented simply by the set of tally marks resulting from this process. I take the tallying procedure to involve the following single step:

> *Tallying procedure*: While there are objects to be tallied that are not yet pointed to, point to such an object and make a tally mark.

The tallying procedure establishes a one-to-one correspondence between objects and marks; thus the resulting set of tally marks is naturally considered as a representation of the cardinality of the objects tallied.

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38 For an excellent discussion of how counting may be seen as a transformation of tallying, see R. L. Goodstein, 'The Meaning of Counting,' in *Essays in the Philosophy of Mathematics* (Leicester: Leicester University Press 1965).

The set of tally marks — what we might call a 'number sign' — is what I will call a 'pure' (that is, an 'arithmetic-free') representation of the cardinality of the collection, if any representation is. Moreover, it is not hard to see how we might explicate counting in terms of tallying. Counting procedures, on any formulation, are committed to the existence of an ordered sequence of representations of cardinality. But tallying provides a means of obtaining this sequence: we can build this sequence up out of the 'number signs' that result from each step in the tallying process. The tallying procedure, in other words, gives rise to a sequence of the following sort:

> *Tallying sequence*: The ordered sequence of marks (taken as pure representations of cardinality):
>
> |, | |, | | |, | | | |, | | | | |, ...

We gesture towards infinity with the ellipses, a gesture which reflects the plausible assumption that the tallying procedure could be continued indefinitely. Under this assumption, the tallying sequence is clearly isomorphic to the intended model of the natural numbers.[39]

Given this isomorphism, each element in the tallying sequence will correspond to the element occupying the same position in the sequence of natural numbers. (That is, the first element in the sequence corresponds to the number 1, the second to the number 2, and so on.) A tallying sequence thus provides us with a non-theoretical means of understanding the terms 'number' and 'successor' as they appear in Casteneda's formulation of counting: a number is any member of the tallying sequence, and the successor of a number is the next member of the tallying sequence. The possibility of such a reconstruction establishes that counting and tallying are, for purposes of determining cardinality, structurally the same procedures. The only difference lies in the representation of the results, and in the distinction between (in counting) drawing members from a previously established ordered sequence and (in tallying) creating such a sequence on the fly. These differences are

39 In fact, for purposes of fulfilling the requirements of Casullo's scenario, participants in the scenario need not have in hand an *infinite* sequence of cardinal representations, since the arithmetical propositions at issue (being epistemically basic propositions) involve only relatively small numbers. A sufficiently large finite sequence, isomorphic to some initial segment of the natural numbers, would do the trick. For conceptual and expository purposes, it is convenient to make the gesture towards infinity, but the arguments to follow would go through on an understanding of the tallying sequence as large, but finite.

merely ones of convenience and conceptualization — we can think of counting as a representationally convenient form of tallying, and tallying as a representationally primitive form of counting.[40]

These representational differences are such that it seems clear that anyone who was in possession of standard counting procedures of the sort required by Casullo's scenario — who was able to perform 'countings to 2,' and so on, and who understood what such procedures were supposed to do — would accept that these procedures were representational variants of tallying procedures as here described. Of course, we don't require of participants in the scenario that they be aware (for example) that an infinite tallying sequence is order-isomorphic to the sequence of natural numbers. We just require that they be capable of seeing that counting and tallying one representational variants, which capability (under the tacit assumption that they possess the usual cognitive capacities of humans) they will clearly possess, since *we* do.

Since counting is merely a variation of tallying, and participants in Casullo's scenario are cognitively equipped to see this, we are within our rights to continue the present discussion at the level of standard tallying procedures. We can now return to the question whether Casullo's scenario is coherent — that is, to the question whether participants in his scenario could find the cardinal aspect of number ('number signs') to be independent of the arithmetical aspect ('number relations'). Reframing the scenario in terms of tallying, the correlate of the participants' performing a 'counting to two' will now be equivalent to performing a 'tally to | |' on a collection of objects, placing the tally marks on (say) a tablet, as follows:

| |

The participants are then to perform another 'tally to | |' on another collection of objects, again placing the tally marks on (say, a different) tablet:

| |

They are then supposed to place the two sets of objects together and perform another tally. On Casullo's scenario, they might come up with the following result:

40 Consider: beings with extremely fine-grained perceptual abilities might not find it necessary to move to symbolic representations of tally marks, if they were, in virtue of these abilities, capable of instantaneously grasping the cardinality represented by a given set of tally marks.

| | | | |

Is this possible? Under the assumptions of the scenario, could the participants find the result of separately tallying two sets to be different from the result of tallying the union of these sets?

My claim is that this is not possible. The key to establishing this claim turns on the fact that tallying does not, as does more sophisticated counting, distinguish between 'accumulations' and 'new starts.' An example will illustrate. Suppose that participants in the scenario perform the first two steps of the process above (the two 'tallyings to | | '), with the following difference: they place both sets of tally marks on the same row of the same tablet, instead of on two different tablets. For future reference, let us call this procedure 'successive tallying.' The first two steps would then be:

| | (At end of first tally)

| | | | (At end of second tally)

When they do this they arrive at an interesting result. Because of the cumulative nature of tally marks, the resulting combination of these marks *already* represents a cardinality, which, when they compare the tallying and natural number sequences, is seen to be the natural number 4. The relevant question to ask, for purposes of investigating whether Casullo's scenario is coherent, is: Under the assumptions of the scenario, could the participants find the cardinality resulting from the successive tallying of two exclusive collections (call this *procedure 1*) to be something *other* than the cardinality resulting from the tallying of the union of the two collections (*procedure 2*)? In what follows, I give two arguments for thinking that the answer to this question is no, and hence for thinking that Casullo's scenario is, after all, incoherent.

The first argument proceeds by a consideration of the relevant differences involved in the two tallying procedures. If the cardinality resulting from procedure 1 were to be something other than the cardinality resulting from procedure 2, there would presumably have to be some possible relevant difference in the two cases, between either the objects being tallied, the carrying out of the procedures, or the procedures themselves. Since the participants are justified in believing that the Stability and Correct Counting conditions hold, they are justified in believing that no relevant difference can derive from either the objects being tallied, or from their carrying out the procedures — they are justified in believing that the objects are stable and that they get it right in both cases. Most crucially, since tallying does not distinguish between accumulations and new starts, they are justified in believing that no relevant difference can be found in the procedures themselves. All there is to the tallying procedure is that each object tallied gets a single tally mark. This will be true whether they perform a tally of one collection and successively append a second tally onto the first, or they perform a tally of the union of the two collections: from the standpoint of tallying, the two procedures are indistinguishable. But if the participants are justified in believing that the objects are relevantly the same (by the Stability condition), that the acts of tallying are relevantly the same (by the Correct Counting condition) and that the procedures are relevantly the same (as above), then there doesn't appear to be any way for them to rationally believe the *results* of the procedures to be different. If the participants rationally believe the result of the first procedure to be N, then they will rationally believe the result of the second procedure to be N, and they (and we) can say this a priori. Contra Casullo's claim, the result of going through the second tallying procedure would not then qualify as an 'open empirical question.'[41] Under the assumptions of the scenario, only one outcome is epistemically possible as regards the result of tallying all the objects together, and this will be represented by whatever 'number sign' results from tallying any exclusive and exhaustive subcollections of the collection, under conditions that the participants append these results to each other.[42]

The second argument proceeds by *reductio*.[43] Suppose Casullo's scenario occurs. At the end of procedure 1 (the successive tally of two

41 It should be clear that this argument holds against Casullo's claim (as does the argument to follow) even if the participants (misguided about one or other of the conditions' holding) don't get it right about how many objects there were — that is, even if the results of their counting procedures aren't, in fact, correct. Casullo's claim is just the claim that *given* some counting results, the arithmetical relations holding between these results is 'an open empirical question.' Whether the counting results are correct is irrelevant to this claim, and to my arguments against it.

42 This argument can also be used to show that, under the assumption that the conditions *in fact* hold, only one outcome is genuinely possible. Just replace all references to justified belief with references to knowledge (or to what knowledge entails — namely, truth). The argument then goes: Since the Stability and Correct Counting conditions hold, no relevant difference can derive from either the objects being tallied, or from the participants carrying out the procedures — the objects are stable and the participants get it right in both cases. Most crucially, since tallying does not distinguish between accumulations and new starts, no relevant difference can be found in the procedures themselves... [as such] there doesn't appear to be any way for the results of the procedures to be different. If the result of the first procedure is known, then so is the result of the second procedure, and the participants (and we) can say this a priori.

43 Many thanks to Eric Hiddleston for the heart of this argument.

disjoint collections, S1 and S2) the participants end up with the following tally:

| | | |

Let T1 be the set of marks in this tally. At the end of procedure 2 (the tally of the union of these two collections, S1 ∪ S2) the participants end up with the following tally:

| | | | |

Let T2 be the set of marks in this tally. Since the participants are justified in believing that the Stability condition holds, they are justified in believing that the second tally is of S1 ∪ S2, the same sets as in the first tally, and since they are justified in believing that the Correct Counting condition holds, they are justified in believing that both tallies are correct. Now, in order for participants to be in possession of the tallying procedure, they must understand that in any correct tally there will be a one-to-one correspondence between the tally marks and the objects in the set tallied. So, the participants are justified in believing that there is a one-to-one correspondence between T2 and S1 ∪ S2. Unfortunately for Casullo, they are also justified in believing there to be a one-to-one correspondence between T1 and S1 ∪ S2. In any (justifiably believed correct) tally the participants make, they will take each tally mark to bear the following relation R to exactly one member of the set tallied: _ was marked while I was pointing at _. They will take each member of T1 to bear R to exactly one object in either S1 or S2, and to no other object. And they will take each member of S1 to have R born to it by exactly one member of T1, and the same for each member of S2. So, they will (just as we do) justifiably believe there to be a one-to-one correspondence between T1 and S1 ∪ S2. But they also justifiably believe there to be a one-to-one correspondence between T2 and S1 ∪ S2, so (just as we do) they will justifiably believe there to be a one-to-one correspondence between T1 and T2.[44] But according to Casullo's scenario, they don't

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44 Here I am assuming that participants in the scenario could easily see that the composition of two one-to-one correspondences (where the range of the first is the domain of the second) will itself be a one-to-one correspondence. This seems right, since they can surely see (as we do) that this follows straightforwardly from even a non-technical understanding of the notion of a one-to-one correspondence (from the 'transitivity of matching,' as it were). In addition, attention to what participants would *do* with the results of counting/tallying procedures indicates their acceptance of compositionality. For example, given their justified beliefs that the Correct Counting and Stability conditions hold, participants would surely justifiably believe

justifiably believe there to be a one-to-one correspondence between T1 and T2. Consequently, the scenario is epistemically impossible.[45]

The cumulative nature of tally marks means that counting, when performed as a tallying procedure, *defines its own arithmetical relations*. We can establish, for example, arithmetical relations involving addition (of positive numbers) for a given 'number sign' by drawing one or more conceptual lines in between the various tally marks making up the set and seeing what 'number signs' we can divide the original set into. An example will make this clear. Consider the following set of tally marks:

| | | |

By drawing purely conceptual lines between these tally marks we can determine arithmetical relations holding between the original tally marks and other sets of tally marks:

| | | |

| | ||

|| ||

| |||

From the first of these groupings, we can read off from the sequence of tally marks that | and | and | and | tally marks make |||| tally marks, from the second, that | and | and || tally marks make |||| tally marks; and so on. It is easy to see that such relations are informal versions of the standard arithmetical relations. Replacing the word 'and' with '+,' the word 'make' with '=,' and the various sets of tally marks with their natural number correlates, these purely conceptual divisions correspond to various elementary propositions of arithmetic (namely, 1+1+1+1=4, 1+1+2=4, 2+2=4, 1+3=4). Moreover, since each tally mark is undistin-

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that the *copying* of a given tally onto another tablet would *preserve* the cardinality associated with the collection originally tallied. (After all, Casullo's scenario is not designed to call the results of countings/tallyings into question.) But to justifiably believe this is to justifiably believe that the compositionality of two one-to-one-correspondences (where the range of the first is the domain of the second) is itself a one-to-one correspondence.

45 As with the first argument, this argument can be adapted to show that, under the assumption that the conditions *in fact* hold, only one outcome is genuinely possible. Again, just replace all references to justified belief, and to what the participants 'take' to be the case, with references to knowledge (or truth).

guished from the point of view of being a 'unit,' we can read off various other arithmetical properties (for example, commutativity) from such conceptual groupings of a single set of tally marks. The important point is that we are able to do all this on the basis of purely conceptual manipulations on a single tally, so that arithmetical relations involving the number represented by the tally are neither epistemically nor genuinely 'up for grabs,' as it were, but instead are crucially dependent upon (one is tempted to say, are supervenient on) the cardinal aspect of number.

More would need to be said in order to establish that multiplication could be treated similarly, but I see no barriers to such a development.[46] And it should be clear that I am not suggesting that arithmetic is 'really about' conceptual divisions of concrete (or, for that matter, non-concrete) tally marks; there are obvious problems with such a suggestion, especially as pertaining to arithmetical relations involving large numbers, and to our conception of the numbers as infinite. But even if we cannot verify the arithmetical relations involving numbers represented in any but relatively small tallies, there is no reason to think that matters would be different for tallies representing large numbers, so the above considerations will likely apply to all of the propositions in standard arithmetical theory. In any case, there can be no question that for the epistemically basic propositions with which Casullo is concerned, we have arrived at

46 Roughly: to multiply m by n, we (in tally notation) tally m, n times. That is, (keeping track, perhaps with another tally) we string together n tallies of the number m. For example, to multiply 3 by 2, we write one tally of $|\,|\,|$:

$$|\,|\,|$$

and then append a second tally of $|\,|\,|$, to get

$$|\,|\,|\,|\,|\,|$$

which member of the tallying sequence corresponds to the arabic numeral '6.' Multiplication, it should be clear, just amounts to successive additions of a number to itself, so seeing this relation, too, as (we might say) supervening on the cardinal aspect of number, should not be surprising. And as in the case of addition, we can read off various properties of multiplication from a single tally. For example, we can draw purely conceptual marks in the previous tally as follows:

$$\begin{array}{l} |\,|\quad|\,|\quad|\,| \\ |\,|\,|\qquad|\,|\,| \end{array}$$

to see (after appropriate translations) that $2{\cdot}3 = 6$; that $2{\cdot}3 = 3{\cdot}2$ (multiplication is commutative), and so on.

a final verdict: for these propositions, the arithmetic aspect of number is *not* epistemically (nor is it genuinely) independent of the cardinal aspect. Casullo's disconfirming scenario is, after all, incoherent (and so, by the way, is Mill's). We could not rationally hold both (A1) and (A2) concurrently; we could not keep the results of our standard counting procedures, while rejecting our standard arithmetical propositions: both are drawn with the same marks.[47]

V Revisiting the Irrefutability Argument

What, then, can we say about the status of the Irrefutability Argument? Casullo's extension of Mill's original scenario, which was intended to undermine this argument by establishing that the propositions of arithmetic could be empirically disconfirmed, did not go through. If we have in hand a standard counting (tallying) procedure, and are justified in believing that the Stability and Correct counting conditions hold, we can rest assured that the propositions of arithmetic won't be disconfirmed (or refuted) by experience.

Might there be some *other* disconfirming scenario, which an inductive empiricist could use to undermine the Irrefutability argument? Finding Casullo's scenario incoherent depended upon (what the scenario assumed) the participants' justifiably believing the Stability and Correct Counting conditions to hold. This finding is compatible with the propositions of arithmetic being disconfirmed by scenarios experientially more drastic than the one Casullo envisioned, in which one or the other belief is unjustified. But such scenarios can provide no support for an inductive empiricist account of mathematics. For, as Casullo himself notes, in order to *appropriately* inductively generalize to the propositions of mathematics, we must justifiably believe both conditions to be in place.

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47 Both Casullo and Mill may have been drawn into thinking that the cardinality and arithmetical aspects of number were independent of each other, as a consequence of framing their discussions using the standard (arabic numeral) representations of the natural numbers. Using these numerals as representations of cardinality, it might seem that if we perform one 'counting to two' and then a second 'counting to two,' then we are still in the dark as regards the results of a counting of the objects taken altogether. Certainly placing the results of the first two countings next to each other (in this case, placing the second '2' next to the first '2' to achieve '22') is in no way informative as to the result of the counting of the objects taken as a single collection. But by now it will be seen that this is an artificial feature of any representation of cardinality which, in counting, distinguishes 'new starts' from 'accumulations.'

We might wonder, however, whether scenarios in which one or the other justified belief was dropped could be forwarded in service of a *holist* empiricist account. As mentioned earlier, I am here neglecting scenarios which (only) drop the assumption that the Correct Counting condition is justifiably believed. But it is worth noting that any scenario which (only) drops justified belief in the Stability condition will succumb to the same arguments as those given above against Casullo's scenario. To see this, note that if the Correct Counting condition is justifiably believed to hold, then any supposed instability of objects will presumably *not* extend to the ideas of participants in the scenario. For if it did, then the participants would not be justified in taking themselves to be correctly counting. In order for participants in the scenario to be justified in believing that the Correct Counting condition holds, they must be able to verify that, in counting, they are not, for example, counting the same object *twice*. Plausibly, this means that in order for one participant to be justified in correctly counting, some other participant will have to count the number of times the first participant counts each object in the collection to be counted. If the second participant finds (for example) that the answer is 'no count' or 'two counts,' then the assumption of the scenario (that the participants were justified in believing the Correct Counting condition to hold) would fail. So, in order to maintain this assumption, the actions involving countings must themselves be exempt from (the justified belief in) the failure of the Stability condition. But if so, then these actions *themselves* will provide a collection of objects that are justifiably believed to remain stable throughout counting. The scenario under consideration thus reduces to a variation on Casullo's scenario, and the same arguments legislating against Casullo's scenario legislate against it, as well.

Casullo's disconfirming scenario, and the variation just considered, fail to undermine the Irrefutability argument. But Casullo's challenge to this argument served a useful purpose, for in exploring his extension of Mill's original scenario, two gaps were revealed. First, Casullo's emphasizing that participants in the scenario are justified in believing the Correct Counting and Stability conditions to hold, revealed a gap in the usual arguments against Mill-type disconfirming scenarios, and hence in the support for (P1), the claim that mathematical propositions cannot be experientially disconfirmed. Second, exploration of the assumptions underlying Casullo's scenario revealed that there was plausibly a gap between the levels of cognitive competence associated with counting and addition. Here I hope to have closed the first gap (in the Irrefutability argument) for the case of arithmetical propositions, and shown the second gap to be incapable of grounding Casullo's claim that elementary arithmetical propositions could float free of the results of our standard counting procedures, by showing that (to paraphrase Dedekind) arithmetic is indeed a necessary, or at least a natural, consequence of the simplest arithmetical act, that of *tallying*.[48]

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