Practice questions for Mat 188

Jordan Bell

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These are problems with solutions to help study. I have not proof read the solutions extremely carefully, so I don't promise that they are correct. Neither do I promise that these questions are an indicator of what is on the final exam for the course. These are just what I think are representative questions for the sections of the textbook that the course covered. I am providing this as an aide, and it comes with no warranty and no customer service. I don't write vectors with lines above them or in bold. Also, this is not a practice exam because it is shorter than an actual exam would be.

Problem 1. Find a basis for the row space, column space, and null space of

$$A = \begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix}$$

Solution to problem 1. We row reduce the matrix A to get it in reduced echelon form. Here's the reduced echelon form I get: (of course you actually have to do this, I'm just not writing down all the steps)

$$B = \begin{bmatrix} 1 & 0 & 1 & -\frac{2}{7} \\ 0 & 1 & 1 & \frac{4}{7} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(I'm giving the name B to the reduced echelon form of A.) A basis for the row space of A is the rows with leading variables in the reduced echelon form, and a basis for the column space are the columns in the original matrix that correspond to the columns with leading variables in the reduced echelon form. Thus a basis for the row space of A is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ -\frac{2}{7} \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ \frac{4}{7} \end{bmatrix} \right\}$$

and a basis for the column space is

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix} \right\}.$$

To find a basis for the null space of A, we want to find want to find what an arbitrary vector looks like that satisfies $Av = 0$. Because A is equivalent to B, there are elementary matrices $E_1, \ldots, E_r$ (you should know that you can express each of the three elementary row operations as multiplying on the left by a matrix, and this matrix is called an elementary matrix, although you don't have to know the word elementary matrix) such that $B = E_r \cdots E_1 A$. The elementary matrices are invertible, so $Av = 0$ can be satisfied if and only if $Bv = 0$. i.e. the null space of a matrix is the same as the null space of its reduced echelon form. An arbitrary element of the null space of B looks like

$$v = \begin{bmatrix} \frac{2}{7}x_4 - x_3 \\ -\frac{4}{7}x_4 - x_3 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} \frac{2}{7} \\ -\frac{4}{7} \\ 0 \\ 1 \end{bmatrix}.$$

A basis for the null space of A is

$$\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{2}{7} \\ -\frac{4}{7} \\ 0 \\ 1 \end{bmatrix}.$$

Problem 2. Orthogonally diagonalize

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Solution to problem 2. Find the roots of the characteristic polynomial $\det(A - \lambda I)$. They turn out to be $0, 0, 2$. Now find a basis for the null space of $A - 0I$: this involves row reducing the augmented matrix $A|0$:

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

An arbitrary element of the null space of $A - 0I$ looks like

$$v = \begin{bmatrix} -v_2 \\ v_2 \\ v_3 \end{bmatrix} = v_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + v_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

We are fortunate that these two vectors are orthogonal; they didn't have to be, in which case we would have had to use the Gram-Schmidt process at this point because we want orthogonal eigenvectors.

Now find an eigenvector for $\lambda = 2$. It turns out to be

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

We have to normalize the vectors we've found before we make a matrix P with them as columns. $\|v\| = v \cdot v$.

$$\left\| \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\| = 2, \left\| \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\| = 1, \left\| \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\| = 2.$$

Thus if

$$P = \begin{bmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix}$$

and

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

then

$$A = PDP^T$$

is an orthogonal diagonalization of A.

Problem 3. Some true or false questions: 1. If a matrix has 0 as its only eigenvalue, then it is the zero matrix. 2. If a matrix is orthogonal then it is symmetric. 3. If a matrix is symmetric then it is orthogonal. 4. If a matrix is orthogonal then it is invertible. 5. If A is an $n \times m$ matrix, then the rank of A is equal to the nullity of A.

For showing that something is false there is no recipe for doing it. Often it involves just playing around with matrices until you find an example.

Solution to problem 3. 1. False: $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ has 0 as its only eigenvalue, but it is not the zero matrix.

2. False: $\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ is orthogonal, but it is not symmetric.

3. False: the zero matrix is symmetric but it is not orthogonal, because its columns don't have norm 1.

4. True: A matrix A is orthogonal if and only if $A^T A = I$ so $A^{-1} = A^T$.

5. False: The rank nullity theorem states that the rank of A plus the nullity of A is equal to m. But this isn't enough of an answer yet: we have to give an actual example to show the statement is false. Let $A = [0]$. This is the 1×1 zero matrix. Its rank is 0 and its nullity is 1.

Problem 4. If

$$T \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, T \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, T \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

what is $T \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$?

Solution to problem 4. Let v_1, v_2, v_3 be the three vectors on which T is defined. Since we're given Tv_1, Tv_2, Tv_3, we should write $\begin{bmatrix}1\\0\\0\end{bmatrix}$ in terms of v_1, v_2, v_3. This means we want to solve the augmented matrix $[v_1 v_2 v_3 | e_1]$, where $e_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$.

$$\begin{bmatrix}1 & 0 & 5 & 1\\ 2 & 1 & 2 & 0\\ -1 & 2 & 1 & 0\end{bmatrix} \to \begin{bmatrix}1 & 0 & 5 & 1\\ 0 & 1 & -8 & -2\\ 0 & 2 & 6 & 1\end{bmatrix} \to \begin{bmatrix}1 & 0 & 5 & 1\\ 0 & 1 & -8 & -2\\ 0 & 0 & 22 & 5\end{bmatrix} \to \begin{bmatrix}1 & 0 & 5 & 1\\ 0 & 1 & -8 & -2\\ 0 & 0 & 1 & \frac{5}{22}\end{bmatrix}$$

$$\to \begin{bmatrix}1 & 0 & 0 & -\frac{3}{22}\\ 0 & 1 & 0 & -\frac{4}{22}\\ 0 & 0 & 1 & \frac{5}{22}\end{bmatrix}$$

So

$$\begin{bmatrix}1\\0\\0\end{bmatrix} = -\frac{3}{22}v_1 - \frac{4}{22}v_2 + \frac{5}{22}v_3.$$

Therefore

$$T\begin{bmatrix}1\\0\\0\end{bmatrix} = -\frac{3}{22}T(v_1) - \frac{4}{22}T(v_2) + \frac{5}{22}T(v_3) = \begin{bmatrix}-\frac{3}{22}\\ -\frac{4}{22}\\ \frac{5}{22}\end{bmatrix}.$$

Problem 5. Find the projection of $\begin{bmatrix}3\\1\\-1\end{bmatrix}$ onto the plane $x + 2y - 2z = 0$.

Solution to problem 5. First we have to write the plane as the span of vectors. A point on the plane looks like:

$$\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}-2y+2z\\y\\z\end{bmatrix} = y\begin{bmatrix}-2\\1\\0\end{bmatrix} + z\begin{bmatrix}2\\0\\1\end{bmatrix}.$$

Therefore these two vectors are a basis for the plane. But they aren't orthogonal and they need to be to find an orthogonal projection, so we have to use Gram-Schmidt now. Let $v_1 = \begin{bmatrix}-2\\1\\0\end{bmatrix}$, and define

$$v_2 = \begin{bmatrix}2\\0\\1\end{bmatrix} - \frac{\begin{bmatrix}2\\0\\1\end{bmatrix} \cdot \begin{bmatrix}-2\\1\\0\end{bmatrix}}{\begin{bmatrix}-2\\1\\0\end{bmatrix} \cdot \begin{bmatrix}-2\\1\\0\end{bmatrix}} \begin{bmatrix}-2\\1\\0\end{bmatrix} = \begin{bmatrix}2\\0\\1\end{bmatrix} - \frac{-4}{5}\begin{bmatrix}-2\\1\\0\end{bmatrix} = \begin{bmatrix}\frac{2}{5}\\ \frac{4}{5}\\ 1\end{bmatrix}.$$

Now we can find the orthogonal projection onto the plane. It is equal to

$$\frac{\begin{bmatrix}3\\1\\-1\end{bmatrix}\cdot v_1}{v_1\cdot v_1}v_1+\frac{\begin{bmatrix}3\\1\\-1\end{bmatrix}\cdot v_2}{v_2\cdot v_2}v_2=\frac{-5}{5}v_1+\frac{1}{\frac{9}{5}}v_2=-v_1+\frac{5}{9}v_2=\begin{bmatrix}2\\-1\\0\end{bmatrix}+\frac{1}{9}\begin{bmatrix}2\\4\\5\end{bmatrix}$$

$$=\begin{bmatrix}\frac{20}{9}\\-\frac{5}{9}\\\frac{5}{9}\end{bmatrix}=\frac{1}{9}\begin{bmatrix}20\\-5\\5\end{bmatrix}.$$

This is the orthogonal projection of $\begin{bmatrix}3\\1\\-1\end{bmatrix}$ onto the plane $x+2y-2z=0$.

Problem 6. Give an example of a matrix that's not diagonalizable and show that it's not diagonalizable. Give an example of a matrix that's not orthogonal and show that it's not orthogonal. Is there a 2×2 matrix that is orthogonal and symmetric and is not a multiple of the identity matrix? Give an example of a matrix such that $\|Ax\|\leq\|x\|$ for all vectors x.

Solution to problem 6. $A=\begin{bmatrix}1&1\\0&1\end{bmatrix}$ is an example of a matrix that is not diagonalizable. But you're not just asked to state it, you're asked to show that it's not diagonalizable. So you have to try to diagonalize it and fail. Its eigenvalues are 1 and 1, but when we find the null space of $A-1I$ we find that it is spanned by $\begin{bmatrix}1\\0\end{bmatrix}$, so it is only 1 dimensional even though so there are not two independent eigenvectors.

Let $A=[0]$. The columns of an orthogonal matrix each have norm 1, while the only column of A has norm 0, so A is not an orthogonal matrix.

Finding a matrix that is orthogonal and symmetric just involves playing around trying to find an example.

$$A=\begin{bmatrix}1&0\\0&-1\end{bmatrix}.$$

This is not a multiple of the identity matrix, but it is both orthogonal and symmetric so it's an example. So yes, such a matrix does exist.

Since we're just asked for an example and we're not told what the dimensions of A have to be, we can do the simplest possible case: $n=1$. Let $A=[\frac{1}{2}]$, the 1×1 matrix whose entry is $\frac{1}{2}$. Here $\|x\|$ is just the absolute value of x, and

$$\|Ax\|=|\frac{x}{2}|=\frac{1}{2}|x|=\leq\frac{1}{2}|x|=\frac{1}{2}\|x\|.$$

Problem 7. If A is an $n\times n$ matrix and $\det A=11$, and B is the matrix obtained from A by applying the following steps in order, what is $\det B$? For these to make sense we are assuming that $n\geq 4$, but otherwise I make no assumptions about n.

- Double the third column
- Add twice the first row to the first row
- Swap the first and fourth rows
- Subtract twice the second row from the first row
- Multiply every entry in the matrix by 2

Solution to problem 7. Let AE_1 be A with its third column doubled. $\det E_1 = 2$. (when you do a column operation you write the "elementary matrix" on the right)

Let E_2 be the matrix that adds twice the first row to the first row. This is not the same as adding twice the first row to another row. Here is means multiplying the first row by 2, so $\det E_2 = 3$.

Let E_3 be the matrix that swaps the first and the third rows. $\det E_3 = -1$.

Let E_4 be the matrix that subtracts twice the second row from the first row. This means adding -2 times the second row to the first row, so $\det E_4 = 1$.

So far we have $E_4E_3E_2AE_1$, and the determinant of this is $1 \cdot -1 \cdot 3 \cdot 11 \cdot 2 = -66$. We have the final step of multiplying every entry in the matrix by 2. This is the same as multiplying by $2I$, and $\det(2I) = 2^n$ (not 2). So the determinant of B is $-66 \cdot 2^n$.