Mat 244 Assignment 1 Solutions

Jordan Bell

June 18, 2013

1. \( y_0 = 2, \ h = 0.1 \). \( f(t, y) = 5 - 3\sqrt{y} \). I used Sage (http://www.sagenb.org) to store the values of \( y_0, y_1, y_2, y_3, y_4 \).

\[
y_1 = 2 + (5 - 3\sqrt{2}) \cdot 0.1 = 2.07573593128807.
\]

\[
y_2 = 2.07573593128807 + (5 - 3\sqrt{2.07573593128807}) \cdot 0.1 = 2.14351349571129.
\]

\[
y_3 = 2.14351349571129 + (5 - 3\sqrt{2.14351349571129}) \cdot 0.1 = 2.20429121187995.
\]

\[
y_4 = 2.20429121187995 + (5 - 3\sqrt{2.20429121187995}) \cdot 0.1 = 2.25888554389941.
\]

\[
y_5 = 2.25888554389941 + (5 - 3\sqrt{2.25888554389941}) \cdot 0.1 = 2.30799786503587.
\]

Therefore, using Euler’s method with step size \( h = 0.1 \), our approximation for \( y(0.5) \) is

\[
2.30799786503587.
\]

A good solution doesn’t need to have this many decimal places.

2. The equation is

\[
y' - \frac{1}{2t} y = -\frac{1}{2} + \frac{t^2}{2}.
\]

Multiplying by an integrating factor \( \mu \) this becomes

\[
\mu y' - \frac{\mu}{2t} y = -\frac{1}{2} + \frac{\mu t^2}{2}.
\]

We want

\[
\mu' = -\frac{\mu}{2t},
\]

hence

\[
\ln \mu = \frac{1}{2} \ln t = \ln(t^{-1/2}),
\]

hence

\[
\mu = t^{-1/2}.
\]

Therefore

\[
(y t^{-1/2})' = \frac{1}{2} t^{-1/2} + \frac{1}{2} t^{3/2}.
\]
Integrating gives
\[ yt^{-1/2} = -t^{1/2} + \frac{1}{5}t^{5/2} + C. \]
Since \( y(1) = 0 \), we have
\[ 0 = -1 + \frac{1}{5} + C, \]
so
\[ C = \frac{4}{5}. \]
Therefore the solution of the initial value problem is
\[ y = -t + \frac{1}{5}t^3 + \frac{4}{5}t^{1/2}. \]

3. The equation is
\[ \frac{dy}{y} = \frac{4tdt}{t^2 + 1}. \]
Integrating,
\[ \ln y = 2 \ln(t^2 + 1) + C. \]
Using the initial condition \( y(0) = 2 \) gives
\[ \ln 2 = C. \]
Thus,
\[ \ln y = \ln((t^2 + 1)^2) + \ln 2 = \ln(2(t^2 + 1)^2), \]
so
\[ y = 2(t^2 + 1)^2. \]

4. Here \( M = -y - t^2y^2, N = -2t - t^3y \). So \( M_y = -1 - 2t^2y \) and \( N_t = -2 - 3t^2y \).
If we multiply by an integrating factor \( \mu \) and the equation is then exact, we would have
\[ \mu y M + \mu M_y = \mu t N + \mu N_t. \]
If \( \mu_y = 0 \), then
\[ \mu(M_y - N_t) = \mu t N, \]
so
\[ \frac{\mu_t}{\mu} = \frac{M_y - N_t}{N} = \frac{-1 - 2t^2y + 2 + 3t^2y}{-2t - t^3y} = \frac{1 + t^2y}{-2t - t^3y}, \]
which doesn’t work.
If \( \mu_t = 0 \), then
\[ \mu_y M = \mu(N_t - M_y) \]
so
\[ \frac{\mu_y}{\mu} = \frac{N_t - M_y}{M} = \frac{-2 - 3t^2y + 1 + 2t^2y}{-y - t^2y^2} = \frac{-1 - t^2y}{-y - t^2y^2} = \frac{1}{y}. \]
and integrating we get

\[ \mu = y. \]

Multiplying the original equation by \( y \), it becomes

\[ -y^2 - t^2 y^3 + (-2ty - t^3 y^2) y' = 0. \]

So \( M = -y^2 - t^2 y^3 \) and \( N = -2ty - t^3 y^2 \). We want to find \( \psi \) such that \( \psi_t = M \) and \( \psi_y = N \). Then,

\[ \psi(t, y) = -ty^2 - \frac{t^3 y^3}{3} + g(y). \]

Since \( \psi_y = N \), we have

\[ -2ty - t^3 y^2 + g'(y) = -2ty - t^3 y^2. \]

Hence \( g'(y) = 0 \), and then \( g(y) = 0 \) (we are trying to find some \( \psi(t, y) \) such that if \( y \) solves the ODE then \( \psi(t, y(t)) = C \), and it turns out that it doesn’t matter if we take \( g \) to be 0 or any other particular constant.)

In conclusion, if \( y \) is a solution of the equation

\[ -y - t^2 y^2 + (-2t - t^3 y) y' = 0, \]

then there is a constant \( C \) such that \( y(t) \) satisfies

\[ -ty^2 - \frac{t^3 y^3}{3} = C. \]

5. The IVP is

\[ \frac{dy}{y^2} = 2tdt, \quad y(0) = y_0. \]

If \( y_0 = 0 \) then the solution is \( y(t) = 0 \) with domain \( \mathbb{R} \). Otherwise,

\[ -\frac{1}{y} = t^2 + C. \]

Using the initial condition,

\[ -\frac{1}{y_0} = C. \]

Thus,

\[ -\frac{1}{y} = t^2 - \frac{1}{y_0}. \]

So

\[ \frac{1}{y} = -t^2 + \frac{1}{y_0}, \]

and

\[ y = \frac{1}{-t^2 + \frac{1}{y_0}}. \]
If \( y_0 < 0 \) then the denominator cannot be 0, and the domain of the solution is \( \mathbb{R} \).

If \( y_0 > 0 \), for the denominator to be nonzero we need

\[
t \neq \frac{1}{\sqrt{y_0}}.
\]

Since the solution must be defined at time 0 (since we are given that \( y(0) = y_0 \)), this implies that

\[
|t| < \frac{1}{\sqrt{y_0}},
\]

because \( t \) cannot be equal to \( \frac{1}{\sqrt{y_0}} \) and starts out less than it (i.e. I am excluding the possibility that \( t > \frac{1}{\sqrt{y_0}} \)). If the solution becomes infinite at a certain time, it doesn’t make sense to ask what the value of it is for some later time.

6. \( f(t, y) = t^2 + y^2 \), \( y(0) = 1 \). \( \phi_0 = 1 \).

\[
\phi_1(t) = 1 + \int_0^t s^2 + 1 \, ds = 1 + \frac{t^3}{3} + t.
\]

\[
\phi_2(t) = 1 + \int_0^t s^2 + \left( 1 + \frac{s^3}{3} + s \right)^2 \, ds = 1 + \int_0^t s^2 + 1 + \frac{s^6}{9} + s^2 + \frac{2}{3}s^3 + 2s + \frac{2}{3}s^4 \, ds
\]

\[
= 1 + \int_0^t 1 + 2s + 2s^2 + \frac{2}{3}s^3 + \frac{2}{3}s^4 + \frac{s^6}{9} \, ds
\]

\[
= 1 + t + t^2 + \frac{2}{3}t^3 + \frac{t^4}{6} + \frac{2}{15}t^5 + \frac{t^7}{63}.
\]