Mat 244 Assignment 1 Solutions

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1. $y_0 = 2$, $h = 0.1$. $f(t, y) = 5 - 3\sqrt{y}$. I used Sage (http://www.sagenb.org) to store the values of y_0, y_1, y_2, y_3, y_4.

$$y_1 = 2 + (5 - 3\sqrt{2}) \cdot 0.1 = 2.07573593128807.$$

$$y_2 = 2.07573593128807 + (5 - 3\sqrt{2.07573593128807}) \cdot 0.1 = 2.14351349571129.$$

$$y_3 = 2.14351349571129 + (5 - 3\sqrt{2.14351349571129}) \cdot 0.1 = 2.20429121187995.$$

$$y_4 = 2.20429121187995 + (5 - 3\sqrt{2.20429121187995}) \cdot 0.1 = 2.25888554389941.$$

$$y_5 = 2.25888554389941 + (5 - 3\sqrt{2.25888554389941}) \cdot 0.1 = 2.30799786503587.$$

Therefore, using Euler's method with step size $h = 0.1$, our approximation for $y(0.5)$ is

$$2.30799786503587.$$

A good solution doesn't need to have this many decimal places.

2. The equation is

$$y' - \frac{1}{2t}y = -\frac{1}{2} + \frac{t^2}{2}.$$

Multiplying by an integrating factor μ this becomes

$$\mu y' - \mu\frac{1}{2t}y = -\mu\frac{1}{2} + \mu\frac{t^2}{2}.$$

We want

$$\mu' = -\mu\frac{1}{2t},$$

hence

$$\ln\mu = -\frac{1}{2}\ln t = \ln(t^{-1/2}),$$

hence

$$\mu = t^{-1/2}.$$

Therefore

$$(yt^{-1/2})' = -\frac{1}{2}t^{-1/2} + \frac{1}{2}t^{3/2}.$$

Integrating gives

$$yt^{-1/2} = -t^{1/2} + \frac{1}{5}t^{5/2} + C.$$

Since $y(1) = 0$, we have

$$0 = -1 + \frac{1}{5} + C,$$

so

$$C = \frac{4}{5}.$$

Therefore the solution of the initial value problem is

$$y = -t + \frac{1}{5}t^3 + \frac{4}{5}t^{1/2}.$$

3. The equation is

$$\frac{dy}{y} = \frac{4tdt}{t^2+1}.$$

Integrating,

$$\ln y = 2\ln(t^2+1) + C.$$

Using the initial condition $y(0) = 2$ gives

$$\ln 2 = C.$$

Thus,

$$\ln y = \ln((t^2+1)^2) + \ln 2 = \ln(2(t^2+1)^2),$$

so

$$y = 2(t^2+1)^2.$$

4. Here $M = -y - t^2y^2$, $N = -2t - t^3y$. So $M_y = -1 - 2t^2y$ and $N_t = -2 - 3t^2y$.

If we multiply by an integrating factor μ and the equation is then exact, we would have

$$\mu_y M + \mu M_y = \mu_t N + \mu N_t.$$

If $\mu_y = 0$, then

$$\mu(M_y - N_t) = \mu_t N,$$

so

$$\frac{\mu_t}{\mu} = \frac{M_y - N_t}{N} = \frac{-1 - 2t^2y + 2 + 3t^2y}{-2t - t^3y} = \frac{1 + t^2y}{-2t - t^3y},$$

which doesn't work.

If $\mu_t = 0$, then

$$\mu_y M = \mu(N_t - M_y)$$

so

$$\frac{\mu_y}{\mu} = \frac{N_t - M_y}{M} = \frac{-2 - 3t^2y + 1 + 2t^2y}{-y - t^2y^2} = \frac{-1 - t^2y}{-y - t^2y^2} = \frac{1}{y},$$

and integrating we get

$$\mu = y.$$

Multiplying the original equation by y, it becomes

$$-y^2 - t^2y^3 + (-2ty - t^3y^2)y' = 0.$$

So $M = -y^2 - t^2y^3$ and $N = -2ty - t^3y^2$. We want to find ψ such that $\psi_t = M$ and $\psi_y = N$. Then,

$$\psi(t, y) = -ty^2 - \frac{t^3y^3}{3} + g(y).$$

Since $\psi_y = N$, we have

$$-2ty - t^3y^2 + g'(y) = -2ty - t^3y^2.$$

Hence $g'(y) = 0$, and then $g(y) = 0$ (we are trying to find some $\psi(t, y)$ such that if y solves the ODE then $\psi(t, y(t)) = C$, and it turns out that it doesn't matter if we take g to be 0 or any other particular constant.)

In conclusion, if y is a solution of the equation

$$-y - t^2y^2 + (-2t - t^3y)y' = 0,$$

then there is a constant C such that $y(t)$ satisfies

$$-ty^2 - \frac{t^3y^3}{3} = C.$$

5. The IVP is

$$\frac{dy}{y^2} = 2tdt, \qquad y(0) = y_0.$$

If $y_0 = 0$ then the solution is $y(t) = 0$ with domain $\mathbb{R}$.

Otherwise,

$$-\frac{1}{y} = t^2 + C.$$

Using the initial condition,

$$-\frac{1}{y_0} = C.$$

Thus,

$$-\frac{1}{y} = t^2 - \frac{1}{y_0}.$$

So

$$\frac{1}{y} = -t^2 + \frac{1}{y_0},$$

and

$$y = \frac{1}{-t^2 + \frac{1}{y_0}}.$$

If $y_0 < 0$ then the denominator cannot be 0, and the domain of the solution is $\mathbb{R}$.

If $y_0 > 0$, for the denominator to be nonzero we need

$$t \neq \frac{1}{\sqrt{y_0}}.$$

Since the solution must be defined at time 0 (since we are given that $y(0) = y_0$), this implies that

$$|t| < \frac{1}{\sqrt{y_0}},$$

because t cannot be equal to $\frac{1}{\sqrt{y_0}}$ and starts out less than it (i.e. I am excluding the possibility that $t > \frac{1}{\sqrt{y_0}}$). If the solution becomes infinite at a certain time, it doesn't make sense to ask what the value of it is for some later time.

6. $f(t, y) = t^2 + y^2$, $y(0) = 1$. $\phi_0 = 1$.

$$\phi_1(t) = 1 + \int_0^t s^2 + 1 ds = 1 + \frac{t^3}{3} + t.$$

$$\phi_2(t) = 1 + \int_0^t s^2 + \left(1 + \frac{s^3}{3} + s\right)^2 ds = 1 + \int_0^t s^2 + 1 + \frac{s^6}{9} + s^2 + \frac{2}{3}s^3 + 2s + \frac{2}{3}s^4 ds$$

$$= 1 + \int_0^t 1 + 2s + 2s^2 + \frac{2}{3}s^3 + \frac{2}{3}s^4 + \frac{s^6}{9} ds$$

$$= 1 + t + t^2 + \frac{2}{3}t^3 + \frac{t^4}{6} + \frac{2}{15}t^5 + \frac{t^7}{63}.$$