Mat 244 Assignment 2

Due July 10.

1. [1 point] Check that
   
   \[ y(t) = \frac{1}{\lambda} \int_0^t f(s) \sin(\lambda(t-s))ds \]

   is a solution of the initial value problem

   \[ y''(t) + \lambda^2 y(t) = f(t), \quad \lambda > 0, \quad y(0) = 0, y'(0) = 0. \]

2. [2 points] Solve the initial value problem

   \[ y'' + 4y' + 4y = \frac{e^{-2t}}{t^2}, \quad y(1) = 0, y'(1) = 0. \]

3. [3 points] The Wronskian of two functions \( y_1, y_2 \) is defined by

   \[ W(y_1, y_2)(t) = y_1(t)y'_2(t) - y'_1(t)y_2(t). \]

   Let \( Ly = y'' + py' + qy \). Abel’s formula states that if \( Ly_1 = 0 \) and \( Ly_2 = 0 \), then

   \[ W(y_1, y_2)(t) = e^{\int p(t)dt} \cdot k. \]

   Suppose that \( Ly_1 = 0 \). Let

   \[ y_2(t) = y_1(t) \int \frac{1}{y_1(t)^2} e^{\int p(t)dt} dt. \]

   Show that \( Ly_2 = 0 \). (i.e., if \( y_1 \) is a solution and we define \( y_2 \) in the above way, we want to show that \( y_2 \) is a solution.)

   (There may be more than one way to do this and if you find a solution that doesn’t involve Abel’s formula that’s fine; I recommend solving the question by using Abel’s formula which will result in a first order linear ODE which you can then solve by finding an integrating factor.)

4. [2 points] Solve the initial value problem

   \[ (1 - x^2)y'' - 2xy' + 12y = 0, \quad y(0) = 0, y'(0) = 1. \]

   The solution will actually be a polynomial that you can write out.
5. [3 points] Find a power series solution of the initial value problem

\[ y'' + xy' + 2y = 0, \quad y(0) = 0, \quad y'(0) = 1 \]

and write the power series in terms of \( e^x \).

What I mean by writing the power series in terms of \( e^x \): if the solution were

\[ y(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{n!}, \]

we can write this as

\[ y(x) = x^2 \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!} = x^2 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!} = x^2 e^{-x^2}. \]

6. [2 points] Find the terms up to \( x^6 \) of the solution of the initial value problem

\[ y'' - y' + xy = 0, \quad y(0) = 1, \quad y'(0) = 1. \]

(For example, the terms up to \( x^6 \) of \( \cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \) are \( 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} \).)