1. I solved the question in an inefficient way originally, so my suggestion was not especially helpful.

\[ f = yy^3. \]

\[ \frac{\partial f}{\partial y'} = 3yy^2. \]

Since \( f \) doesn’t involve \( x \), the Euler-Lagrange equation can be written as

\[ \frac{\partial f}{\partial y'} y' - f = c_1, \]

i.e.

\[ 3yy^3 - yy^3 = c_1, \]

or

\[ yy'^3 = c_1, \]

so

\[ y'^3 = \frac{c_1}{y}, \]

so

\[ y' = \frac{c_1}{y^{1/3}} \]

This is a separable ODE:

\[ y^{1/3} dy = c_1 dx \]

Integrating,

\[ y^{4/3} \cdot \frac{3}{4} = c_1 x + c_2, \]

or

\[ y^{4/3} = c_1 x + c_2. \]

So

\[ y = (c_1 x + c_2)^{3/4}. \]

\( y(0) = 0 \) so \( c_2 = 0. \) \( y(L) = h, \) so

\[ h = (c_1 L)^{3/4}, \]
hence
\[ c_1 = \frac{h^{4/3}}{L}. \]

Hence
\[ y = \frac{h}{L^{3/4}} x^{3/4}. \]

2. \( f = \frac{\sqrt{1+y'^2}}{x}. \)
\[ \frac{\partial f}{\partial y} = 0, \quad \frac{\partial f}{\partial y'} = \frac{y'}{x\sqrt{1+y'^2}} \]

The Euler-Lagrange equation is
\[ 0 = \frac{d}{dx} x\frac{y'}{\sqrt{1+y'^2}} \]
so
\[ c_1 = \frac{y'}{x\sqrt{1+y'^2}}. \]

Squaring
\[ c_1^2 = \frac{y'^2}{x^2(1+y'^2)} \]

hence
\[ c_1^2 x^2 + c_1^2 x^2 y'^2 = y'^2 \]
so
\[ y'^2(1 - c_1^2 x^2) = c_1^2 x^2 \]
Hence
\[ y' = \frac{c_1 x}{\sqrt{1 - c_1^2 x^2}} \]
Integrating,
\[ y = -\frac{\sqrt{1 - c_1^2 x^2}}{c_1} + c_2. \]

You could stop there and start using the boundary conditions. But circles come up a lot in calculus of variations and this actually is a circle and it’s easier to work with if we manipulate it before substituting:
\[ y_1 - c_2 = -\frac{\sqrt{1 - c_1^2 x^2}}{c_1} \]
so
\[ c_1^2 (y - c_2)^2 = 1 - c_1^2 x^2. \]
Dividing by \( c_1^2 \),
\[ x^2 + (y - c_2)^2 = \frac{1}{c_1^2}. \]
\[ y(1) = 0 \text{ and } y(2) = 1. \] It follows that \( c_2 = 2 \), and then \( c_1 = \frac{1}{\sqrt{5}} \). So

\[ x^2 + (y - 2)^2 = 5. \]

Thus, to satisfy the boundary conditions it follows that the extremal is

\[ y = -\sqrt{5 - x^2} + 2. \]

3. \( F = y'^2 + \lambda y \).

\[ \frac{\partial F}{\partial y} = \lambda, \quad \frac{\partial F}{\partial y'} = 2y'. \]

The Euler-Lagrange equation is

\[ \lambda = \frac{d}{dx}(2y') \]

so

\[ y' = \frac{\lambda}{2} x + c_1. \]

Integrating,

\[ y = \frac{\lambda}{4} x^2 + c_1 x + c_2. \]

As \( y(1) = 3 \) and \( y(4) = 24 \),

\[ 3 = \frac{\lambda}{4} + c_1 + c_2 \]

and

\[ 24 = 4\lambda + 4c_1 + c_2. \]

And \( J(y) = 36 \) implies

\[ \frac{21\lambda}{4} + \frac{15c_1}{2} + 3c_2 = 36. \]

The solution of this system of three equations is

\[ c_1 = 2, c_2 = 0, \lambda = 4. \]

Hence

\[ y = x^2 + 2x. \]

4. The eigenvalues are \(-1 + i\) and \(-1 - i\). The eigenvectors respectively

\[ P = \begin{pmatrix} 1 & 1 \\ 2 - i & 2 + i \end{pmatrix} \text{ and } P^{-1} = \frac{1}{2i} \begin{pmatrix} 2 + i & -1 \\ -2 + i & 1 \end{pmatrix} \text{, and } D = \begin{pmatrix} -1 + i & 0 \\ 0 & -1 - i \end{pmatrix}. \]

\[ A = PDP^{-1} \]
\[ \exp(At) = P \begin{pmatrix} e^{-1+i}t & 0 \\ 0 & e^{-1-i}t \end{pmatrix} P^{-1} = e^{-t} P \begin{pmatrix} \cos t + i \sin t & 0 \\ 0 & \cos t - i \sin t \end{pmatrix} P^{-1}. \]

This takes some work to multiply out. It comes out to the following:

\[ e^{-t} \begin{pmatrix} \cos t + 2 \sin t & -\sin t \\ 5 \sin t & \cos t - 2 \sin t \end{pmatrix} \]

Hence

\[ x(t) = e^{-t} \begin{pmatrix} \cos t + 2 \sin t & -\sin t \\ 5 \sin t & \cos t - 2 \sin t \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = e^{-t} \begin{pmatrix} \cos t + 3 \sin t \\ -\cos t + 7 \sin t \end{pmatrix} \]

**5.** The eigenvalues are \(-1\) and \(-\frac{1}{4}\). The eigenvectors are \(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\) and \(\begin{pmatrix} -4 \\ 3 \end{pmatrix}\).

Since the eigenvalues are distinct and both negative, the phase portrait is a node.

**6. a)** The eigenvalues of the matrix are

\[ \frac{5}{4} \pm \frac{\sqrt{3}}{2} \sqrt{\alpha} \]

b) If \(\alpha\) is negative the matrix has complex eigenvalues and the phase portrait is a spiral. If \(\alpha = 0\), the matrix has a repeated positive root, and the phase portrait is an improper node. If \(\alpha\) is positive, then there are three possibilities: either both eigenvalues are positive (node), one is positive and one is zero (so there will be an equilibrium line, i.e. a line on which there is no movement), or one is positive and one is negative (saddle-point). The tipping point is \(\frac{5}{4} = \frac{\sqrt{3}}{2} \sqrt{\alpha}\), so \(\frac{25}{16} = \frac{3 \alpha}{4}\), i.e. \(\alpha = \frac{25}{12}\).

In summary: the two values of \(\alpha\) where the qualitative behavior of the phase portrait changes are \(\alpha = 0\) and \(\alpha = \frac{25}{12}\).

c) You should draw three phase portraits, one less than \(\alpha = 0\) (\(\alpha = -1\) is a good choice), one greater than 0 and less than \(\frac{25}{12}\) (\(\alpha = 1\) is a good choice), and one greater than \(\frac{25}{12}\) (\(\alpha = 3\) is a good choice). The first will be a spiral, the second will be a node, and the third will be a saddle-point.