Assignment 3 solutions

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1. I solved the question in an inefficient way originally, so my suggestion was not especially helpful.

$f = yy'^3$.

$$\frac{\partial f}{\partial y'} = 3yy'^2.$$

Since f doesn't involve x, the Euler-Lagrange equation can be written as

$$\frac{\partial f}{\partial y'} y' - f = c_1,$$

i.e.

$$3yy'^3 - yy'^3 = c_1,$$

or

$$yy'^3 = c_1,$$

so

$$y'^3 = \frac{c_1}{y}$$

so

$$y' = \frac{c_1}{y^{1/3}}$$

This is a separable ODE:

$$y^{1/3} dy = c_1 dx$$

Integrating,

$$y^{4/3} \cdot \frac{3}{4} = c_1 x + c_2,$$

or

$$y^{4/3} = c_1 x + c_2.$$

So

$$y = (c_1 x + c_2)^{3/4}.$$

$y(0) = 0$ so $c_2 = 0$. $y(L) = h$, so

$$h = (c_1 L)^{3/4},$$

hence

$$c_1 = \frac{h^{4/3}}{L}.$$

Hence

$$y = \frac{h}{L^{3/4}} x^{3/4}.$$

2. $f = \frac{\sqrt{1+y'^2}}{x}$.

$$\frac{\partial f}{\partial y} = 0, \frac{\partial f}{\partial y'} = \frac{y'}{x\sqrt{1+y'^2}}$$

The Euler-Lagrange equation is

$$0 = \frac{d}{dx} \frac{y'}{x\sqrt{1+y'^2}}$$

so

$$c_1 = \frac{y'}{x\sqrt{1+y'^2}}.$$

Squaring

$$c_1^2 = \frac{y'^2}{x^2(1+y'^2)}$$

hence

$$c_1^2 x^2 + c_1^2 x^2 y'^2 = y'^2$$

so

$$y'^2(1 - c_1^2 x^2) = c_1^2 x^2$$

Hence

$$y' = \frac{c_1 x}{\sqrt{1 - c_1^2 x^2}}$$

Integrating,

$$y = -\frac{\sqrt{1 - c_1^2 x^2}}{c_1} + c_2.$$

You could stop there and start using the boundary conditions. But circles come up a lot in calculus of variations and this actually is a circle and it's easier to work with if we manipulate it before substituting:

$$y_1 - c_2 = -\frac{\sqrt{1 - c_1^2 x^2}}{c_1}$$

so

$$c_1^2 (y - c_2)^2 = 1 - c_1^2 x^2.$$

Dividing by c_1^2,

$$x^2 + (y - c_2)^2 = \frac{1}{c_1^2}.$$

$y(1) = 0$ and $y(2) = 1$. It follows that $c_2 = 2$, and then $c_1 = \frac{1}{\sqrt{5}}$. So

$$x^2 + (y-2)^2 = 5.$$

Thus, to satisfy the boundary conditions it follows that the extremal is

$$y = -\sqrt{5 - x^2} + 2.$$

3. $F = y'^2 + \lambda y$.

$$\frac{\partial F}{\partial y} = \lambda, \frac{\partial F}{\partial y'} = 2y'.$$

The Euler-Lagrange equation is

$$\lambda = \frac{d}{dx}(2y')$$

so

$$y' = \frac{\lambda}{2}x + c_1.$$

Integrating,

$$y = \frac{\lambda}{4}x^2 + c_1 x + c_2.$$

As $y(1) = 3$ and $y(4) = 24$,

$$3 = \frac{\lambda}{4} + c_1 + c_2$$

and

$$24 = 4\lambda + 4c_1 + c_2.$$

And $J(y) = 36$ implies

$$\frac{21\lambda}{4} + \frac{15c_1}{2} + 3c_2 = 36.$$

The solution of this system of three equations is

$$c_1 = 2, c_2 = 0, \lambda = 4.$$

Hence

$$y = x^2 + 2x.$$

4. The eigenvalues are $-1 + i$ and $-1 - i$. The eigenvectors respectively

$$\begin{pmatrix} 1 \\ 2-i \end{pmatrix}, \begin{pmatrix} 1 \\ 2+i \end{pmatrix}.$$

$P = \begin{pmatrix} 1 & 1 \\ 2-i & 2+i \end{pmatrix}$ and $P^{-1} = \frac{1}{2i}\begin{pmatrix} 2+i & -1 \\ -2+i & 1 \end{pmatrix}$, and $D = \begin{pmatrix} -1+i & 0 \\ 0 & -1-i \end{pmatrix}$.

$$A = PDP^{-1}$$

so

$$\exp(At) = P \begin{pmatrix} e^{(-1+i)t} & 0 \\ 0 & e^{(-1-i)t} \end{pmatrix} P^{-1} = e^{-t} P \begin{pmatrix} \cos t + i \sin t & 0 \\ 0 & \cos t - i \sin t \end{pmatrix} P^{-1}.$$

This takes some work to multiply out. It comes out to the following:

$$e^{-t} \begin{pmatrix} \cos t + 2 \sin t & -\sin t \\ 5 \sin t & \cos t - 2 \sin t \end{pmatrix}$$

Hence

$$x(t) = e^{-t} \begin{pmatrix} \cos t + 2 \sin t & -\sin t \\ 5 \sin t & \cos t - 2 \sin t \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = e^{-t} \begin{pmatrix} \cos t + 3 \sin t \\ -\cos t + 7 \sin t \end{pmatrix}$$

5. The eigenvalues are -1 and $-\frac{1}{4}$. The eigenvectors are $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -4 \\ 3 \end{pmatrix}$. Since the eigenvalues are distinct and both negative, the phase portrait is a node.

6. a) The eigenvalues of the matrix are

$$\frac{5}{4} \pm \frac{\sqrt{3}}{2} \sqrt{\alpha}$$

b) If α is negative the matrix has complex eigenvalues and the phase portrait is a spiral. If $\alpha = 0$, the matrix has a repeated positive root, and the phase portrait is an improper node. If α is positive, then there are three possibilities: either both eigenvalues are positive (node), one is positive and one is zero (so there will be an equilibrium line, i.e. a line on which there is no movement), or one is positive and one is negative (saddle-point). The tipping point is $\frac{5}{4} = \frac{\sqrt{3}}{2}\sqrt{\alpha}$, so $\frac{25}{16} = \frac{3\alpha}{4}$, i.e. $\alpha = \frac{25}{12}$

In summary: the two values of α where the qualitative behavior of the phase portrait changes are $\alpha = 0$ and $\alpha = \frac{25}{12}$

c) You should draw three phase portraits, one less than $\alpha = 0$ ($\alpha = -1$ is a good choice), one greater than 0 and less than $\frac{25}{12}$ ($\alpha = 1$ is a good choice), and one greater than $\frac{25}{12}$ ($\alpha = 3$ is a good choice). The first will be a spiral, the second will be a node, and the third will be a saddle-point.