UNIVERSITY OF TORONTO

Final exam

Term: Summer 2013

Student ID Information

Last name: _____ **First name:** _____

Student ID #: _____

Course Code:	**Mat 244**
Course Title:	Introduction to Ordinary Differential Equations
Instructor:	Jordan Bell
Date of Test:	
Time Period: Start:	
End:	
Duration of Test:	3 hours
Number of Test Pages: (including this cover sheet)	17 pages
Additional Materials Allowed:	**Scientific calculator**

Grade Table	
Question	Score
1	/5
2	/5
3	/6
4	/6
5	/7
6	/10
7	/6
8	/6
9	/6
10	/9
11	/8
12	/8
13	/12
14	/6
Σ	/100

1. (5 points) Euler's method

For the initial value problem

$$y' = t + y^2, \qquad y(0) = 1,$$

use Euler's method with step size $h = 0.1$ to approximate $y(0.3)$.

2. (5 points) Linear equations

(a) [1 point] If μ is an integrating factor for the differential equation

$$y' + p(t)y = g(t),$$

state what $\mu' y$ is equal to.

(b) [4 points] Solve the initial value problem

$$y' = -\frac{4}{t}y + \frac{1}{t}, \qquad y(1) = \frac{5}{4},$$

and state the domain of the solution.

3. (6 points) Separable equations

For $y_0 \in \mathbb{R}$, solve the initial value problem

$$y' = e^{y-t}, \qquad y(0) = y_0,$$

and state the domain of the solution.

4. (6 points) Exact equations

Solve the initial value problem

$$-y(1+xy)+xy'=0, \qquad y(1)=1,$$

and state the domain of the solution. Get an explicit expression for y.

5. (7 points) Integral equations

(a) [1 point] Write the initial value problem

$$y' = ty, \qquad y(0) = 1$$

as an integral equation.

(b) [2 points] Write the Picard iterates $\phi_0, \phi_1, \phi_2, \phi_3$ for the intial value problem in (a).

(c) [4 points] The sequence of Picard iterates ϕ_n for the initial value problem in (a) converges to a function ϕ. State what ϕ is.

6. (10 points) Variation of parameters

(a) [4 points] Let y_1, y_2 be two solutions of

$$y'' + p(t)y' + q(t)y = 0,$$

and let W be their Wronskian. Derive a differential equation that W satisfies.

(b) [6 points] Solve the initial value problem

$$y'' - 2y' = 12t - 10, \qquad y(0) = 0, y'(0) = 1.$$

If y is the solution of this initial value problem, how does y behave as $t \to \infty$?

7. (6 points) Power series solutions

Find the terms of degree less than or equal to 6 in the power series solution of the initial value problem

$$y'' + x^2 y' + 2y = 0, \qquad y(0) = 1, y'(0) = 1.$$

8. (6 points) Power series solutions

Find the power series solution of the initial value problem

$$y'' + xy = 0, \qquad y(0) = 1, y'(0) = 1.$$

9. (6 points) Calculus of variations

Find the stationary curves of the functional

$$I(y) = \int_{x_1}^{x_2} \left(y^2 - (y')^2 - 2y\cosh x\right) dx.$$

(Fact: $(\cosh x)' = \sinh x, (\sinh x)' = \cosh x$.)

10. (9 points) Calculus of variations

Find the extremal of the functional

$$I(y) = \int_0^{\pi} ((y')^2 - y^2)dx$$

such that $y(0) = 0, y(\pi) = 1$, and

$$\int_0^{\pi} y dx = 1.$$

11. (8 points) Homogeneous systems of equations

Find the general solution of

$$x' = \begin{pmatrix} 1 & -1 \\ 5 & -3 \end{pmatrix} x,$$

and draw a phase portrait for the system.

12. (8 points) Homogeneous systems of equations

Find the general solution of

$$x' = \begin{pmatrix} 2 & \frac{3}{2} \\ -\frac{3}{2} & -1 \end{pmatrix} x,$$

and draw a phase portrait for the system.

13. (12 points) Inhomogeneous systems of equations

(a) [3 points] Fact: If $\Psi(t)$ is a fundamental matrix for the equation $x' = Ax$, then $\Psi' = A\Psi$. Show that

$$x(t) = \Psi(t)\Psi^{-1}(0)x_0 + \Psi(t)\int_0^t \Psi^{-1}(s)g(s)ds$$

is a solution of the initial value problem

$$x'(t) = Ax(t) + g(t), \qquad x(0) = x_0.$$

(b) [9 points] Find the solution of the initial value problem

$$x'(t) = \begin{pmatrix} 2 & -1 \\ -5 & -2 \end{pmatrix} x(t) + \begin{pmatrix} 6e^t \\ -12e^t \end{pmatrix}, \qquad x(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

More space for your solution to the last question. (If you are using this page for rough work or to answer another question make it clear; otherwise I will assume it is for the last question.)

14. (6 points) Matrix exponentials

If A is a square matrix (not necessarily 2×2), the exponential of A is defined by

$$\exp(A) = \sum_{n=0}^{\infty} \frac{A^n}{n!}.$$

Fact: If $D = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ then $\exp(D) = \begin{pmatrix} e^a & 0 \\ 0 & e^b \end{pmatrix}$.

(a) [1 point] Show that if $x(t) = \exp(At)$ then $x(t)$ is a solution of $x' = Ax$.

(b) [1 point] Show that if A is a square matrix (not necessarily 2×2) and $A = P^{-1}DP$, then

$$\exp(At) = P^{-1} \exp(Dt) P.$$

(c) [4 points] Let $A = \begin{pmatrix} 13 & 30 \\ -5 & -12 \end{pmatrix}$. Calculate $\exp(A)$.

More space. If you write here make it very clear what question the work is for. Otherwise I will assume it is rough work and will not mark it.