Final exam  
Term:  Summer 2013

Student ID Information

Last name: ___________________________  First name: ___________________________

Student ID #: ___________________________

Course Code: Mat 244
Course Title: Introduction to Ordinary Differential Equations
Instructor: Jordan Bell
Date of Test: 
Time Period: Start:   
End:   
Duration of Test: 3 hours
Number of Test Pages: 17 pages (including this cover sheet)
Additional Materials Allowed: Scientific calculator

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1. (5 points) Euler’s method
For the initial value problem
\[ y' = t + y^2, \quad y(0) = 1, \]
use Euler’s method with step size \( h = 0.1 \) to approximate \( y(0.3) \).
2. (5 points) Linear equations

(a) [1 point] If $\mu$ is an integrating factor for the differential equation

$$y' + p(t)y = g(t),$$

state what $\mu'y$ is equal to.

(b) [4 points] Solve the initial value problem

$$y' = -\frac{4}{t}y + \frac{1}{t}, \quad y(1) = \frac{5}{4},$$

and state the domain of the solution.
3. (6 points) Separable equations

For \( y_0 \in \mathbb{R} \), solve the initial value problem

\[ y' = e^{y-t}, \quad y(0) = y_0, \]

and state the domain of the solution.
4. (6 points) Exact equations

Solve the initial value problem

\[-y(1 + xy) + xy' = 0, \quad y(1) = 1,\]

and state the domain of the solution. Get an explicit expression for \( y \).
5. (7 points) Integral equations
   
   (a) [1 point] Write the initial value problem
   
   \[ y' = ty, \quad y(0) = 1 \]

   as an integral equation.
   
   (b) [2 points] Write the Picard iterates \( \phi_0, \phi_1, \phi_2, \phi_3 \) for the initial value problem in (a).
   
   (c) [4 points] The sequence of Picard iterates \( \phi_n \) for the initial value problem in (a) converges to a function \( \phi \). State what \( \phi \) is.
6. (10 points) Variation of parameters
   (a) [4 points] Let \( y_1, y_2 \) be two solutions of
   \[
   y'' + p(t)y' + q(t)y = 0,
   \]
   and let \( W \) be their Wronskian. Derive a differential equation that \( W \) satisfies.
   (b) [6 points] Solve the initial value problem
   \[
   y'' - 2y' = 12t - 10, \quad y(0) = 0, y'(0) = 1.
   \]
   If \( y \) is the solution of this initial value problem, how does \( y \) behave as \( t \to \infty \)?
7. (6 points) Power series solutions

Find the terms of degree less than or equal to 6 in the power series solution of the initial value problem

\[ y'' + x^2 y' + 2y = 0, \quad y(0) = 1, y'(0) = 1. \]
8. (6 points) Power series solutions

Find the power series solution of the initial value problem

\[ y'' + xy = 0, \quad y(0) = 1, y'(0) = 1. \]
9. **(6 points) Calculus of variations**

Find the stationary curves of the functional

\[ I(y) = \int_{x_1}^{x_2} \left( y^2 - (y')^2 - 2y \cosh x \right) \, dx. \]

(Fact: \( \cosh x' = \sinh x, (\sinh x)' = \cosh x. \))
10. (9 points) Calculus of variations

Find the extremal of the functional

\[ I(y) = \int_0^\pi ((y')^2 - y^2) \, dx \]

such that \( y(0) = 0 \), \( y(\pi) = 1 \), and

\[ \int_0^\pi y \, dx = 1. \]
11. (8 points) Homogeneous systems of equations

Find the general solution of

\[ x' = \begin{pmatrix} 1 & -1 \\ 5 & -3 \end{pmatrix} x, \]

and draw a phase portrait for the system.
12. (8 points) Homogeneous systems of equations

Find the general solution of

$$x' = \begin{pmatrix} 2 & \frac{3}{2} \\ \frac{3}{2} & -1 \end{pmatrix} x,$$

and draw a phase portrait for the system.
13. **(12 points) Inhomogeneous systems of equations**

(a) [3 points] Fact: If $\Psi(t)$ is a fundamental matrix for the equation $x' = Ax$, then $\Psi' = A\Psi$. Show that

$$x(t) = \Psi(t)\Psi^{-1}(0)x_0 + \Psi(t) \int_0^t \Psi^{-1}(s)g(s)ds$$

is a solution of the initial value problem

$$x'(t) = Ax(t) + g(t), \quad x(0) = x_0.$$

(b) [9 points] Find the solution of the initial value problem

$$x'(t) = \begin{pmatrix} 2 & -1 \\ -5 & 2 \end{pmatrix} x(t) + \begin{pmatrix} 6e^t \\ -12e^t \end{pmatrix}, \quad x(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$
More space for your solution to the last question. (If you are using this page for rough work or to answer another question make it clear; otherwise I will assume it is for the last question.)
14. (6 points) Matrix exponentials

If $A$ is a square matrix (not necessarily $2 \times 2$), the exponential of $A$ is defined by

$$
\exp(A) = \sum_{n=0}^{\infty} \frac{A^n}{n!}.
$$

Fact: If $D = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ then $\exp(D) = \begin{pmatrix} e^a & 0 \\ 0 & e^b \end{pmatrix}$.

(a) [1 point] Show that if $x(t) = \exp(At)$ then $x(t)$ is a solution of $x' = Ax$.

(b) [1 point] Show that if $A$ is a square matrix (not necessarily $2 \times 2$) and $A = P^{-1}DP$, then

$$
\exp(At) = P^{-1}\exp(Dt)P.
$$

(c) [4 points] Let $A = \begin{pmatrix} 13 & 30 \\ -5 & -12 \end{pmatrix}$. Calculate $\exp(A)$. 
More space. If you write here make it very clear what question the work is for. Otherwise I will assume it is rough work and will not mark it.