1. Solve the initial value problem

\[ 3y' + \frac{1}{t}y = 2, \quad y(1) = 2. \]

**Solution** Rewrite as

\[ y' + \frac{1}{3t}y = \frac{2}{3}. \]

Multiply by \( \mu \)

\[ y'\mu + \frac{1}{3t}y\mu = \frac{2}{3}\mu. \]

We want the left hand side to be equal to \((y\mu)' = y'\mu + y\mu'\). So we want

\[ \frac{1}{3t}y\mu = y\mu', \]

hence

\[ \frac{\mu'}{\mu} = \frac{1}{3t}. \]

Integrating,

\[ \ln \mu = \frac{1}{3} \ln t = \ln(t^{1/3}). \]

Hence

\[ \mu = t^{1/3}. \]

Thus

\[ (yt^{1/3})' = \frac{2}{3}t^{1/3}. \]

Integrating,

\[ yt^{1/3} = \frac{1}{2}t^{4/3} + C. \]

As \( y(1) = 2 \), we have

\[ 2 = \frac{1}{2} + C, \]

so \( C = \frac{3}{2} \), and hence the solution is

\[ y = \frac{t}{2} + \frac{3}{2}t^{-1/3}. \]
2. Solve the initial value problem

\[ y' = \frac{1}{2}ty^3(1 + t^2)^{-1/2}, \quad y(0) = 1, \]

and state the domain of the solution.

**Solution** Write this as

\[ \frac{dy}{y^3} = \frac{1}{2}t(1 + t^2)^{-1/2}dt. \]

Integrate both sides to obtain

\[ y^{-2} \frac{1}{-2} = \frac{1}{2} \sqrt{1 + t^2} + C. \]

Then

\[ y^{-2} = -\sqrt{1 + t^2} + K. \]

As \( y(0) = 1, \)

\[ 1 = -1 + K, \]

so \( K = 2. \) Then

\[ y^2 = \frac{1}{-\sqrt{1 + t^2} + 2}, \]

and hence

\[ y = \frac{1}{\sqrt{-1 + t^2 + 2}}. \]

The quantity in the outer square root must be positive, so the domain of the solution is

\[ |t| < \sqrt{3}. \]

3. For the initial value problem

\[ y' = \frac{4 - ty}{1 + y^2}, \quad y(0) = -2. \]

approximate \( y(0.2) \) using Euler’s method with \( h = 0.1. \) Use a calculator.

**Solution** For \( f(t, y) = \frac{4 - ty}{1 + y^2} \) the ODE is \( y'(t) = f(t, y(t)). \) \( t_0 = 0 \)

\( y_0 = -2. \)

\[ y_1 = y_0 + f(t_0, y_0)h = -2 + f(0, -2) \cdot 0.1 = -1.92. \]

\( t_1 = t_0 + h = 0.1. \) Then

\[ y_2 = y_1 + f(t_1, y_1)h = -1.92 + f(0.1, -1.92) \cdot 0.1 = -1.83. \]

\( y_2 \) is the approximation Euler’s method gives for \( y(2 \cdot h) = y(0.2). \) So using Euler’s method with step size \( h = 0.1, \) the approximation for \( y(0.2) \) is

\[ -1.83. \]
4. Write the Picard iterates $\phi_0, \phi_1, \phi_2$ for the initial value problem
\[ y' = t^2 + ty^2, \quad y(0) = 1. \]

**Solution.** Here $f(t, y) = t^2 + ty^2$. $\phi_0 = y(0) = 1$.

\[ \phi_1 = y(0) + \int_0^t f(s, \phi_0) ds = 1 + \int_0^t s^2 + s ds = 1 + \frac{t^3}{3} + \frac{t^2}{2}. \]

Then

\[ \phi_2 = y(0) + \int_0^t f(s, \phi_1) ds \]
\[ = 1 + \int_0^t s^2 + s \left( 1 + \frac{s^3}{3} + \frac{s^2}{2} \right)^2 ds \]
\[ = 1 + \int_0^t s^2 + s \left( 1 + \frac{s^6}{9} + \frac{s^4}{4} + 2 \frac{s^3}{3} + 2 \frac{s^2}{2} + 2 \frac{s^3}{3} \cdot \frac{s^2}{2} \right) ds \]
\[ = 1 + \int_0^t s^2 + s \frac{s^7}{9} + \frac{s^5}{4} + \frac{2}{3} s^4 + s^3 + \frac{s^6}{3} ds \]
\[ = 1 + \frac{t^3}{3} + \frac{t^2}{2} + \frac{t^6}{4} + \frac{2t^5}{15} + \frac{t^4}{4} + \frac{t^7}{21}. \]

5. Solve the initial value problem
\[ 2ty^3 + y^4 + (ty^3 - 2)y' = 0, \quad y(0) = 1. \]

**Solution** Let $M = 2ty^3 + y^4$ and $N = ty^3 - 2$. Then $M_y = 6ty^2 + 4y^3$ and $N_t = y^3$, so the equation is not exact. Multiply by $\mu$:
\[ \mu M + \mu N y' = 0. \]

If the equation were now exact we would have $(\mu M)_y = (\mu N)_t$, i.e.
\[ \mu_y M + \mu M_y = \mu_t N + \mu N_t. \]

If $\mu_y = 0$ then
\[ \mu (M_y - N_t) = \mu_t N, \]
so
\[ \frac{\mu_t}{\mu} = \frac{M_y - N_t}{N} = \frac{6ty^2 + 4y^3 - y^3}{ty^3 - 2} = \frac{6ty^2 + 3y^3}{ty^3 - 2}. \]

We can’t cancel factors to make this just a function of $t$. So say $\mu_t = 0$, then
\[ \mu_y M = \mu (N_t - M_y), \]
and
\[ \frac{\mu_y}{\mu} = \frac{N_t - M_y}{M} = \frac{y^3 - 6ty^2 - 4y^3}{2ty^3 + y^4} = \frac{-6ty^2 - 3y^3}{2ty^3 + y^4} = \frac{-3}{y}. \]
Integrating,
\[ \ln \mu = -3 \ln y = \ln (y^{-3}). \]
Hence \( \mu = y^{-3} \). So multiply the original ODE by \( \mu = y^{-3} \):
\[ 2t + y + (t - 2y^{-3})y' = 0. \]
This is now an exact ODE. Now let \( M = 2t + y \) and \( N = t - 2y^{-3} \). Integrate \( M \) with respect to \( t \) to get
\[ \phi = t^2 + ty + g(y). \]
Since \( \phi_y = N \) we have
\[ t + g'(y) = t - 2y^{-3}, \]
so
\[ g'(y) = -2y^{-3}, \]
and hence
\[ g(y) = y^{-2}. \]
Thus
\[ \phi(t, y) = t^2 + ty + y^{-2}. \]
Thus the solution of the ODE satisfies
\[ t^2 + ty + y^{-2} = C. \]
Since \( y(0) = 1 \),
\[ 1 = C, \]
so the solution satisfies
\[ t^2 + ty + y^{-2} = 1. \]