Mat 244 Practice Test 1 and Solutions

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1. Solve the initial value problem

$$3y' + \frac{1}{t}y = 2, \qquad y(1) = 2.$$

Solution Rewrite as

$$y' + \frac{1}{3t}y = \frac{2}{3}.$$

Multiply by μ

$$y'\mu + \frac{1}{3t}y\mu = \frac{2}{3}\mu.$$

We want the left hand side to be equal to $(y\mu)' = y'\mu + y\mu'$. So we want

$$\frac{1}{3t}y\mu = y\mu',$$

hence

$$\frac{\mu'}{\mu} = \frac{1}{3t}.$$

Integrating,

$$\ln \mu = \frac{1}{3}\ln t = \ln(t^{1/3}).$$

Hence

$$\mu = t^{1/3}.$$

Thus

$$(yt^{1/3})' = \frac{2}{3}t^{1/3}.$$

Integrating,

$$yt^{1/3} = \frac{1}{2}t^{4/3} + C.$$

As $y(1) = 2$, we have

$$2 = \frac{1}{2} + C,$$

so $C = \frac{3}{2}$, and hence the solution is

$$y = \frac{t}{2} + \frac{3}{2}t^{-1/3}.$$

2. Solve the initial value problem

$$y' = \frac{1}{2}ty^3(1+t^2)^{-1/2}, \qquad y(0) = 1,$$

and state the domain of the solution.

Solution Write this as

$$\frac{dy}{y^3} = \frac{1}{2}t(1+t^2)^{-1/2}dt.$$

Integrate both sides to obtain

$$y^{-2}\frac{1}{-2} = \frac{1}{2}\sqrt{1+t^2} + C.$$

Then

$$y^{-2} = -\sqrt{1+t^2} + K.$$

As $y(0) = 1$,

$$1 = -1 + K,$$

so $K = 2$. Then

$$y^2 = \frac{1}{-\sqrt{1+t^2}+2},$$

and hence

$$y = \frac{1}{\sqrt{-\sqrt{1+t^2}+2}}$$

The quantity in the outer square root must be positive, so the domain of the solution is

$$|t| < \sqrt{3}.$$

3. For the initial value problem

$$y' = \frac{4-ty}{1+y^2}, \qquad y(0) = -2.$$

approximate $y(0.2)$ using Euler's method with $h = 0.1$. Use a calculator.

Solution For $f(t,y) = \frac{4-ty}{1+y^2}$ the ODE is $y'(t) = f(t, y(t))$. $t_0 = 0$ and $y_0 = -2$.

$$y_1 = y_0 + f(t_0, y_0)h = -2 + f(0,-2)\cdot 0.1 = -1.92.$$

$t_1 = t_0 + h = 0.1$. Then

$$y_2 = y_1 + f(t_1, y_1)h = -1.92 + f(0.1, -1.92)\cdot 0.1 = -1.83.$$

y_2 is the approximation Euler's method gives for $y(2\cdot h) = y(0.2)$. So using Euler's method with step size $h = 0.1$, the approximation for $y(0.2)$ is

$$-1.83.$$

4. Write the Picard iterates ϕ_0, ϕ_1, ϕ_2 for the initial value problem

$$y' = t^2 + ty^2, \qquad y(0) = 1.$$

Solution. Here $f(t, y) = t^2 + ty^2$. $\phi_0 = y(0) = 1$.

$$\phi_1 = y(0) + \int_0^t f(s, \phi_0) ds = 1 + \int_0^t s^2 + s ds = 1 + \frac{t^3}{3} + \frac{t^2}{2}.$$

Then

$$\begin{aligned}
\phi_2 &= y(0) + \int_0^t f(s, \phi_1) ds \\
&= 1 + \int_0^t s^2 + s\left(1 + \frac{s^3}{3} + \frac{s^2}{2}\right)^2 ds \\
&= 1 + \int_0^t s^2 + s\left(1 + \frac{s^6}{9} + \frac{s^4}{4} + 2\frac{s^3}{3} + 2\frac{s^2}{2} + 2\frac{s^3}{3} \cdot \frac{s^2}{2}\right) ds \\
&= 1 + \int_0^t s^2 + s + \frac{s^7}{9} + \frac{s^5}{4} + \frac{2}{3}s^4 + s^3 + \frac{s^6}{3} ds \\
&= 1 + \frac{t^3}{3} + \frac{t^2}{2} + \frac{t^8}{72} + \frac{t^6}{24} + \frac{2t^5}{15} + \frac{t^4}{4} + \frac{t^7}{21}.
\end{aligned}$$

5. Solve the initial value problem

$$2ty^3 + y^4 + (ty^3 - 2)y' = 0, \qquad y(0) = 1.$$

Solution Let $M = 2ty^3 + y^4$ and $N = ty^3 - 2$. Then $M_y = 6ty^2 + 4y^3$ and $N_t = y^3$, so the equation is not exact. Multiply by μ:

$$\mu M + \mu N y' = 0.$$

If the equation were now exact we would have $(\mu M)_y = (\mu N)_t$, i.e.

$$\mu_y M + \mu M_y = \mu_t N + \mu N_t.$$

If $\mu_y = 0$ then

$$\mu(M_y - N_t) = \mu_t N,$$

so

$$\frac{\mu_t}{\mu} = \frac{M_y - N_t}{N} = \frac{6ty^2 + 4y^3 - y^3}{ty^3 - 2} = \frac{6ty^2 + 3y^3}{ty^3 - 2}.$$

We can't cancel factors to make this just a function of t. So say $\mu_t = 0$, then

$$\mu_y M = \mu(N_t - M_y),$$

and

$$\frac{\mu_y}{\mu} = \frac{N_t - M_y}{M} = \frac{y^3 - 6ty^2 - 4y^3}{2ty^3 + y^4} = \frac{-6ty^2 - 3y^3}{2ty^3 + y^4} = -\frac{3}{y}.$$

Integrating,

$$\ln \mu = -3 \ln y = ln(y^{-3}).$$

Hence $\mu = y^{-3}$. So multiply the original ODE by $\mu = y^{-3}$:

$$2t + y + (t - 2y^{-3})y' = 0.$$

This is now an exact ODE. Now let $M = 2t + y$ and $N = t - 2y^{-3}$. Integrate M with respect to t to get

$$\phi = t^2 + ty + g(y).$$

Since $\phi_y = N$ we have

$$t + g'(y) = t - 2y^{-3},$$

so

$$g'(y) = -2y^{-3},$$

and hence

$$g(y) = y^{-2}.$$

Thus

$$\phi(t, y) = t^2 + ty + y^{-2}.$$

Thus the solution of the ODE satisfies

$$t^2 + ty + y^{-2} = C.$$

Since $y(0) = 1$,

$$1 = C,$$

so the solution satisfies

$$t^2 + ty + y^{-2} = 1.$$