Mat 244 Practice Test 3

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You should be able to do all of the questions in this test given enough time. This is certainly longer than the actual test will be. I wanted to give you more worked out examples for calculus of variations.

1. Find the extremals of the functional

$$\int_{x_1}^{x_2} y^2 + y'^2 - 2y \sin x dx.$$

Solution. $f = y^2 + y'^2 - 2y \sin x$. So

$$\frac{\partial f}{\partial y} = 2y - 2\sin x, \qquad \frac{\partial f}{\partial y'} = 2y'.$$

Hence the Euler-Lagrange equation is

$$2y - 2\sin x = 2y''$$

i.e.

$$y'' - y = -\sin x.$$

This is a second order ODE. The solution of the homogeneous equation is

$$c_1 e^x + c_2 e^{-x}.$$

Now we use variation of parameters. $y_1 = e^x, y_2 = e^{-x}$. The Wronskian is $W = -2$. $g = -\sin x$.

$$u_1 = -\int \frac{g y_2}{W} dx = \frac{-1}{2} \int \sin x e^{-x} dx$$

$$u_2 = \int \frac{g y_1}{W} dx = \frac{1}{2} \int \sin x e^{x} dx$$

To compute u_1, u_2 we need to use integration by parts. I won't write this out; it is something that you should be able to skillfully do if you have sufficient time, and you should have absolutely no problem remembering how to do integration by parts. You don't have to do it in your head; unless it's obvious to me, I manually set u and dv and write out the integration by parts formula.

$$u_1 = \frac{1}{4}e^{-x}(\cos x + \sin x)$$

$$u_2 = \frac{1}{4}e^{x}(-\cos x + \sin x)$$

So the extremals of the functional $\int_{x_1}^{x_2} y^2 + y'^2 - 2y \sin x dx$ are

$$y = c_1 e^x + c_2 e^{-x} + \frac{1}{4}(\cos x + \sin x) + \frac{1}{4}(-\cos x + \sin x)$$

$$= c_1 e^x + c_2 e^{-x} + \frac{\sin x}{2}.$$

2. Find the extremals of the functional

$$I(y) = \int_{x_1}^{x_2} \frac{1 + (y(x))^2}{(y'(x))^2} dx$$

This takes longer than I would want for a test question, but if I gave you enough time I would expect you to be able to do something like this level of difficulty.

Solution. $f = \frac{1+y^2}{y'^2}$. Since f doesn't depend on x, the Euler-Lagrange equation for this functional is

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'} y' - f\right) = 0$$

which can also be written

$$\frac{\partial f}{\partial y'} y' - f = c_1.$$

Now,

$$\frac{\partial f}{\partial y'} = -\frac{2(1+y^2)}{y'^3}.$$

Hence

$$\frac{2(1+y^2)}{y'^2} + \frac{1+y^2}{y'^2} = c_1$$

so

$$2 + 2y^2 + 1 + y^2 = c_1 y'^2$$

so

$$1 + y^2 = c_1 y'^2$$

so (in this step I change c_1, which is logically acceptable untl we have introduced c_2)

$$y' = c_1 \sqrt{1 + y^2}$$

i.e.

$$\frac{dy}{c_1 \sqrt{1 + y^2}} = dx.$$

To integrate the left hand side, we use the substitution $y = \tan\theta$. We get

$$\frac{1}{c_1}\frac{\sec^2\theta d\theta}{\sqrt{\sec^2\theta}} = dx$$

so

$$\frac{1}{c_1}\sec\theta d\theta = dx.$$

The integral of $\sec\theta$ is $\ln(\sec\theta + \tan\theta)$ (I would give this to fact to you). So,

$$\frac{1}{c_1}\ln(\sec\theta + \tan\theta) = x + c_2$$

or (so the new c_2 is the old c_2 times c_1)

$$\ln(\sec\theta + \tan\theta) = c_1 x + c_2$$

What is $\sec\theta$? It is $\sqrt{1+y^2}$. This comes from drawing a right triangle, with angle θ, adjacent side 1, and opposite side y, so hypotenuse is $\sqrt{1+y^2}$.

Thus

$$\ln(\sqrt{1+y^2} + y) = c_1 x + c_2$$

so

$$\sqrt{1+y^2} + y = e^{c_1 x + c_2}$$

so

$$\sqrt{1+y^2} = e^{c_1 x + c_2} - y$$

so squaring,

$$1 + y^2 = e^{2(c_1 x + c_2)} - 2ye^{c_1 x + c_2} + y^2$$

so

$$1 = e^{2(c_1 x + c_2)} - 2ye^{c_1 x + c_2}$$

so

$$y = \frac{1}{2}\left(e^{c_1 x + c_2} - e^{-(c_1 x + c_2)}\right).$$

This is precisely $\sec(c_1 x + c_2)$; but you don't need to remember what sec is, and the previous line would be an acceptable answer.

3. Find the extremal of $I(y) = \int_0^\pi y'^2 dx$, for $y(0) = 0$ and $y(\pi) = 0$, subject to

$$J(y) = \int_0^\pi y^2 dx = 1.$$

Here $f = y'^2$ and $g = y^2$. $F = y'^2 + \lambda y^2$. Even though F doesn't involve x and therefore there is a special form of the Euler-Lagrange equation that we can use, which is often helpful, we will not use

$$\frac{\partial F}{\partial y'}y' - F = c_1,$$

but instead

$$\frac{\partial F}{\partial y} = \frac{d}{dx}\frac{\partial F}{\partial y'}$$

This gives us

$$2\lambda y = \frac{d}{dx}2y'$$

so

$$y'' - \lambda y = 0.$$

Whether or not the roots of the characteristic equation as positive depends on whether λ is positive or negative. But definitely the roots are

$$\pm\sqrt{\lambda}.$$

So the general solution of this ODE is

$$y = c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x}.$$

(λ may be negative, so the square root could be complex)

We know that $y(0) = 0$. So $c_1 = -c_2$, hence

$$y = c_1(e^{\sqrt{\lambda}x} - e^{-\sqrt{\lambda}x}).$$

But we also know that $y(\pi) = 0$. Check the following: if λ is positive then $y(\pi)$ cannot be 0. So λ is negative; say $\lambda = -\mu$ for μ positive. Then

$$y = c_1(e^{\sqrt{-\mu}x} - e^{-\sqrt{-\mu}x}) = c_1(e^{i\sqrt{\mu}x} - e^{-i\sqrt{\mu}x})$$

or

$$y = c\sin(\sqrt{\mu}x).$$

But this can be zero if and only if $\sqrt{\mu}$ is an integer. So

$$y = c\sin(mx),$$

for m a positive integer (it's a positive integer because it is $\sqrt{\mu}$ and by this we mean just the positive square root).

Then we have to use the fact that $J(y) = 1$. So

$$\int_0^\pi c^2 \sin^2(mx)dx = 1$$

i.e.

$$c^2 \int_0^\pi \frac{1}{2} - \frac{1}{2}\cos(2mx)dx = 1$$

so

$$c^2\frac{\pi}{2} - \frac{c^2\sin(2m\pi)}{4m} = 1$$

But $\sin(2m\pi) = 0$, so

$$c^2\frac{\pi}{2} = 1.$$

Hence

$$c = \pm\sqrt{\frac{2}{\pi}}.$$

Therefore the extremals are

$$y = \pm\sqrt{\frac{2}{\pi}}\sin(mx),$$

for m a positive integer.