Solutions of Test 1

Jordan Bell

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1. To have \((y\mu)' = \mu y' - \frac{2}{t}\mu y\), we need \(\mu'\mu = -\frac{2}{t}\mu y\) and so

\[
\frac{\mu'}{\mu} = -\frac{2}{t}.
\]

Integrating,

\[
\ln \mu = -2 \ln t = \ln(t^{-2}).
\]

So \(\mu = t^{-2}\). Since the left hand side of the equation is equal to \((y\mu)'\), we have \((yt^{-2})' = te^t\).

Then we integrate (integrate \(te^t\) by parts) and get

\[
yt^{-2} = te^t - e^t + C.
\]

As \(y(1) = 1\), we have

\[
1 = e^1 - e^1 + C,
\]

so \(C = 1\). Thus the solution is

\[
y = t^3e^t - t^2e^t + Ct^2.
\]

2. \(M = t^5y^5\) and \(N = t^6y^4 + t\). If we want \(\mu M + \mu Ny' = 0\) to be exact, then

\[
(\mu M)_y = (\mu N)_t,
\]

i.e.

\[
\mu_y M + \mu M_y = \mu_N t + \mu N_t.
\]

Try an integrating factor that is only a function of \(t\), so \(\mu_y = 0\). Then

\[
\mu M_y = \mu_N t + \mu N_t,
\]

so

\[
\frac{\mu_t}{\mu} = \frac{M_y - N_t}{N} = \frac{-t^5y^4 - 1}{t^6y^4 + t} = \frac{-1}{t}.
\]

Integrating,

\[
\ln \mu = - \ln t = \ln(t^{-1}).
\]
So $\mu = t^{-1}$. Multiplying by this the equation becomes exact, and our new $M$ and $N$ are $M = t^4y^5$ and $N = t^5y^4 + 1$. Then integrating $M$ with respect to $t$ gives $\psi$,

$$\psi = \frac{t^5y^5}{5} + g(y).$$

As $\psi_y = N$,

$$t^5y^4 + g'(y) = t^5y^4 + 1.$$

Thus

$$g'(y) = 1,$$

and $g(y) = y$. Therefore $\psi = \frac{t^5y^5}{5} + y$. Therefore a solution of the differential equation satisfies

$$\frac{t^5y^5}{5} + y = C.$$

Using the initial condition $y(0) = 1$,

$$0 + 1 = C.$$

So $C = 1$. Hence a solution satisfies

$$\frac{t^5y^5}{5} + y = 1.$$

3. Here $f(t, y) = t - y^2$. The initial value problem written as an integral equation is

$$y(t) = 1 + \int_0^t f(t, y(s))ds = 1 + \int_0^t s - y(s)^2ds.$$

$$\phi_0 = 1.$$

$$\phi_1 = 1 + \int_0^t s - 1 ds = 1 + \frac{t^2}{2} - t.$$

$$\phi_2 = 1 + \int_0^t s - \left(1 + \frac{s^2}{2} - s\right)^2 ds = 1 + \int_0^t s - \left(1 + \frac{s^4}{4} + s^2 - 2s - s^2\right) ds$$

$$= 1 + \int_0^t -1 + 3s - 2s^2 + s^3 - \frac{s^4}{4} ds = 1 - t + \frac{3}{2}t^2 - \frac{2}{3}t^3 + \frac{t^4}{4} - \frac{t^5}{20}.$$  

4. $f(t, y) = t - y^2$, $y_0 = 1$ and $t_0 = 0$. Then,

$$y_1 = y_0 + f(t_0, y_0) \cdot h = 1 + f(0, 1) \cdot 0.1 = 1 - 0.1 = 0.9$$

and $t_1 = t_0 + h = 0 + 0.1 = 0.1$. Next,

$$y_2 = y_1 + f(t_1, y_1) \cdot h = 0.9 + f(0.1, 0.9) \cdot 0.1 = 0.829$$

and $t_2 = t_1 + h = 0.2$. 

2
5. Separating the variables,

\[ 1 + y \frac{dy}{dt} = \frac{3}{2} t^2 \, dt. \]

Integrating,

\[ y + \frac{y^2}{2} = \frac{t^3}{2} + C. \]

Using \( y(0) = 1, \)

\[ 1 + \frac{1}{2} = C, \]

so \( C = \frac{3}{2}. \) Thus

\[ y + \frac{y^2}{2} = \frac{t^3}{2} + \frac{3}{2}, \]

so

\[ 2y + y^2 = t^3 + 3, \]

or

\[ y^2 + 2y - t^3 - 3 = 0. \]

Using the quadratic formula,

\[ y = \frac{-2 \pm \sqrt{4 + 4t^3 + 12}}{2}. \]

(The solution will in fact be the positive one of the two possibilities.) For the expression in the square root to be nonnegative, we need

\[ 16 + 4t^3 \geq 0, \]

or

\[ t^3 \geq -4. \]

So the domain of the solution of the initial value problem is \([-4^{1/3}, \infty), \) which is \((-1.587 \ldots, \infty). \)