Solutions of Test 1

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1. To have $(y\mu)' = \mu y' - \frac{2}{t}\mu y$, we need $\mu' y = -\frac{2}{t}\mu y$ and so

$$\frac{\mu'}{\mu} = -\frac{2}{t}.$$

Integrating,

$$\ln \mu = -2 \ln t = \ln(t^{-2}).$$

So $\mu = t^{-2}$. Since the left hand side of the equation is equal to $(y\mu)'$, we have

$$(yt^{-2})' = te^t.$$

Then we integrate (integrate te^t by parts) and get

$$yt^{-2} = te^t - e^t + C.$$

As $y(1) = 1$, we have

$$1 = e^1 - e^1 + C,$$

so $C = 1$. Thus the solution is

$$y = t^3 e^t - t^2 e^t + Ct^2.$$

2. $M = t^5 y^5$ and $N = t^6 y^4 + t$. If we want $\mu M + \mu N y' = 0$ to be exact, then

$$(\mu M)_y = (\mu N)_t,$$

i.e.

$$\mu_y M + \mu M_y = \mu_t N + \mu N_t.$$

Try an integrating factor that is only a function of t, so $\mu_y = 0$. Then

$$\mu M_y = \mu_t N + \mu N_t,$$

so

$$\frac{\mu_t}{\mu} = \frac{M_y - N_t}{N} = \frac{-t^5 y^4 - 1}{t^6 y^4 + t} = \frac{-1}{t}.$$

Integrating,

$$\ln \mu = -\ln t = \ln(t^{-1}).$$

So $\mu = t^{-1}$. Multiplying by this the equation becomes exact, and our new M and N are $M = t^4y^5$ and $N = t^5y^4 + 1$. Then integrating M with respect to t gives ψ,

$$\psi = \frac{t^5y^5}{5} + g(y).$$

As $\psi_y = N$,

$$t^5y^4 + g'(y) = t^5y^4 + 1.$$

Thus

$$g'(y) = 1,$$

and $g(y) = y$. Therefore $\psi = \frac{t^5y^5}{5} + y$. Therefore a solution of the differential equation satisfies

$$\frac{t^5y^5}{5} + y = C.$$

Using the initial condition $y(0) = 1$,

$$0 + 1 = C.$$

So $C = 1$. Hence a solution satisfies

$$\frac{t^5y^5}{5} + y = 1.$$

3. Here $f(t, y) = t - y^2$. The initial value problem written as an integral equation is

$$y(t) = 1 + \int_0^t f(s, y(s))ds = 1 + \int_0^t s - y(s)^2 ds.$$

$\phi_0 = 1$.

$$\phi_1 = 1 + \int_0^t s - 1ds = 1 + \frac{t^2}{2} - t.$$

$$\phi_2 = 1 + \int_0^t s - \left(1 + \frac{s^2}{2} - s\right)^2 ds = 1 + \int_0^t s - \left(1 + \frac{s^4}{4} + s^2 + s^2 - 2s - s^3\right) ds$$

$$= 1 + \int_0^t -1 + 3s - 2s^2 + s^3 - \frac{s^4}{4} ds = 1 - t + \frac{3}{2}t^2 - \frac{2}{3}t^3 + \frac{t^4}{4} - \frac{t^5}{20}.$$

4. $f(t, y) = t - y^2$. $y_0 = 1$ and $t_0 = 0$. Then,

$$y_1 = y_0 + f(t_0, y_0) \cdot h = 1 + f(0, 1) \cdot 0.1 = 1 - 0.1 = 0.9$$

and $t_1 = t_0 + h = 0 + 0.1 = 0.1$. Next,

$$y_2 = y_1 + f(t_1, y_1) \cdot h = 0.9 + f(0.1, 0.9) \cdot 0.1 = 0.829$$

and $t_2 = t_1 + h = 0.2$.

5. Separating the variables,

$$1 + ydy = \frac{3}{2}t^2 dt.$$

Integrating,

$$y + \frac{y^2}{2} = \frac{t^3}{2} + C.$$

Using $y(0) = 1$,

$$1 + \frac{1}{2} = C,$$

so $C = \frac{3}{2}$. Thus

$$y + \frac{y^2}{2} = \frac{t^3}{2} + \frac{3}{2},$$

so

$$2y + y^2 = t^3 + 3,$$

or

$$y^2 + 2y - t^3 - 3 = 0.$$

Using the quadratic formula,

$$y = \frac{-2 \pm \sqrt{4 + 4t^3 + 12}}{2}.$$

(The solution will in fact be the positive one of the two possibilities.) For the expression in the square root to be nonnegative, we need

$$16 + 4t^3 \geq 0,$$

or

$$t^3 \geq -4.$$

So the domain of the solution of the initial value problem is $[-4^{1/3}, \infty)$, which is $(-1.587\ldots, \infty)$.