Solutions of Test 2

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1. The characteristic equation is $r^2 + 3r - 10 = 0$, and solving gives $r = -5, 2$. So solutions of the homogeneous equation are $y_1 = e^{-5t}$ and $y_2 = e^{2t}$. Calculate the Wronskian: we get $W = 7e^{-3t}$.

To find $u_1$ we’ll end up doing integration by parts (going from third to fourth line).

$$u_1 = -\int \frac{y_2 g}{W} dt = -\int \frac{e^{2t} 7te^t}{7e^{-3t}} dt = -\int te^{6t} dt = -\frac{te^{6t}}{6} + \frac{e^{6t}}{36}.$$  

To find $u_2$ we’ll end up doing integration by parts (going from third to fourth line).

$$u_2 = \int \frac{y_1 g}{W} dt = \int \frac{e^{-5t} 7te^t}{7e^{-3t}} dt = \int te^{-t} dt = -te^{-t} - e^{-t}.$$  

Thus we can write the solution of the initial value problem as

$$y = c_1 e^{-5t} + c_2 e^{2t} + e^{-5t} \left( -\frac{te^t}{6} + \frac{e^{6t}}{36} \right) + e^{2t} \left( -te^{-t} - e^{-t} \right) = c_1 e^{-5t} + c_2 e^{2t} - \frac{7}{6} te^t - \frac{35}{36} e^t.$$  

If we want to choose $c_1$ and $c_2$ to satisfy the initial conditions, we’ll need to evaluate $y(0)$ and $y'(0).$
\[ y(0) = c_1 + c_2 - \frac{35}{36}, \]
\[ y'(t) = -5c_1 e^{-5t} + 2c_2 e^{2t} - \frac{7}{6} e^t - \frac{7}{6} t e^t - \frac{35}{36} e^t, \]
\[ y'(0) = -5c_1 + 2c_2 - \frac{7}{6} - \frac{35}{36} \]

Using the initial conditions we get \( c_1 = \frac{-2}{7} \) and \( c_2 = \frac{2}{7} \). Thus the solution of the initial value problem is
\[ y(t) = -\frac{2}{7} e^{-5t} + \frac{2}{7} e^{2t} - \frac{7}{6} t e^t - \frac{35}{36} e^t. \]

2. The first equation gives
\[ u_1' = -\frac{y_2}{y_1} u_2'. \]

Putting this into the second equation gives
\[ -y_1 y_2 u_2' + y_1' y_2 u_2 = g. \]

Multiplying by \( y_1 \) gives
\[ -y_1 y_2 u_2' + y_1 y_2' u_2 = gy_1, \]

i.e.,
\[ u_2'(y_1 y_2' - y_1' y_2) = gy_1, \]
or,
\[ u_2' W = gy_1. \]

Thus
\[ u_2 = \int \frac{gy_1}{W} dt. \]

Since \( u_2' W = gy_1 \), using \( u_1' = -\frac{y_2}{y_1} u_2' \) gives us
\[ u_1' = -\frac{y_2}{y_1} \frac{gy_1}{W} = -\frac{gy_2}{W}, \]
and thus
\[ u_1 = -\int \frac{gy_2}{W} dt. \]

3. Use the fact that \( y_2'' = -p y_2' - q y_2 \) and \( y_1'' = -p y_1' - q y_1 \). Then
\[ \frac{y_1 y_2'' - y_1'' y_2}{W} = \frac{y_1(-p y_2' - q y_2) - (-p y_1' - q y_1)y_2}{W} \]
\[ = -\frac{-p y_1 y_2' - q y_1 y_2 + p y_1' y_2 + q y_1 y_2}{W} = \frac{p y_1 y_2' - p y_1 y_2}{W} = p. \]
And
\[ \frac{y_1'y_2' - y_1'y_2'}{W} = \frac{y_1'(-py_2' - qy_2) - (-py_1' - qy_1)y_2'}{W} = \frac{-py_1'y_2' - qy_1'y_2 + py_1'y_2' + qy_1'y_2'}{W} = \frac{qW}{W} = q. \]

4. (a) \[ L(f + g) = (f + g)'' + p(f + g)' + q(f + g) \]
\[ = f'' + g'' + pf' + qf + bg' + qg \]
\[ = Lf + LG. \]

(b) \( y = y_1 + iy_2 \). On the one hand, \( Ly = 0 = 0 + 0i \). On the other hand, \( Ly = L(y_1 + iy_2) = Ly_1 + iLiy_2 \).

Since \( Ly_1 \) is real and \( Ly_2 \) is real, it follows that \( Ly_1 = 0 \) and \( Ly_2 = 0 \). (Since \( y_1 \) and \( y_2 \) are the real and imaginary parts of something, they are both real valued, and \( p \) and \( q \) are both real valued so \( L \) applied to a real valued function gives a real valued function.)

5. Let \( y = \sum_{n=0}^{\infty} a_n x^n \). The differential equation is
\[ \sum_{n=0}^{\infty} n(n - 1)a_n x^{n-2} - 2 \sum_{n=0}^{\infty} na_n x^n + 14 \sum_{n=0}^{\infty} a_n = 0. \]

This can be written as
\[ \sum_{n=0}^{\infty} (n + 2)(n + 1)a_{n+2} x^n - 2 \sum_{n=0}^{\infty} na_n x^n + 14 \sum_{n=0}^{\infty} a_n = 0, \]

hence as
\[ \sum_{n=0}^{\infty} ((n + 2)(n + 1)a_{n+2} - 2na_n + 14a_n) x^n = 0. \]

Therefore, for \( n \geq 0 \) we have
\[ (n + 2)(n + 1)a_{n+2} - 2na_n + 14a_n = 0, \]
or,
\[ a_{n+2} = \frac{2(n - 7)}{(n + 2)(n + 1)} a_n. \]

As \( y(0) = 0 \), \( a_0 = 0 \), and since the recurrence relation moves up by twos, \( a_2 = 0 \), and \( a_4 = 0 \), etc.

As \( y'(0) = 1 \), \( a_1 = 1 \), and so
\[ a_3 = \frac{2(-6)}{3 \cdot 2} a_1 = -2. \]
\begin{align*}
a_5 &= \frac{2(-4)}{5 \cdot 4} \cdot a_3 = \frac{-8}{20} \cdot -2 = \frac{4}{5} \\
a_7 &= \frac{2(-2)}{7 \cdot 6} \cdot a_5 = \frac{-4}{42} \cdot \frac{4}{5} = \frac{-16}{210} = \frac{-8}{105}.
\end{align*}

From then on all the coefficients are 0, because there will be a $7 - 7$ factor in $a_9$ so $a_9$ is 0 and thus all the further odd terms are 0.

Therefore
\[ y(x) = x - 2x^3 + \frac{4}{5}x^5 - \frac{8}{105}x^7. \]

6. (a) We use $\frac{1}{1-r} = \sum_{n=0}^{\infty} r^n$.

\[ \frac{1}{1+x^3} = \sum_{n=0}^{\infty} (-x^3)^n. \]

(b) To find the terms of degree \( \leq 10 \) in $\frac{1}{1+x^3}$, we take the terms of degree \( \leq 10 \) in $\frac{1}{1+x^3}$ and then square it, but don’t bother to write terms that are higher than degree 10:

\[ (1 - x^3 + x^6 - x^9)(1 - x^3 + x^6 - x^9) = 1 - x^3 + x^6 - x^9 - x^3 + x^6 - x^9 + x^6 - x^9 - x^3 + O(x^{11}). \]

Collecting, the terms of degree \( \leq 10 \) in $\frac{1}{1+x^3}$ are

\[ 1 - 2x^3 + 3x^6 - 4x^9. \]

(c) The radius of convergence is the distance from 0 to the nearest root of the denominator. The denominator is $x^2 + 2x - 3 = (x + 3)(x - 1)$, so the roots are $x = -3, 1$. The distances of these roots to 0 are 3 and 1, hence the radius of convergence is 1.

7. (a) The characteristic equation is

\[ r^2 - (2\alpha - 1)r + \alpha(\alpha - 1) = 0. \]

The roots of this are

\[ r = \frac{2\alpha - 1 \pm \sqrt{4\alpha^2 - 4\alpha + 1 - 4\alpha(\alpha - 1)}}{2} = \frac{2\alpha - 1 \pm 1}{2} = \alpha - 1, \alpha. \]

If $\alpha < 0$, then both roots are negative, and hence any solution of the homogeneous ODE tends to 0 as $t \to \infty$. (And for $\alpha \geq 0$: one solution of the equation is $e^{\alpha t}$, and if $\alpha \geq 0$ then this does not tend to 0 as $t \to \infty$.)

(b) Let $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$, $y' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots$.

\[ e^x = 1 + x + \frac{x^2}{2} + \cdots, \] so $e^{\alpha x} = 1 + x^2 + \frac{x^4}{2} + \cdots$. 
So the equation is
\[ \sum_{i=1}^{\infty} a_i x^i = (a_0 + a_1 x + \frac{a_2 x^2}{2} + \cdots) \left(1 + x^2 + \frac{x^4}{2} + \cdots\right) \]

We can write this as (I don’t bother to write any terms higher than \(x^4\))
\[ \sum_{i=1}^{\infty} a_i x^i = a_0 + a_1 x + \frac{a_2 x^2}{2} + \frac{a_3 x^3}{3} + \frac{a_4 x^4}{4} + \cdots \]

Comparing powers of \(x\), we get
\[ a_1 = a_0 \]
\[ 2a_2 = a_1 \]
\[ 3a_3 = a_2 + a_0 \]
\[ 4a_4 = a_3 + a_1 \]

Working these out we get
\[ a_2 = \frac{a_0}{2}, \quad a_3 = \frac{a_0}{2}, \quad a_4 = \frac{3a_0}{8} \]

Since \(y(0) = 1\), \(a_0 = 1\). Thus,
\[ y = 1 + x + \frac{x^2}{2} + \frac{x^3}{2} + \frac{3x^4}{8} + \cdots \]