Solutions of Test 2

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1. The characteristic equation is $r^2+3r-10=0$, and solving gives $r=-5,2$. So solutions of the homogeneous equation are $y_1=e^{-5t}$ and $y_2=e^{2t}$. Calculate the Wronskian: we get $W=7e^{-3t}$.

To find u_1 we'll end up doing integration by parts (going from third to fourth line).

$$\begin{aligned} u_1 &= -\int \frac{y_2 g}{W} dt \\ &= -\int \frac{e^{2t} 7te^t}{7e^{-3t}} dt \\ &= -\int te^{6t} dt \\ &= -\frac{te^{6t}}{6} + \frac{e^{6t}}{36}. \end{aligned}$$

To find u_2 we'll end up doing integration by parts (going from third to fourth line).

$$\begin{aligned} u_2 &= \int \frac{y_1 g}{W} dt \\ &= \int \frac{e^{-5t} 7te^t}{7e^{-3t}} dt \\ &= \int te^{-t} dt \\ &= -te^{-t} - e^{-t}. \end{aligned}$$

Thus we can write the solution of the initial value problem as

$$\begin{aligned} y &= c_1 e^{-5t} + c_2 e^{2t} + e^{-5t}\left(-\frac{te^{6t}}{6} + \frac{e^{6t}}{36}\right) + e^{2t}\left(-te^{-t} - e^{-t}\right) \\ &= c_1 e^{-5t} + c_2 e^{2t} - \frac{7}{6} te^t - \frac{35}{36} e^t. \end{aligned}$$

If we want to choose c_1 and c_2 to satisfy the initial conditions, we'll need to evaluate $y(0)$ and $y'(0)$.

$$y(0) = c_1 + c_2 - \frac{35}{36}.$$

$$y'(t) = -5c_1e^{-5t} + 2c_2e^{2t} - \frac{7}{6}e^t - \frac{7}{6}te^t - \frac{35}{36}e^t$$

$$y'(0) = -5c_1 + 2c_2 - \frac{7}{6} - \frac{35}{36}$$

Using the initial conditions we get $c_1 = -\frac{2}{7}$ and $c_2 = \frac{2}{7}$. Thus the solution of the initial value problem is

$$y(t) = -\frac{2}{7}e^{-5t} + \frac{2}{7}e^{2t} - \frac{7}{6}te^t - \frac{35}{36}e^t.$$

2. The first equation gives

$$u_1' = -\frac{y_2}{y_1}u_2'.$$

Putting this into the second equation gives

$$-y_1'\frac{y_2}{y_1}u_2' + y_2'u_2' = g.$$

Multiplying by y_1 gives

$$-y_1'y_2u_2' + y_1y_2'u_2' = gy_1,$$

i.e.,

$$u_2'(y_1y_2' - y_1'y_2) = gy_1,$$

or,

$$u_2'W = gy_1.$$

Thus

$$u_2 = \int \frac{gy_1}{W}dt.$$

Since $u_2'W = gy_1$, using $u_1' = -\frac{y_2}{y_1}u_2'$ gives us

$$u_1' = -\frac{y_2}{y_1}\frac{gy_1}{W} = -\frac{gy_2}{W},$$

and thus

$$u_1 = -\int \frac{gy_2}{W}dt.$$

3. Use the fact that $y_2'' = -py_2' - qy_2$ and $y_1'' = -py_1' - qy_1$. Then

$$-\frac{y_1y_2'' - y_1''y_2}{W} = -\frac{y_1(-py_2' - qy_2) - (-py_1' - qy_1)y_2}{W}$$

$$= -\frac{-py_1y_2' - qy_1y_2 + py_1'y_2 + qy_1y_2}{W} = \frac{py_1y_2' - py_1'y_2}{W} = p.$$

And

$$\frac{y_1'y_2''-y_1''y_2'}{W}=\frac{y_1'(-py_2'-qy_2)-(-py_1'-qy_1)y_2'}{W}$$
$$=\frac{-py_1'y_2'-qy_1'y_2+py_1'y_2'+qy_1y_2'}{W}=\frac{qW}{W}=q.$$

4. (a)

$$\begin{aligned}L(f+g) &= (f+g)''+p(f+g)'+q(f+g)\\ &= f''+g''+p(f'+g')+qf+qg\\ &= f''+pf'+qf+g''+pg'+qg\\ &= Lf+Lg.\end{aligned}$$

(b) $y=y_1+iy_2$. On the one hand, $Ly=0=0+0i$. On the other hand,

$$Ly=L(y_1+iy_2)=Ly_1+iLy_2.$$

Since Ly_1 is real and Ly_2 is real, it follows that $Ly_1=0$ and $Ly_2=0$. (Since y_1 and y_2 are the real and imaginary parts of something, they are both real valued, and p and q are both real valued so L applied to a real valued function gives a real valued function.)

5. Let $y=\sum_{n=0}^{\infty}a_nx^n$. The differential equation is

$$\sum_{n=0}^{\infty}n(n-1)a_nx^{n-2}-2\sum_{n=0}^{\infty}na_nx^n+14\sum_{n=0}^{\infty}a_n=0.$$

This can be written as

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-2\sum_{n=0}^{\infty}na_nx^n+14\sum_{n=0}^{\infty}a_n=0,$$

hence as

$$\sum_{n=0}^{\infty}\left((n+2)(n+1)a_{n+2}-2na_n+14a_n\right)x^n=0.$$

Therefore, for $n\geq 0$ we have

$$(n+2)(n+1)a_{n+2}-2na_n+14a_n=0,$$

or,

$$a_{n+2}=\frac{2(n-7)}{(n+2)(n+1)}a_n.$$

As $y(0)=0$, $a_0=0$, and since the recurrence relation moves up by twos, $a_2=0$, and $a_4=0$, etc.

As $y'(0)=1$, $a_1=1$, and so

$$a_3=\frac{2(-6)}{3\cdot 2}a_1=-2.$$

$$a_5 = \frac{2(-4)}{5 \cdot 4} \cdot a_3 = \frac{-8}{20} \cdot -2 = \frac{4}{5}.$$

$$a_7 = \frac{2(-2)}{7 \cdot 6} \cdot a_5 = \frac{-4}{42} \cdot \frac{4}{5} = \frac{-16}{210} = \frac{-8}{105}.$$

From then on all the coefficients are 0, because there there will be a $7-7$ factor in a_9 so a_9 is 0 and thus all the further odd terms are 0.

Therefore

$$y(x) = x - 2x^3 + \frac{4}{5}x^5 - \frac{8}{105}x^7.$$

6. (a) We use $\frac{1}{1-r} = \sum_{n=0}^{\infty} r^n$.

$$\frac{1}{1+x^3} = \sum_{n=0}^{\infty} (-x^3)^n.$$

(b) To find the terms of degree ≤ 10 in $\frac{1}{(1+x^3)^2}$, we take the terms of degree ≤ 10 in $\frac{1}{1+x^3}$ and then square it, but don't bother to write terms that are higher than degree 10:

$$\begin{aligned} \left(1 - x^3 + x^6 - x^9\right)\left(1 - x^3 + x^6 - x^9\right) &= 1 - x^3 + x^6 - x^9 \\ & \quad -x^3 + x^6 - x^9 \\ & \quad +x^6 - x^9 \\ & \quad -x^9 + O(x^{11}). \end{aligned}$$

Collecting, the terms of degree ≤ 10 in $\frac{1}{(1+x^3)^2}$ are

$$1 - 2x^3 + 3x^6 - 4x^9.$$

(c) The radius of convergence is the distance from 0 to the nearest root of the denominator. The denominator is $x^2 + 2x - 3 = (x+3)(x-1)$, so the roots are $x = -3, 1$. The distances of these roots to 0 are 3 and 1, hence the radius of convergence is 1.

7. (a) The characteristic equation is

$$r^2 - (2\alpha - 1)r + \alpha(\alpha - 1) = 0.$$

The roots of this are

$$r = \frac{2\alpha - 1 \pm \sqrt{4\alpha^2 - 4\alpha + 1 - 4\alpha(\alpha - 1)}}{2} = \frac{2\alpha - 1 \pm 1}{2} = \alpha - 1, \alpha.$$

If $\alpha < 0$, then both roots are negative, and hence any solution of the homogeneous ODE tends to 0 as $t \to \infty$. (And for$\alpha \geq 0$: one solution of the equation is $e^{\alpha t}$, and if $\alpha \geq 0$ then this does not tend to 0 as $t \to \infty$.)

(b) Let $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$, $y' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots$.

$e^x = 1 + x + \frac{x^2}{2} + \cdots$, so $e^{x^2} = 1 + x^2 + \frac{x^4}{2} + \cdots$.

So the equation is

$$a_1+2a_2x+3a_3x^2+4a_4x^3+5a_5x^4+\cdots = \left(1+x^2+\frac{x^4}{2}+\cdots\right)\left(a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+\cdots\right)$$

We can write this as (I don't bother to write any terms higher than x^4)

$$a_1+2a_2x+3a_3x^2+4a_4x^3+5a_5x^4+\cdots = a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+a_0x^2+a_1x^3+a_2x^4+\frac{a_0}{2}x^4+\cdots$$

Comparing powers of x, we get

$$a_1 = a_0$$

$$2a_2 = a_1$$

$$3a_3 = a_2 + a_0$$

$$4a_4 = a_3 + a_1$$

Working these out we get

$$a_2 = \frac{a_0}{2}, a_3 = \frac{a_0}{2}, a_4 = \frac{3a_0}{8}.$$

Since $y(0) = 1$, $a_0 = 1$. Thus,

$$y = 1 + x + \frac{x^2}{2} + \frac{x^3}{2} + \frac{3x^4}{8} + \cdots .$$