Solutions of Test 3

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1. \( f = \frac{y^2}{x^2} \). Then

\[
\frac{\partial f}{\partial y} = 0, \quad \frac{\partial f}{\partial y'} = \frac{2y'}{x^3}.
\]

Hence the Euler-Lagrange equation is

\[
0 = \frac{d}{dx} \left( \frac{2y'}{x^3} \right),
\]

so

\[
\frac{2y'}{x^3} = c_1.
\]

This is a separable equation:

\[
2dy = c_1 x^3 dx.
\]

Integrating both sides gives

\[
2y = \frac{c_1 x^4}{4} + c_2,
\]

or

\[
y = c_1 x^4 + c_2.
\]

Using \( y(0) = 1 \) and \( y(1) = -1 \) gives us \( c_1 = 1 \) and \( c_2 = -2 \). Hence

\[
y = -2x^4 + 1.
\]

2. \( F = y^2 - y^2 + \lambda y \).

\[
\frac{\partial F}{\partial y} = -2y + \lambda, \quad \frac{\partial F}{\partial y'} = 2y'
\]

\[-2y + \lambda = 2y'' \text{ so } 2y'' + 2y = \lambda
\]

If you are lucky you can solve this by looking at it (if you remember that \( \cos \) and \( \sin \) solve \( y'' + y = 0 \)). But you can solve this by finding the homogeneous solutions and then using variation of parameters also.

\[
y = c_1 \cos x + c_2 \sin x + \frac{\lambda}{2}
\]

Since \( y(0) = 0 \) and \( y(\pi) = 1 \), we get \( c_1 + \frac{\lambda}{2} = 0 \) and \(-c_1 + \frac{\lambda}{2} = 1\). This gives us \( \lambda = 1 \) and so \( c_1 = -\frac{1}{2} \).

So \( y = -\frac{\cos x}{2} + c_2 \sin x + \frac{1}{2} \). Now we use the final condition to figure out \( c_2 \).
\[
\int_{0}^{\pi} y \, dx = 2c_2 + \frac{\pi}{2}, \quad \text{(you aren’t expected to be see this just in your head, it takes two lines that I didn’t type up)}
\]
So \(2c_2 + \frac{\pi}{2} = 1\), hence \(c_2 = \frac{1}{2} - \frac{\pi}{4}\).

Hence
\[
y = -\cos x + \left( \frac{1}{2} - \frac{\pi}{4} \right) \sin x + \frac{1}{2}.
\]

3. The eigenvalues of \(A\) are 1 and 2. The eigenvector for the eigenvalue 1 is \((-2)\) and the eigenvector for the eigenvalue 2 is \((1\ -1)\). Thus
\[
A = PDP^{-1}
\]
where \(P = \begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix}, \ D = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \) and \(P^{-1} = \begin{pmatrix} -1 & -1 \\ -1 & -2 \end{pmatrix}.\)

Then
\[
\exp(A) = P \exp(D)P^{-1} = \begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} e & 0 \\ 0 & e^2 \end{pmatrix} \begin{pmatrix} -1 & -1 \\ -1 & -2 \end{pmatrix},
\]
which is equal to
\[
\exp(A) = \begin{pmatrix} 2e - e^2 & 2e - 2e^2 \\ -e + e^2 & -e + 2e^2 \end{pmatrix}.
\]

4. \(A = \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}\). The eigenvalues of \(A\) are 2, 1, and the eigenvectors are \((0)\) and \((1\ 1)\). The general solution is
\[
x = c_1 e^{2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + c_2 e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix}.
\]

Since both eigenvalues are positive, the origin is a node. I will soon upload a scan of the phase portrait.

5. \(A = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix}\). The eigenvalues of \(A\) are \(-1 \pm 2i\). Thus we know that the phase portrait is a spiral going towards the origin. What is the direction of rotation of it? Test the point \((1\ 0)\).

\[
A \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.
\]

Thus, at the point \((1\ 0)\), the tangent vector to the spiral is \((-1\ 1)\). This implies that the spiral is rotating counter-clockwise. I will soon upload a scan of the phase portrait.
But we still have to find the general solution. The eigenvector for \(-1 + 2i\) is 
\(\begin{pmatrix} 2i \\ 1 \end{pmatrix}\). So one solution of \(x' = Ax\) is

\[ e^{(-1+2i)t} \begin{pmatrix} 2i \\ 1 \end{pmatrix} = e^{-t}(\cos(2t) + i \sin(2t)) \begin{pmatrix} 2i \\ 1 \end{pmatrix} = e^{-t} \begin{pmatrix} -2 \sin(2t) + 2i \cos(2t) \\ \cos(2t) + i \sin(2t) \end{pmatrix}. \]

The real part of this is \(e^{-t} \begin{pmatrix} -2 \sin(2t) \\ \cos(2t) \end{pmatrix}\) and the imaginary part of this is \(e^{-t} \begin{pmatrix} 2 \cos(2t) \\ \sin(2t) \end{pmatrix}\). Thus the general solution is

\[ x = c_1 e^{-t} \begin{pmatrix} -2 \sin(2t) \\ \cos(2t) \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 2 \cos(2t) \\ \sin(2t) \end{pmatrix}. \]