Solutions of Test 3

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1. $f = \frac{y'^2}{x^3}$. Then

$$\frac{\partial f}{\partial y} = 0, \qquad \frac{\partial f}{\partial y'} = \frac{2y'}{x^3}.$$

Hence the Euler-Lagrange equation is

$$0 = \frac{d}{dx}\left(\frac{2y'}{x^3}\right),$$

so

$$\frac{2y'}{x^3} = c_1.$$

This is a separable equation:

$$2dy = c_1 x^3 dx.$$

Integrating both sides gives

$$2y = \frac{c_1 x^4}{4} + c_2,$$

or

$$y = c_1 x^4 + c_2.$$

Using $y(0) = 1$ and $y(1) = -1$ gives us $c_1 = 1$ and $c_2 = -2$. Hence

$$y = -2x^4 + 1.$$

2. $F = y'^2 - y^2 + \lambda y$.
$\frac{\partial F}{\partial y} = -2y + \lambda$, $\frac{\partial F}{\partial y'} = 2y'$
$-2y + \lambda = 2y''$ so $2y'' + 2y = \lambda$

If you are lucky you can solve this by looking at it (if you remember that cos and sin solve $y'' + y = 0$). But you can solve this by finding the homogeneous solutions and then using variation of parameters also.

$y = c_1 \cos x + c_2 \sin x + \frac{\lambda}{2}$

Since $y(0) = 0$ and $y(\pi) = 1$, we get $c_1 + \frac{\lambda}{2} = 0$ and $-c_1 + \frac{\lambda}{2} = 1$. This gives us $\lambda = 1$ and so $c_1 = -\frac{1}{2}$.

So $y = -\frac{\cos x}{2} + c_2 \sin x + \frac{1}{2}$. Now we use the final condition to figure out c_2.

$\int_0^\pi y dx = 2c_2 + \frac{\pi}{2}$ (you aren't expected to be see this just in your head, it takes two lines that I didn't type up)

So $2c_2 + \frac{\pi}{2} = 1$, hence $c_2 = \frac{1}{2} - \frac{\pi}{4}$.

Hence

$$y = -\cos x + \left(\frac{1}{2} - \frac{\pi}{4}\right)\sin x + \frac{1}{2}.$$

3. The eigenvalues of A are 1 and 2. The eigenvector for the eigenvalue 1 is $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ and the eigenvector for the eigenvalue 2 is $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Thus

$$A = PDP^{-1}$$

where $P = \begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix}$, $D = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, and $P^{-1} = \begin{pmatrix} -1 & -1 \\ -1 & -2 \end{pmatrix}$.

Then

$$\exp(A) = P\exp(D)P^{-1} = \begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} e & 0 \\ 0 & e^2 \end{pmatrix}\begin{pmatrix} -1 & -1 \\ -1 & -2 \end{pmatrix},$$

which is equal to

$$\exp(A) = \begin{pmatrix} 2e - e^2 & 2e - 2e^2 \\ -e + e^2 & -e + 2e^2 \end{pmatrix}.$$

4. $A = \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}$. The eigenvalues of A are 2, 1, and the eigenvectors are $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. The general solution is

$$x = c_1 e^{2t}\begin{pmatrix} 0 \\ 1 \end{pmatrix} + c_2 e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Since both eigenvalues are positive, the origin is a node. I will soon upload a scan of the phase portrait.

5. $A = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix}$. The eigenvalues of A are $-1 \pm 2i$. Thus we know that the phase portrait is a spiral going towards the origin. What is the direction of rotation of it? Test the point $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

$$A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Thus, at the point $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, the tangent vector to the spiral is $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$. This implies that the spiral is rotating counter-clockwise. I will soon upload a scan of the phase portrait.

But we still have to find the general solution. The eigenvector for $-1+2i$ is $\begin{pmatrix} 2i \\ 1 \end{pmatrix}$. So one solution of $x' = Ax$ is

$$e^{(-1+2i)t}\begin{pmatrix} 2i \\ 1 \end{pmatrix} = e^{-t}(\cos(2t) + i\sin(2t))\begin{pmatrix} 2i \\ 1 \end{pmatrix} = e^{-t}\begin{pmatrix} -2\sin(2t) + 2i\cos(2t) \\ \cos(2t) + i\sin(2t) \end{pmatrix}.$$

The real part of this is $e^{-t}\begin{pmatrix} -2\sin(2t) \\ \cos(2t) \end{pmatrix}$ and the imaginary part of this is $e^{-t}\begin{pmatrix} 2\cos(2t) \\ \sin(2t) \end{pmatrix}$. Thus the general solution is

$$x = c_1 e^{-t}\begin{pmatrix} -2\sin(2t) \\ \cos(2t) \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 2\cos(2t) \\ \sin(2t) \end{pmatrix}.$$