SOME SOLUTIONS FOR MAY 15 RECOMMENDED PROBLEMS

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1.1 21. a) It's always helpful to give things names. Let $C(t)$ be the quantity in grams of the chemical at time t. We have

$$\frac{dC}{dt}(t) = 0.01 \cdot 300 - \frac{C(t)}{1000000} \cdot 300;$$

the units of $\frac{dC}{dt}$ are grams/hour. 0.01 is the concentration of the chemical in the incoming liquid, and $\frac{C(t)}{1000000}$ is the concentration of the chemical in the pond, which is spilling out as the new water is added.

This is a differential equation for the amount of the chemical in the pond at any time.

b) There are two methods of answering this and you should be comfortable with both. On the one hand, the water in the pond is being replaced with the incoming water, which has concentration 0.01 g/gallon of the chemical. No matter how much of the chemical was initially in the pond, the pond is going to look more and more like the incoming water as time goes on. This is analogous to the process by which old bills circulate and are removed from the money supply when they hit a bank. Eventually all the currency is the new bill. Since the pond has 1000000 gallons of water, if the water has 0.01 g/gallon of the chemical then the total amount of the chemical would be 10000 g.

On the other hand, when there is some equilibrium in the pond we will have $\frac{dC}{dt}(t) = 0$, and then using the differential equation in a) we would have

$$0 = 0.01 \cdot 300 - \frac{C(t)}{1000000} \cdot 300,$$

hence

$$C(t) = 0.01 \cdot 1000000 = 10000.$$

22. The volume of a spherical raindrop of radius $r(t)$ is

$$V(t) = \frac{4}{3}\pi r(t)^3.$$

(You *should* memorize the formula for volume, but I would give this to you on a test.) The surface area of the spherical raindrop is

$$A(t) = 4\pi r(t)^2;$$

again, you should memorize this (but you can remember the surface area as the derivative with respect to r of volume). We are given that for some constant α,

$$\frac{dV}{dt}(t) = -\alpha A(t).$$

This is not a sufficient answer; we want an equation that involves only V. But using the formulas for $A(t)$ and $V(t)$ we have

$$A(t) = 4\pi r(t)^2 = 4\pi \left(\frac{3V}{4\pi}\right)^{2/3}.$$

Thus the volume satisfies the following differential equation:

$$\frac{dV}{dt}(t) = -\alpha \cdot 4\pi \left(\frac{3V}{4\pi}\right)^{2/3},$$

and if we put all the constants together as β this is

$$\frac{dV}{dt}(t) = -\beta V(t)^{2/3}.$$

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