Functions of bounded variation and differentiability

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1 Functions of bounded variation

We say that a function $f : A \to \mathbb{R} \cup \{\infty\}$, $A \subset \mathbb{R}$, is increasing if $x \leq y$ implies $F(x) \leq F(y)$, namely if f is order preserving.

Let $a < b$ be real. For a function $F : [a, b] \to \mathbb{R}$, define $V_F : [a, b] \to [0, \infty]$ by

$$V_F(x) = \sup_{N, a=t_0<t_1<\cdots<t_N=b} \sum_j |F(t_j) - F(t_{j-1})|,$$

called the **variation of** F. It is apparent that V_F is increasing. If V_F is bounded, we say that F has **bounded variation**. V_F being bounded is equivalent to $V_F(b) < \infty$. If F is increasing then

$$V_F(x) = F(x) - F(a),$$

so in particular an increasing function has bounded variation.

Define $P_F : [a, b] \to [0, \infty]$ by

$$P_F(x) = \sup_{N, a=t_0<t_1<\cdots<t_N=b} \sum_{F(t_j) \geq F(t_{j-1})} F(t_j) - F(t_{j-1}),$$

called the **positive variation of** F, and define $N_F : [a, b] \to [0, \infty]$ by

$$N_F(x) = \sup_{N, a=t_0<t_1<\cdots<t_N=b} \sum_{F(t_j) \leq F(t_{j-1})} -(F(t_j) - F(t_{j-1})),$$

called the **negative variation of** F. It is apparent that P_F and N_F are increasing.

We now prove the **Jordan decomposition theorem**. It shows in particular that if F has bounded variation then P_F and N_F are bounded.

Theorem 1 (Jordan decomposition theorem)**.** *If $F : [a,b] \to \mathbb{R}$ has bounded variation, then for all $x \in [a,b]$,*

$$V_F(x) = P_F(x) + N_F(x).$$

and

$$F(x) - F(a) = P_F(x) - N_F(x).$$

Proof. For $\epsilon > 0$ there is some L and some $a = r_0 < t_1 < \cdots < r_L = x$ for which

$$\sum_{F(r_j) \geq F(r_{j-1})} F(r_j) - F(r_{j-1}) > P_F(x) - \epsilon,$$

and there is some M and some $a = s_0 < s_1 < \cdots < s_M = x$ for which

$$\sum_{F(s_j) \leq F(s_{j-1})} -(F(s_j) - F(s_{j-1})) > N_F(x) - \epsilon.$$

Let $a = t_0 < t_1 < \cdots < t_N = x$ with $\{t_0, \ldots, t_N\} = \{r_0, \ldots, r_L\} \cup \{s_0, \ldots, s_M\}$. As $\{r_0, \ldots, r_L\} \subset \{t_0, \ldots, t_N\}$,

$$\sum_{F(t_j) \geq F(t_{j-1})} F(t_j) - F(t_{j-1}) \geq \sum_{F(r_j) \geq F(r_{j-1})} F(r_j) - F(r_{j-1})$$

and as $\{s_0, \ldots, s_M\} \subset \{t_0, \ldots, t_N\}$,

$$\sum_{F(t_j) \leq F(t_{j-1})} -(F(t_j) - F(t_{j-1})) \geq \sum_{F(s_j) \leq F(s_{j-1})} -(F(s_j) - F(s_{j-1})).$$

Hence

$$V_F(x) \geq \sum_j |F(t_j) - F(t_{j-1})| > P_F(x) + N_F(x) - 2\epsilon,$$

and as this is true for all $\epsilon > 0$ it follows that $V_F(x) \geq P_F(x) + N_F(x)$. And $\sup(f + g) \leq \sup f + \sup g$, so $V_F(x) \leq P_F(x) + N_F(x)$ and therefore $V_F(x) = P_F(x) + N_F(x)$. Now,

$$\begin{aligned} F(x) - F(a) &= \sum_j F(t_j) - F(t_{j-1}) \\ &= \sum_{F(t_j) \geq F(t_{j-1})} F(t_j) - F(t_{j-1}) - \sum_{F(t_j) \leq F(t_{j-1})} -(F(t_j) - F(t_{j-1})), \end{aligned}$$

which implies

$$|F(x) - F(a) - P_F(x) + N_F(x)| < 2\epsilon,$$

whence $F(x) - F(a) - P_F(x) + N_F(x) = 0$. □

The Jordan decomposition theorem tells us that if F has bounded variation then

$$F(x) = (P_F(x) - F(a)) - N_F(x),$$

and as $x \mapsto P_F(x) - F(a)$ and $x \mapsto N_F(x)$ are increasing, this shows that F is the difference of two increasing functions.

The following says that a function of bounded variation is continuous at a point if and only if its variation is continuous at that point.[1]

Theorem 2. *If $F : [a,b] \to \mathbb{R}$ has bounded variation, then F is continuous at x if and only if V_F is continuous at x.*

Theorem 3. *If $F : [a,b] \to \mathbb{R}$ has bounded variation then there are at most countably many $x \in [a,b]$ at which F is not continuous.*

Proof. According to the Jordan decomposition theorem, $V_F = P_F + N_F$, so it suffices to prove that if $f : [a,b] \to \mathbb{R}$ is increasing then there are at most countably many $x \in [a,b]$ at which f is not continuous. Let $f(a^-) = f(a)$ and for $a < x \leq b$ let

$$f(x^-) = \lim_{y \to x, y < x} f(y),$$

and let $f(b^+) = f(b)$ and for $a \leq x < b$ let

$$f(x^+) = \lim_{y \to x, y > x} f(y);$$

this makes sense because f is increasing, and also because f is increasing we have $f(x^-) \leq f(x) \leq f(x^+)$. Let E be the set of those $x \in [a,b]$ at which f is not continuous. If $x \in E$, then $f(x^-) < f(x^+)$ and hence there is some $r_x \in (f(x^-), f(x^+)) \cap \mathbb{Q}$. If $x, y \in E$, $x < y$, then as $x < y$ we have $f(x^+) \leq f(y^-)$, and as $x, y \in E$, $f(x^-) < r_x < f(x^+)$ and $f(y^-) < r_y < f(y^+)$, so $r_x < r_y$. Therefore $x \mapsto r_x$ is one-to-one $E \to \mathbb{Q}$, showing that E is countable. □

2 Coverings

The following is the **rising sum lemma**, due to F. Riesz.[2] (We don't use the rising sun lemma elsewhere in these notes, and instead use the Vitali covering theorem, stated next.)

Lemma 4 (Rising sun lemma)**.** *Let $G : [a,b] \to \mathbb{R}$ be continuous and let E be the set of those $x \in (a,b)$ for which there is some $x < y \leq b$ satisfying $G(y) > G(x)$. G is open, and if G is nonempty then G is the union of countably many disjoint $(a_k, b_k) \subset [a,b]$. If $a_k > a$ then $G(b_k) = G(a_k)$, and if $a_k = a$ then $G(b_k) \geq G(a_k)$.*

[1] V. I. Bogachev, *Measure Theory*, volume 1, p. 333, Proposition 5.2.2; `http://individual.utoronto.ca/jordanbell/notes/helly.pdf`, p. 6, Theorem 9.

[2] Elias M. Stein and Rami Shakarchi, *Real Analysis*, p. 118, Lemma 3.2.

Proof. If $x_0 \in E$, there is some $x_0 < y_0 \leq b$ with $G(y_0) > G(x_0)$. Writing $\epsilon = G(y_0) - G(x_0)$, as G is continuous there is some $\delta > 0$, $(x_0 - \delta, x_0 + \delta) \subset [a, b]$, such that if $|x - x_0| < \delta$ then $|G(x) - G(x_0)| < \epsilon$, so

$$\begin{aligned} G(y_0) - G(x) &= \epsilon + G(x_0) - G(x) \\ &\geq \epsilon - |G(x) - G(x_0)| \\ &> 0. \end{aligned}$$

Thus if $x \in (x_0 - \delta, x_0 + \delta)$ then $G(y_0) > G(x)$, which shows that E is open.

Suppose now that E is nonempty, and for $x \in E$ let

$$A_x = \inf\{t \in \mathbb{R} : (t, x) \subset E\}, \quad B_x = \sup\{t \in \mathbb{R} : (x, t) \subset E\}.$$

As E is open, there is some $\delta_x > 0$ such that $(x - \delta_x, x + \delta_x) \subset E$, so $A_x \leq x - \delta_x < x$ and $B_x \geq x + \delta_x > x$. Furthermore, as E is open it follows that $A_x \notin E$ and $B_x \notin E$. For $x, y \in E$, either $(A_x, B_x) \cap (A_y, B_y) = \emptyset$ or $(A_x, B_x) = (A_y, B_y)$, and as (A_x, B_x) contains at least one rational number,

$$E = \bigcup_{x \in E \cap \mathbb{Q}} (A_x, B_x).$$

As $E \cap \mathbb{Q}$ is countable, there are pairwise disjoint $(a_k, b_k) \subset [a, b]$, $a_k \notin E, b_k \notin E$, $k \in I$, such that

$$E = \bigcup_{k \in I} (a_k, b_k).$$

For $k \in I$, suppose by contradiction that $G(b_k) < G(a_k)$. Let

$$C_k = \left\{ c \in (a_k, b_k) : G(c) = \frac{G(a_k) + G(b_k)}{2} \right\},$$

which is nonempty by the intermediate value theorem. Let $c_k = \sup C_k$, and because G is continuous, $c_k \in C_k$. $c_k = b_k$ would imply $G(b_k) = \frac{G(a_k)+G(b_k)}{2}$, contradicting $G(b_k) < G(a_k)$; hence $c_k \in (a_k, b_k) \subset E$. Then because $c_k \in E$, there is some $c_k < d \leq b$ satisfying $G(d) > G(c_k)$. If $d > b_k$ then as $b_k \in (a, b) \setminus E$ it holds that $G(d) \leq G(b_k) < G(c_k) < G(d)$, a contradiction, and if $d = b_k$ then $G(d) = G(b_k) < G(c_k) < G(d)$, a contradiction; hence $d < b_k$. As $G(d) > G(c_k) > G(b_k)$, by the intermediate value theorem there is some $c \in (d, b_k)$ such that $G(c) = G(c_k)$. But then we have $c \in C_k$ and $c > c_k$, contradicting $c_k = \sup C_k$. Therefore,

$$G(b_k) \geq G(a_k).$$

If $a_k \neq a$ then $a_k \in (a, b) \setminus E$, which means that there is no $a_k < y \leq b$ satisfying $G(y) > G(a_k)$. Hence $G(b_k) \leq G(a_k)$, which shows that for $a_k \neq a$, we have $G(b_k) = G(a_k)$. $\square$

Let λ be Lebesgue measure on the Borel σ-algebra of $\mathbb{R}$ and let λ^* be Lebesgue outer measure on $\mathbb{R}$.

A **Vitali covering** of a set $E \subset \mathbb{R}$ is a collection $\mathcal{V}$ of closed intervals such that for $\epsilon > 0$ and for $x \in E$ there is some $I \in \mathcal{V}$ with $x \in I$ and $0 < \lambda(I) < \epsilon$. The following is the **Vitali covering theorem**.[3]

Theorem 5 (Vitali covering theorem)**.** *Let U be an open set in $\mathbb{R}$ with $\lambda(U) < \infty$, let $E \subset U$, and let $\mathcal{V}$ be a Vitali covering of E each interval of which is contained in U. Then for any $\epsilon > 0$ there are pairwise disjoint $I_1, \ldots, I_n \in \mathcal{V}$ such that*

$$\lambda^* \left(E \setminus \bigcup_{j=1}^{n} I_j \right) < \epsilon.$$

3 Differentiability

Let $F : [a, b] \to \mathbb{R}$ be a function. The **Dini derivatives** of F are the following. $D^- F(x) : (a, b] \to \mathbb{R} \cup \{\infty\}$ is defined by

$$D^- F(x) = \limsup_{h \to 0, h < 0} \frac{F(x+h) - F(x)}{h},$$

$D_- F(x) : (a, b] \to \mathbb{R} \cup \{-\infty\}$ is defined by

$$D_- F(x) = \liminf_{h \to 0, h < 0} \frac{F(x+h) - F(x)}{h},$$

$D^+ F(x) : [a, b) \to \mathbb{R} \cup \{\infty\}$ is defined by

$$D^+ F(x) = \limsup_{h \to 0, h > 0} \frac{F(x+h) - F(x)}{h},$$

$D_+ F(x) : [a, b) \to \mathbb{R} \cup \{-\infty\}$ is defined by

$$D_+ F(x) = \liminf_{h \to 0, h > 0} \frac{F(x+h) - F(x)}{h}.$$

For $x \in [a, b]$, the **upper derivative of F at x** is

$$\overline{D} F(x) = \limsup_{h \to 0, h \neq 0} \frac{F(x+h) - F(x)}{h},$$

and the **lower derivative of F at x** is

$$\underline{D} F(x) = \liminf_{h \to 0, h \neq 0} \frac{F(x+h) - F(x)}{h}.$$

[3] `http://individual.utoronto.ca/jordanbell/notes/vitali.pdf`

Let

$$\mathscr{L} = \{x \in (a, b] : D^- F(x) = D_- F(x)\},$$
$$\mathscr{R} = \{x \in [a, b) : D^+ F(x) = D_+ F(x)\}.$$

For $x \in \mathscr{L}$, the **left-derivative of F at x** is

$$F'_-(x) = D^- F(x) = D_- F(x),$$

and for $x \in \mathscr{R}$, the **right-derivative of F at x** is

$$F'_+(x) = D^+ F(x) = D_+ F(x).$$

For $x \in (a, b)$, for F to be **differentiable at x** means that

$$-\infty < D_- F(x) = D^- F(x) = D_+ F(x) = D^+ F(x) < \infty.$$

We prove that the set of points at which F is left-differentiable and right-differentiable but $F'_-(x) \neq F'_+(x)$ is countable.[4]

Lemma 6. *$\{x \in \mathscr{L} \cap \mathscr{R} : F'_-(x) \neq F'_+(x)\}$ is countable.*

Proof. Let $\mathbb{Q} = \{r_k : k \geq 1\}$, $r_k \neq r_j$ for $k \neq j$, and let

$$E = \{x \in \mathscr{L} \cap \mathscr{R} : F'_-(x) < F'_+(x)\},$$

For $x \in E$, as $F'_-(x) < F'_+(x)$ there is a minimal k with $F'_-(x) < r_k < F'_+(x)$. As $r_k > F'_-(x)$, there is a minimal m such that $r_m < x$ and for all $t \in (r_m, x)$, $\frac{F(t)-F(x)}{t-x} < r_k$ and hence $F(t) - F(x) > r_k(t - x)$. Likewise, as $r_k < F'_+(x)$, there is a minimal n such that $r_n > x$ and for all $t \in (x, r_n)$, $\frac{F(t)-F(x)}{t-x} > r_k$ and hence $F(t) - F(x) > r_k(t - x)$. Hence

$$F(t) - F(x) > r_k(t - x), \qquad t \in (r_m, r_n), t \neq x. \tag{1}$$

Now for distinct $x, y \in E$ suppose by contradiction that $(k(x), m(x), n(x)) = (k(y), m(y), n(y))$. As $x, y \in (r_m, r_n)$, using (1) with $t = y$ and $t = x$ we get

$$F(y) - F(x) > r_k(y - x), \qquad F(x) - F(y) > r_k(x - y),$$

yielding $r_k(x - y) < F(x) - F(y) < r_k(x - y)$, a contradiction. Therefore $x \mapsto (k(x), m(x), n(x))$ is one-to-one $E \to \mathbb{N}^3$, for $\mathbb{N}$ the positive integers, which shows that E is countable.

We similarly prove that

$$\{x \in \mathscr{L} \cap \mathscr{R} : F'_-(x) > F'_+(x)\}$$

is countable. □

[4] V. I. Bogachev, *Measure Theory*, volume 1, p. 332, Lemma 5.1.3.

4 Differentiability of increasing functions

We now use the Vitali covering lemma to prove that the Dini derivatives of an increasing function are finite almost everywhere.[5]

Lemma 7. *Let* $F : [a, b] \to \mathbb{R}$ *be an increasing function and let*

$$A^- = \{x \in (a, b] : D^- F(x) = \infty\}, \ A_- = \{x \in (a, b] : D_- F(x) = -\infty\},$$

$$A^+ = \{x \in [a, b) : D^+ F(x) = \infty\}, \ A_+ = \{x \in [a, b) : D_+ F(x) = -\infty\}.$$

Then

$$\lambda^*(A^-) = 0, \lambda^*(A_-) = 0, \lambda^*(A^+) = 0, \lambda^*(A_-) = 0.$$

Proof. Because F is increasing, for any $h \neq 0$, $\frac{F(x+h)-F(x)}{h} \geq 0$, and therefore $A_- = \emptyset$ and $A_+ = \emptyset$. Suppose by contradiction that

$$\lambda^*(A^-) = \alpha > 0.$$

As $\alpha > 0$, there is some $r > 0$ satisfying

$$\frac{r\alpha}{2} > F(b) - F(a).$$

For x^-, because $D^- F(x) = \infty$ there is an increasing sequence $t_{x,k} \in [a, b]$ that tends to x such that for each $k \geq 1$,

$$\frac{F(x) - F(t_{x,k})}{x - t_{x,k}} \geq r. \tag{2}$$

Let

$$\mathcal{V} = \{[t_{x,k}, x] : x \in A^-, k \geq 1\},$$

which is a Vitali covering of A^-, and so by the Vitali covering theorem there are pairwise disjoint $[t_{x_j,k_j}, x_j] \in \mathcal{V}$, $1 \leq j \leq n$, such that

$$\lambda^* \left(A^- \setminus \bigcup_{j=1}^{n} [t_{x_j,k_j}, x_j] \right) < \frac{\alpha}{2}$$

and then

$$\lambda^*(A^-) \leq \lambda^* \left(A^- \setminus \bigcup_{j=1}^{n} [t_{x_j,k_j}, x_j] \right) + \lambda \left(\bigcup_{j=1}^{n} [t_{x_j,k_j}, x_j] \right),$$

[5]Russell A. Gordon, *The Integrals of Lebesgue, Denjoy, Perron, and Henstock*, p. 55, Lemma 4.8.

hence

$$\begin{aligned}\sum_{j=1}^{n} \lambda([t_{x_j,k_j}, x_j]) &= \lambda\left(\bigcup_{j=1}^{n}[t_{x_j,k_j}, x_j]\right) \\ &\geq \lambda^*(A^-) - \lambda^*\left(A^- \setminus \bigcup_{j=1}^{n}[t_{x_j,k_j}, x_j]\right) \\ &> \alpha - \frac{\alpha}{2}.\end{aligned}$$

That is,

$$\sum_{j=1}^{n}(x_j - t_{x_j,k_j}) > \frac{\alpha}{2}.$$

Now, by (2), $F(x_j) - F(t_{x_j,k_j}) \geq r(x_j - t_{x_j,k_j})$, so

$$\sum_{j=1}^{n}(F(x_j) - F(t_{x_j,k_j})) \geq \sum_{j=1}^{n} r(x_j - t_{x_j,k_j}) > \frac{r\alpha}{2} > F(b) - F(a).$$

But because the intervals $[t_{x_j,k_j}, x_j]$ are pairwise disjoint and F is increasing, $\sum_{j=1}^{n}(F(x_j) - F(t_{x_j,k_j})) \leq F(b) - F(a)$, contradicting the above inequality. Therefore $\lambda^*(A^-) = 0$.

Suppose by contradiction that

$$\lambda^*(A^+) = \alpha > 0.$$

As $\alpha > 0$, there is some $r > 0$ satisfying

$$\frac{r\alpha}{2} > F(b) - F(a).$$

For $x \in A^+$, because $D^+F(x) = \infty$ there is a decreasing sequence $t_{x,k} \in [a, b]$ that tends to x such that for each $k \geq 1$,

$$\frac{F(t_{x,k}) - F(x)}{t_{x,k} - x} \geq r. \tag{3}$$

Let

$$\mathcal{V} = \{[x, t_{x,k}] : x \in A^+, k \geq 1\},$$

which is a Vitali covering of A^+, and so by the Vitali covering theorem there are pairwise disjoint $[x_j, t_{x_j,k_j}] \in \mathcal{V}$, $1 \leq j \leq n$, such that

$$\lambda^*\left(A^+ \setminus \bigcup_{j=1}^{n}[x_j, t_{x_j,k_j}]\right) < \frac{\alpha}{2}$$

and then

$$\lambda^*(A^+) \leq \lambda^*\left(A^+ \setminus \bigcup_{j=1}^{n}[x_j, t_{x_j,k_j}]\right) + \lambda\left(\bigcup_{j=1}^{n}[x_j, t_{x_j,k_j}]\right),$$

hence

$$\begin{aligned}\sum_{j=1}^{n}\lambda([x_j, t_{x_j,k_j}]) &= \lambda\left(\bigcup_{j=1}^{n}[x_j, t_{x_j,k_j}]\right)\\ &\geq \lambda^*(A^+) - \lambda^*\left(A^+ \setminus \bigcup_{j=1}^{n}[x_j, t_{x_j,k_j}]\right)\\ &> \alpha - \frac{\alpha}{2}.\end{aligned}$$

That is,

$$\sum_{j=1}^{n}(t_{x_j,k_j} - x_j) > \frac{\alpha}{2}.$$

Now, by (3), $F(t_{x_j,k_j}) - F(x_j) \geq r(t_{x_j,k_j} - x_j)$, so

$$\sum_{j=1}^{n}(F(t_{x_j,k_j}) - F(x_j)) \geq \sum_{j=1}^{n} r(t_{x_j,k_j} - x_j) > \frac{r\alpha}{2} > F(b) - F(a).$$

But because the intervals $[x_j, t_{x_j,k_j}]$ are pairwise disjoint and F is increasing, $\sum_{j=1}^{n}(F(t_{x_j,k_j}) - F(x_j)) \leq F(b) - F(a)$, contradicting the above inequality. Therefore $\lambda^*(A^+) = 0$. □

We now prove that an increasing function is differentiable almost everywhere.[6]

Theorem 8. *Let $F : [a, b] \to \mathbb{R}$ be increasing and let*

$$E = \{x \in (a, b) : -\infty < D_-F(x) = D^-F(x) = D_+F(x) = D^+F(x) < \infty\}.$$

Then $\lambda^*([a, b] \setminus E) = 0$.

Proof. Let

$$A = \{x \in (a, b) : D_+F(x) < D^+F(x)\},$$

and suppose by contradiction that $\lambda^*(A) > 0$. Since

$$A = \bigcup_{p,q\in\mathbb{Q},p<q} \{x \in (a, b) : D_+F(x) < p < q < D^+F(x)\},$$

[6] Russell A. Gordon, *The Integrals of Lebesgue, Denjoy, Perron, and Henstock*, p. 55, Theorem 4.9.

which is a union of countably many sets, there are some $p, q \in \mathbb{Q}$, $p < q$, such that $\lambda^*(B) = \beta > 0$,

$$B = \{x \in (a, b) : D_+F(x) < p < q < D^+F(x)\}.$$

Let $\epsilon > 0$. There is an open set $U \subset (a, b)$ with $B \subset U$ and $\lambda(U) < \lambda^*(B) + \epsilon = \beta + \epsilon$. For $x \in B$, because $D_+F(x) < p$ and because x belongs to the open set U, there is a sequence $t_{x,k} \in (x, x+1/k)$, $[x, t_{x,k}] \subset U$, such that for each $k \geq 1$,

$$\frac{F(t_{x,k}) - F(x)}{t_{x,k} - x} < p.$$

Then

$$\mathcal{V} = \{[x, t_{x,k}] : x \in B, k \geq 1\}$$

is a Vitali covering of B, so by the Vitali covering theorem there are pairwise disjoint $[x_j, t_{x_j,k_j}] \in \mathcal{V}$, $1 \leq j \leq m$, such that

$$\lambda^* \left(B \setminus \bigcup_{j=1}^{m} [x_j, t_{x_j,k_j}] \right) < \epsilon,$$

and then, as the intervals $[x_j, t_{x_j,k_j}]$ are pairwise disjoint and are all contained in U,

$$\begin{aligned}
\sum_{j=1}^{m} (F(t_{x_j,k_j}) - F(x_j)) &< \sum_{j=1}^{m} p(t_{x_j,k_j} - x_j) \\
&= p \sum_{j=1}^{m} \lambda([x_j, t_{x_j,k_j}]] \\
&= p\lambda \left(\bigcup_{j=1}^{m} [x_j, t_{x_j,k_j}] \right) \\
&\leq p\lambda(U) \\
&< p(\beta + \epsilon).
\end{aligned}$$

Let $C = B \cap \bigcup_{j=1}^{n} (x_j, t_{x_j,k_j})$, for which

$$\beta = \lambda^*(B) \leq \lambda^*(C) + \lambda^* \left(B \setminus \bigcup_{j=1}^{n} [x_j, t_{x_j,k_j}] \right) < \lambda^*(C) + \epsilon,$$

so

$$\lambda^*(C) > \beta - \epsilon.$$

For $y \in C$ there is some i for which $y \in (x_i, t_{x_i,k_i})$, and because $D^+F(y) > q$ there is a sequence $u_{y,l} \in (y, y + 1/l)$, $[y, u_{y,l}] \subset (x_i, t_{x_i,k_i})$, such that for each $l \geq 1$,

$$\frac{F(u_{y,l}) - F(y)}{u_{y,l} - y} > q.$$

Then

$$\mathcal{W} = \{[y, u_{y,l}] : y \in B, l \geq 1\}$$

is a Vitali covering of C, so by the Vitali covering theorem there are pairwise disjoint $[y_j, u_{y_j,l_j}] \in \mathcal{W}$, $1 \leq j \leq n$, such that

$$\lambda^* \left(C \setminus \bigcup_{j=1}^{n} [y_j, u_{y_j,l_j}] \right) < \epsilon,$$

so

$$\lambda^*(C) \leq \lambda^* \left(C \setminus \bigcup_{j=1}^{n} [y_j, u_{y_j,l_j}] \right) + \lambda \left(\bigcup_{j=1}^{n} [y_j, u_{y_j,l_j}] \right) < \epsilon + \sum_{j=1}^{n} \lambda([y_j, u_{y_j,l_j}]),$$

and then

$$\sum_{j=1}^{n} (F(u_{y_j,l_j}) - F(y_j)) > \sum_{j=1}^{n} q(u_{y_j,l_j} - y_j) > q(\lambda^*(C) - \epsilon) > q(\beta - 2\epsilon).$$

Now for $1 \leq i \leq m$ let $\pi_i = \{1 \leq j \leq n : [y_j, u_{y_j,l_j}] \subset (x_i, t_{x_i,k_i})\}$. Because F is increasing, if $j \in \pi_i$ then $F(u_{y_j,l_j}) - F(y_j) \leq F(t_{x_i,k_i}) - F(x_i)$, and because each $[y_j, u_{y_j,l_j}]$ is contained in some (x_i, t_{x_i,k_i}),

$$\begin{aligned} q(\beta - 2\epsilon) &< \sum_{j=1}^{n} (F(u_{y_j,l_j}) - F(y_j)) \\ &= \sum_{i=1}^{m} \sum_{j \in \pi_i} (F(u_{y_j,l_j}) - F(y_j)) \\ &\leq \sum_{i=1}^{m} (F(t_{x_i,k_i}) - F(x_i)); \end{aligned}$$

the last inequality also uses that the intervals $[y_j, u_{y_j,l_j}]$ are pairwise disjoint. But we have found $\sum_{i=1}^{m} (F(t_{x_i,k_i}) - F(x_i)) < p(\beta + \epsilon)$, so $q(\beta - 2\epsilon) < p(\beta + \epsilon)$. As this is true for all $\epsilon > 0$, it holds that $q\beta \leq p\beta$, and as $\beta > 0$ we get $q \leq p$, contradicting that $p < q$. Therefore $\lambda^*(A) = 0$. $\square$