$C^k$ spaces and spaces of test functions

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1 Notation

Let $N$ denote the set of nonnegative integers. For $\alpha \in \mathbb{N}^n$, we write

$$|\alpha| = \alpha_1 + \cdots + \alpha_n,$$

and

$$\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}.$$  \hfill (1.1)

We denote by $B_r(x)$ the open ball with center $x$ and radius $r$.

2 Open sets

Let $\Omega$ be an open subset of $\mathbb{R}^n$ and let $k$ be either a nonnegative integer or $\infty$. We define $C^k(\Omega)$ to be the set of those functions $f : \Omega \to \mathbb{C}$ such that for each $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq k$, the derivative $\partial^\alpha f$ exists and is continuous. We write $C(\Omega) = C^0(\Omega)$.  \hfill (2.1)

One proves that there is a sequence of compact sets $K_j$ such that each $K_j$ is contained in the interior of $K_{j+1}$ and $\Omega = \bigcup_{j=1}^\infty K_j$; we call this an exhaustion of $\Omega$ by compact sets. For $f \in C^k(\Omega)$, we define

$$p_{k,N}(f) = \sup_{|\alpha| \leq \min(k,N)} \sup_{x \in K_N} |(\partial^\alpha f)(x)|;$$

this definition makes sense for $k = \infty$. If $f$ is a nonzero element of $C^k(\Omega)$, then there is some $x \in \Omega$ for which $f(x) \neq 0$ and then there is some $N$ for which $x \in K_N$, and hence $p_{k,N}(f) \geq \sup_{y \in K_N} |f(y)| \geq |f(x)| > 0$. Thus, $p_{k,N}$ is a separating family of seminorms on $C^k(\Omega)$. Those sets of the form

$$V_{k,N} = \left\{ f \in C^k(\Omega) : p_{k,N}(f) < \frac{1}{N} \right\}$$

form a local basis at 0 for a topology on $C^k(\Omega)$, and because $p_{k,N}$ is a separating family of seminorms, with this topology $C^k(\Omega)$ is a locally convex space.  \hfill (2.2)

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Because $p_{k,N}$ is a countable separating family of seminorms, this topology is metrizable. We prove in the following theorem that $C(\Omega)$ is a Fréchet space.

**Theorem 1.** If $\Omega$ is an open subset of $\mathbb{R}^n$, then $C(\Omega)$ is a Fréchet space.

**Proof.** Let $f_i \in C(\Omega)$ be a Cauchy sequence. That is, for every $N$ there is some $i_N$ such that if $i, j \geq i_N$ then

$$f_i - f_j \in V_{0,N} = \left\{ f \in C(\Omega) : \sup_{x \in K_N} |f(x)| < \frac{1}{N} \right\}.$$  

For each $x \in \Omega$, eventually $x \in K_N$. If $x \in K_N$ and $i, j \geq i_N$, then

$$|f_i(x) - f_j(x)| < \frac{1}{N}.$$  

Therefore, $f_i(x)$ is a Cauchy sequence in $\mathbb{C}$ and hence converges to some $f(x) \in \mathbb{C}$. We have thus defined a function $f : \Omega \rightarrow \mathbb{C}$. We shall prove that $f \in C(\Omega)$ and that $f_i \rightarrow f$ in $C(\Omega)$.

Let $K$ be a compact subset of $\Omega$, let $\epsilon > 0$, and let $N$ be large enough both so that $K \subseteq K_N$ and so that $N > \frac{1}{\epsilon}$. For $i, j \geq i_N$,

$$\sup_{x \in K_N} |f_i(x) - f_j(x)| < \frac{1}{N} \leq \epsilon.$$  

Let $i \geq i_N$ and $x \in K_N$. There is some $j_x$ such that $j \geq j_x$ implies that $|f_j(x) - f(x)| < \epsilon$, and hence for $j \geq \max(i_N, j_x)$,

$$|f_i(x) - f(x)| \leq |f_i(x) - f_j(x)| + |f_j(x) - f(x)| < \epsilon + \epsilon.$$  

This shows that for $i \geq i_N$,

$$\sup_{x \in K} |f_i(x) - f(x)| \leq \sup_{x \in K_N} |f_i(x) - f(x)| \leq 2\epsilon.$$  

We have proved that for any compact subset $K$ of $\Omega$, we have $\sup_{x \in K} |f_i(x) - f(x)| \rightarrow 0$ as $i \rightarrow \infty$.

Let $x \in \Omega$, let $\epsilon > 0$, and let $N$ be large enough both so that $x$ lies in the interior of $K_N$ and so that $N > \frac{1}{\epsilon}$. Because $\sup_{x \in K_N} |f_i(x) - f(x)| \rightarrow 0$ as $i \rightarrow \infty$, there is some $i_0$ so that $i \geq i_0$ implies

$$\sup_{x \in K_N} |f_i(x) - f(x)| < \epsilon.$$  

Let $i = \max(i_0, i_N)$. Because $f_i$ is continuous, there is some $\delta > 0$ so that $|x - y| < \delta$ implies that $|f_i(x) - f_i(y)| < \epsilon$; take $\delta$ small enough so that the open ball with center $x$ and radius $\delta$ is contained in $K_N$. For $|y - x| < \delta$,

$$|f(x) - f(y)| \leq |f(x) - f_i(x)| + |f_i(x) - f_i(y)| + |f_i(y) - f(y)|$$

$$\leq \sup_{z \in K_N} |f(z) - f_i(z)| + \frac{1}{N} \sup_{z \in K_N} |f(z) - f_i(z)|$$

$$< \epsilon + \epsilon + \epsilon.$$  

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This shows that $f$ is continuous at $x$ and $x$ was an arbitrary point in $\Omega$, hence $f \in C(\Omega)$.

We have already established that for any compact subset $K$ of $\Omega$, we have $\sup_{x \in K} |f_i(x) - f(x)| \to 0$ as $i \to \infty$. Thus, for any $N$, there is some $j_N$ so that if $i \geq j_N$ then $\sup_{x \in K_N} |f_i(x) - f(x)| < \frac{1}{N}$. In other words, if $i \geq j_N$, then $p_{0,N}(f_i - f) < \frac{1}{N}$, i.e. $f_i - f \in V_{0,N}$, showing that $f_i \to f$ in $C(\Omega)$.

**Theorem 3.** If $\Omega$ is an open subset of $\mathbb{R}^n$ and $k$ is a positive integer, then $C^k(\Omega)$ is a Fréchet space.

**Proof.** We have proved in Theorem 1 that $C(\Omega) = C^0(\Omega)$ is a Fréchet space. We assume that $C^{k-1}(\Omega)$ is a Fréchet space, and using this induction hypothesis we shall prove that $C^k(\Omega)$ is a Fréchet space.

Let $f_i \in C^k(\Omega)$ be a Cauchy sequence in $C^k(\Omega)$. $f_i$ is in particular a Cauchy sequence in the Fréchet space $C(\Omega)$, hence there is some $g \in C(\Omega)$ such that $f_i \to g$ in $C(\Omega)$. We shall prove that $g \in C^k(\Omega)$ and that $f_i \to g$ in $C^k(\Omega)$.

For each $1 \leq p \leq n$ we have $\partial_p f_i \in C^{k-1}(\Omega)$, and $\partial_p f_i$ is a Cauchy sequence in $C^{k-1}(\Omega)$. Because $C^{k-1}(\Omega)$ is a Fréchet space, for each $p$ there is some $g_p \in C^{k-1}(\Omega)$ such that $\partial_p f_i \to g_p$ in $C^{k-1}(\Omega)$. Fix $p$, and let $\alpha \in \mathbb{N}^n$ have $p$th entry 1 and all other entries 0. Then, fix $x \in \Omega$, and take $N$ large enough so that $x$ lies in the interior of $K_N$. For each $i$, define $F_i(t) = f(x + t\alpha)$, for which

$$F_i'(t) = (\nabla f)(x + t\alpha) \cdot \alpha = (\partial_p f_i)(x + t\alpha).$$

For nonzero $\tau$ small enough so that the line segment from $x$ to $x + \tau \alpha$ is contained in $K_N$,

$$F_i(\tau) - F_i(0) = \int_0^\tau F_i'(t)dt,$$

i.e.

$$f_i(x + \tau \alpha) - f_i(x) = \int_0^\tau (\partial_p f_i)(x + t\alpha)dt.$$  

Because $f_i \to g$ in $C(\Omega)$ and $\partial_p f_i \to g_p$ in $C(\Omega)$, we have $\sup_{y \in K_N} |f_i(y) - g(y)| \to 0$ and $\sup_{y \in K_N} |(\partial_p f_i)(y) - g_p(y)| \to 0$, from which it follows that

$$g(x + \tau \alpha) - g(x) = \int_0^\tau g_p(x + t\alpha)dt,$$

or

$$\frac{g(x + \tau \alpha) - g(x)}{\tau} = \frac{1}{\tau} \int_0^\tau g_p(x + t\alpha)dt.$$  

As $\tau$ tends to 0, the right hand side tends to $g_\alpha(x)$, showing that $(\partial_p g)(x) = g_p(x)$. But $x$ was an arbitrary point in $\Omega$, so $\partial_p g = g_p \in C^{k-1}(\Omega)$. Thus, for each $1 \leq p \leq n$ we have $\partial_\alpha g \in C^{k-1}(\Omega)$, from which it follows that $g \in C^k(\Omega)$.

**Theorem 2.** If $\Omega$ is an open subset of $\mathbb{R}^n$, then $C^\infty(\Omega)$ is a Fréchet space.

**Proof.** Let $f_i \in C^\infty(\Omega)$ be a Cauchy sequence in $C^\infty(\Omega)$. Thus, for each $k$, $f_i$ is a Cauchy sequence in $C^k(\Omega)$, and so by Theorem 2 there is some $g_k \in C^k(\Omega)$ for which $f_i \to g_k$ in $C^k(\Omega)$. Define $g = g_0$, and check that $g_0 = g_1 = g_2 = \cdots$, and hence that $g \in C^\infty(\Omega)$.
3 Closed sets

Let \( \Omega \) be an open subset of \( \mathbb{R}^n \) such that \( \overline{\Omega} \) is compact, i.e. \( \Omega \) is a bounded open subset of \( \mathbb{R}^n \). If \( k \) is a nonnegative integer, let \( C^k(\overline{\Omega}) \) be those elements \( f \) of \( C^k(\Omega) \) such that for each \( \alpha \in \mathbb{N}^n \) with \( |\alpha| \leq k \), the function \( \partial^\alpha f \) is continuous \( \Omega \to \mathbb{C} \) and can be extended to a continuous function \( \overline{\Omega} \to \mathbb{C} \); if there is such a continuous function \( \overline{\Omega} \to \mathbb{C} \) it is unique, and it thus makes sense to talk about the value of \( \partial^\alpha f \) at points in \( \partial \Omega \), and thus to write \( \partial^\alpha f : \overline{\Omega} \to \mathbb{C} \). We write \( C(\overline{\Omega}) = C^0(\overline{\Omega}) \). For \( f \in C^k(\overline{\Omega}) \), we define
\[
\|f\|_k = \sup_{|\alpha| \leq k} \sup_{x \in \overline{\Omega}} |(\partial^\alpha f)(x)|.
\]
It is straightforward to check that this is a norm on \( C^k(\overline{\Omega}) \).

**Theorem 4.** If \( \Omega \) is a bounded open subset of \( \mathbb{R}^n \), then \( C(\overline{\Omega}) \) is a Banach space.

**Proof.** Let \( f_i \in C(\overline{\Omega}) \) be a Cauchy sequence. Thus, \( f_i : \overline{\Omega} \to \mathbb{C} \) are continuous, and for any \( \epsilon > 0 \) there is some \( i_\epsilon \) such that if \( i, j \geq i_\epsilon \) then
\[
\sup_{x \in \overline{\Omega}} |f_i(x) - f_j(x)| < \epsilon.
\]
Then, for each \( x \in \overline{\Omega} \) we have that \( f_i(x) \) is a Cauchy sequence in \( \mathbb{C} \) and hence converges to some \( f(x) \in \mathbb{C} \), thus defining a function \( f : \overline{\Omega} \to \mathbb{C} \). For \( x \in \overline{\Omega} \) and \( \epsilon > 0 \), because \( f_i(x) \to f(x) \), there is some \( j_x \) such that \( j \geq j_x \) implies that
\[
|f_i(x) - f(x)| < \epsilon.
\]

This shows that \( \sup_{x \in \overline{\Omega}} |f_i(x) - f(x)| \to 0 \) as \( i \to \infty \).

Fix \( x \in \Omega \) and let \( \epsilon > 0 \). What we just proved shows that there is some \( i_\epsilon \) for which \( i \geq i_\epsilon \) implies that \( \sup_{z \in \overline{\Omega}} |f_i(z) - f(z)| < \epsilon \). As \( f_i : \overline{\Omega} \to \mathbb{C} \) is continuous, there is some \( \delta > 0 \) such that for \( y \in B_\delta(x) \cap \overline{\Omega} \), we have \( |f_i(x) - f_i(y)| < \epsilon \). Then, for \( y \in B_\delta(x) \cap \overline{\Omega} \),
\[
|f(x) - f(y)| \leq |f(x) - f_i(x)| + |f_i(x) - f_i(y)| + |f_i(y) - f(y)| < \epsilon + \epsilon + \epsilon.
\]
This proves that \( f \) is continuous at \( x \), and because \( x \) was an arbitrary point in \( \overline{\Omega} \), we have that \( f \in C(\overline{\Omega}) \).

**Theorem 5.** If \( \Omega \) is a bounded open subset of \( \mathbb{R}^n \) and \( k \) is a positive integer, then \( C^k(\overline{\Omega}) \) is a Banach space.

**Proof.** We proved in Theorem 4 that \( C(\overline{\Omega}) = C^0(\overline{\Omega}) \) is a Banach space. We assume that \( C^{k-1}(\overline{\Omega}) \) is a Banach space, and using this induction hypothesis we shall prove that \( C^k(\overline{\Omega}) \) is a Banach space.
Let \( f_i \in C^k(\Omega) \) be a Cauchy sequence. In particular, \( f_i \) is a Cauchy sequence in \( C(\Omega) \), and because \( C(\Omega) \) is a Banach space, there is some \( g \in C(\Omega) \) for which \( \| f_i - g \|_0 \to 0 \). For each \( 1 \leq p \leq n \) we have \( \partial_p f_i \in C^{k-1}(\Omega) \). Because \( C^{k-1}(\Omega) \) is a Banach space, for each \( p \) there is some \( g_p \in C^{k-1}(\Omega) \) for which \( \| \partial_p f_i - g_p \|_{k-1} \to 0 \).

Let \( \alpha \in \mathbb{N}^n \) have \( p \)th entry 1 and all other entries 0, and let \( x \in \Omega \). For nonzero \( \tau \) small enough so that the line segment from \( x \) to \( x + \tau \alpha \) is contained in \( \Omega \),

\[
    f_i(x + \tau \alpha) - f_i(x) = \int_0^\tau (\partial_p f_i)(x + t\alpha)dt.
\]

Because \( \| f_i - g \|_0 \to 0 \) and \( \| \partial_p f_i - g_p \|_0 \to 0 \) (the latter because \( \| \partial_p f_i - g_p \|_{k-1} \to 0 \)), we obtain

\[
    g(x + \tau \alpha) - g(x) = \int_0^\tau g_p(x + t\alpha)dt,
\]

or

\[
    \frac{g(x + \tau \alpha) - g(x)}{\tau} = \frac{1}{\tau} \int_0^\tau g_p(x + t\alpha)dt.
\]

As \( \tau \) tends to 0 the right hand side tends to \( g_p(x) \), which shows that \( (\partial_p g)(x) = g_p(x) \). We did this for all \( x \in \Omega \), and so \( \partial_p g = g_p \in C^{k-1}(\Omega) \). Because this is true for each \( 1 \leq p \leq n \), we obtain \( g \in C^k(\Omega) \).

If \( \Omega \) is a bounded open subset of \( \mathbb{R}^n \), then

\[
    C^\infty(\Omega) = \bigcap_{k=0}^{\infty} C^k(\Omega).
\]

It can be proved that \( C^\infty(\Omega) \) is the projective limit of the Banach spaces \( C^k(\Omega) \), \( k = 0, 1, \ldots \). A projective limit of a countable projective system of Banach spaces is a Fréchet space, and thus \( C^\infty(\Omega) \) is a Fréchet space.

4 Test functions

Let \( \Omega \) be an open subset of \( \mathbb{R}^n \). If \( f : \Omega \to \mathbb{C} \) is a function, the support of \( f \) is the closure of the set \( \{ x \in \Omega : f(x) \neq 0 \} \). We denote the support of \( f \) by \( \text{supp} f \). If \( \text{supp} f \) is a compact set, we say that \( f \) has compact support, and we denote by \( C^\infty_c(\Omega) \) the set of all elements of \( C^\infty(\Omega) \) with compact support. We write \( \mathcal{D}(\Omega) = C^\infty_c(\Omega) \).

For \( f \in \mathcal{D}(\Omega) \), we define

\[
    \| f \|_N = \sup_{|\alpha| \leq N} \sup_{x \in \Omega} |(\partial^\alpha f)(x)|.
\]

If \( K \) is a compact subset of \( \Omega \), we define

\[
    \mathcal{D}(K) = \{ f \in C^\infty_c(\Omega) : \text{supp} f \subseteq K \}.
\]

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The restriction of these norms to $\mathcal{D}(K)$ are norms, in particular seminorms. Hence, with the topology for which a local basis at 0 is the collection of sets of the form \( \{ f \in \mathcal{D}(K) : \| f \|_N < 1/N \} \), we have that $\mathcal{D}(K)$ is a locally convex space, and because there are countably many seminorms $\| \cdot \|_N$, the space is metrizable. One checks that the topology on $\mathcal{D}(K)$ is equal to the subspace topology it inherits from $C^\infty(\Omega)$.

Theorem 6. If $\Omega$ is an open subset of $\mathbb{R}^n$ and $K$ is a compact subset of $\Omega$, then $\mathcal{D}(K)$ is a closed subspace of the Fréchet space $C^\infty(\Omega)$.

Proof. Let $f_i \in \mathcal{D}(K)$, $f \in C^\infty(\Omega)$, and suppose that $f_i \to f$ in $C^\infty(\Omega)$. If $x \in \Omega \setminus K$, then $f_i(x) = 0$. There is some $K_N$ that contains $K$, and the fact that $f_i \to f$ gives us in particular that

\[
|f(x)| = |0 - f(x)| = |f_i(x) - f(x)| \leq \sup_{y \in K_N} |f_i(y) - f(y)| \to 0,
\]

hence $f(x) = 0$. This shows that supp $f \subseteq K$, and hence that $f \in \mathcal{D}(K)$.

Let $K_j$ be an exhaustion of $\Omega$ by compact sets. Check that $\mathcal{D}(K_j)$ is a closed subspace of $\mathcal{D}(K_{j+1})$ and that the inclusion $\mathcal{D}(K_j) \hookrightarrow \mathcal{D}(K_{j+1})$ is a homeomorphism onto its image. We define the following topology on the set $\mathcal{D}(\Omega)$. Let $\mathcal{B}$ be the collection of all convex balanced subsets $V$ of $\mathcal{D}(U)$ such that for all $j$, the set $V \cap \mathcal{D}(K_j)$ is open in $\mathcal{D}(K_j)$. (To be balanced means that $\alpha V \subseteq V$ if $|\alpha| \leq 1$.) We define $\mathcal{T}$ be the collection of all subsets $U$ of $\mathcal{D}(\Omega)$ such that $x_0 \in U$ implies that there is some $V \in \mathcal{B}$ for which $x_0 + V \subseteq U$. We check that $\mathcal{T}$ is a topology on $\mathcal{D}(\Omega)$, which we call the strict inductive limit topology. One proves\(^4\) that with this topology, $\mathcal{D}(\Omega)$ is a locally convex space. With the strict inductive limit topology, we call the locally convex space $\mathcal{D}(\Omega)$ the strict inductive limit of the Fréchet spaces $\mathcal{D}(K_1) \hookrightarrow \mathcal{D}(K_2) \hookrightarrow \cdots$, and write

\[
\mathcal{D}(\Omega) = \lim_{\longrightarrow} \mathcal{D}(K_j).
\]

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