

The nonlinear Schrödinger equation is Hamiltonian

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Let

$$H(u) = \int \frac{1}{2} |\nabla u|^2 + \frac{2\mu}{p+1} |u|^{p+1} dx.$$

We have

$$H(u + \epsilon v) = \int \frac{1}{2} \nabla(u + \epsilon v) \nabla(\bar{u} + \epsilon \bar{v}) + \frac{2\mu}{p+1} (u + \epsilon v)^{\frac{p+1}{2}} (\bar{u} + \epsilon \bar{v})^{\frac{p+1}{2}}.$$

We have, for $v \in T_u$,

$$\begin{aligned} dH(u)v &= \left. \frac{d}{d\epsilon} H(u + \epsilon v) \right|_{\epsilon=0} \\ &= \int \frac{1}{2} (\nabla v \nabla \bar{u} + \nabla u \nabla \bar{v}) \\ &\quad + \frac{2\mu}{p+1} \left(\frac{p+1}{2} u^{\frac{p-1}{2}} v \bar{u}^{\frac{p+1}{2}} + \frac{p+1}{2} u^{\frac{p+1}{2}} \bar{u}^{\frac{p-1}{2}} \bar{v} \right) dx \\ &= \Re \int \nabla \bar{v} \nabla u + 2\mu |u|^{p-1} u \bar{v} dx \\ &= \Re \int (-\Delta u + 2\mu |u|^{p-1} u) \bar{v} dx. \end{aligned}$$

Define

$$\omega(v_1, v_2) = -2\Im \int v_1 \bar{v}_2 dx.$$

The Hamiltonian vector field X_H of H is defined, for $v \in T_u$, by

$$\omega(X_H(u), v) = dH(u)v.$$

Therefore we have for $v \in T_u$ that

$$\begin{aligned} -2\Im \int X_H(u) \bar{v} dx &= \Re \int (-\Delta u + 2\mu |u|^{p-1} u) \bar{v} dx \\ &= \Im \int (-i\Delta u + 2i\mu |u|^{p-1} u) \bar{v} dx. \end{aligned}$$

It follows that

$$-2X_H(u) = -i\Delta u + 2i\mu|u|^{p-1}u,$$

i.e.,

$$X_H(u) = \frac{i}{2}\Delta u - i\mu|u|^{p-1}u.$$

The flow $S(t)$ of the Hamiltonian vector field X_H satisfies, for any u_0 and for $u(t) = S(t)u_0$,

$$u_t(t) = X_H(u(t)) = \frac{i}{2}\Delta u(t) - i\mu|u(t)|^{p-1}u(t).$$

This equation can be written as

$$iu_t(t) + \frac{1}{2}\Delta u(t) = \mu|u(t)|^{p-1}u(t).$$