

Projective limits of topological vector spaces

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1 Definitions

Let (I, \preceq) be a *directed poset*: if $i \in I$ then $i \preceq i$; if $i, j, k \in I$, $i \preceq j$, and $j \preceq k$, then $i \preceq k$: if $i, j \in I$, $i \preceq j$, and $j \preceq i$, then $i = j$; if $i, i \in I$, then there is some $k \in I$ such that $i \preceq k$ and $j \preceq k$.¹

A *projective system* in a category \mathfrak{C} is a set $\{X_i : i \in I\}$ of objects in \mathfrak{C} and a set $\{\phi_{ij} : X_i \rightarrow X_j : i, j \in I, i \succeq j\}$ of morphisms, such that if $i \in I$ then $\phi_{ii} = \text{id}_{X_i}$ and such that if $i, j, k \in I$ and $k \preceq j \preceq i$ then

$$\begin{array}{ccc} X_i & \xrightarrow{\phi_{ik}} & X_k \\ & \searrow \phi_{ij} & \nearrow \phi_{jk} \\ & X_j & \end{array}$$

commutes. If Y is an object in \mathfrak{C} and for each $i \in I$, $\psi_i : Y \rightarrow X_i$ is a morphism, we say that the morphisms ψ_i are *compatible* with the projective system $\{X_i, \phi_{ij}, I\}$ if $\phi_{ij} \circ \psi_i = \psi_j$ whenever $j \preceq i$, that is,

$$\begin{array}{ccc} X_i & \xrightarrow{\phi_{ij}} & X_j \\ & \swarrow \psi_i & \nwarrow \psi_j \\ & Y & \end{array}$$

commutes.

Suppose that X is an object in \mathfrak{C} and the morphisms $\phi_i : X \rightarrow X_i$ are compatible with the projective system $\{X_i, \phi_{ij}, I\}$. We say that X is a *projective limit* of the projective system if whenever Y is an object in \mathfrak{C} and there are

¹I am using the first chapter of L. Ribes and Z. Zalesskii, *Profinite Groups*, which has a clean presentation of projective and direct limits, and also Paul Garrett's notes *Functions on circles* and *Basic categorical constructions*, which are on his homepage. I have not used them, but J. L. Taylor, *Notes on locally convex topological vector spaces* looks readable and comprehensive.

morphisms $\psi_i : Y \rightarrow X_i$ that are compatible with the projective system, there is one and only one morphism $\psi : Y \rightarrow X$ such that

$$\phi_i \circ \psi = \psi_i$$

for all $i \in I$. This is a definition of a projective limit by a *universal property*, but it is not clear which projective systems have projective limits; but any object and morphisms that satisfy this property will be called a projective limit.

2 Direct products of topological vector spaces

A *topological vector space* is a vector space V over \mathbb{C} that is a Hausdorff topological space, and such that addition $V \times V \rightarrow V$ is continuous and scalar multiplication $\mathbb{C} \times V \rightarrow V$ is continuous.

Let I be a set and let V_i be topological vector spaces, where $\alpha_i : V_i \times V_i \rightarrow V_i$ is the addition map and $\mu_i : \mathbb{C} \times V_i \rightarrow V_i$ is the scalar multiplication map. Let

$$V = \prod_{i \in I} V_i = \left\{ v : I \rightarrow \bigcup_{i \in I} V_i : \text{if } i \in I \text{ then } v_i \in V_i \right\},$$

the Cartesian product of the sets V_i , and let V have the product topology. V is a Hausdorff topological space. For each $i \in I$, define $p_i : V \rightarrow V_i$ by $p_i(v) = v_i$, the projection from V to V_i ; p_i is continuous.

Define $\alpha : V \times V \rightarrow V$ by $\alpha(v, w)_i = \alpha_i(v_i, w_i)$, and define $\mu : \mathbb{C} \times V \rightarrow V$ by $\mu(\lambda, v)_i = \mu_i(\lambda, v_i)$. V is a vector space with addition α and scalar multiplication μ . Let U_i be an open set in V_i and let $U = p_i^{-1}(U_i)$. Let $W_j = \alpha_i^{-1}(U_i)$ for $j = i$ and $W_j = V_j \times V_j$ for $j \neq i$, and let $W = \prod_{j \in I} W_j$: $W = \alpha^{-1}(U)$ and W is an open set in $V \times V$. Since $\alpha^{-1}(U)$ is an open set in $V \times V$ for each subbasic open set U in V , it follows that α is continuous. Moreover, let U_i be an open set in V_i and let $U = p_i^{-1}(U_i)$. Let $W_j = \alpha_i^{-1}(U_i)$ for $j = i$ and $W_j = \mathbb{C} \times V_j$ for $j \neq i$, and let $W = \prod_{j \in I} W_j$: $W = \mu_i^{-1}(U)$ and W is an open set in $\mathbb{C} \times V$. It follows that μ is continuous. Therefore V is a topological vector space.²

3 Projective limits of topological vector spaces

Let \mathfrak{C} be the category of topological vector spaces: an object of \mathfrak{C} is a topological vector space, and a morphism $V \rightarrow W$ is a linear map that is continuous. Let $\{V_i, \phi_{ij}, I\}$ be a projective system in \mathfrak{C} . Let

$$X = \prod_{j \in I} V_j,$$

and let $p_i : X \rightarrow V_i$ be the projection map. Define

$$V = \left\{ v \in X : \text{if } i, j \in I \text{ and } j \preceq i \text{ then } \phi_{ij}(p_i(v)) = p_j(v) \right\}.$$

²Paul Garrett in his notes gives a diagrammatic proof that the direct product of topological vector spaces is a topological vector space.

V is a Hausdorff topological space with the subspace topology. Since both p_i and ϕ_{ij} are linear, it follows that V is a vector subspace of X . Let $\alpha : X \times X \rightarrow X$ be addition and $\mu : \mathbb{C} \times X \rightarrow X$ be scalar multiplication, and let $\iota : V \hookrightarrow X$ be the inclusion map. ι is continuous, so $\iota \times \iota : V \times V \rightarrow X \times X$ is continuous, and hence $\alpha \circ (\iota \times \iota) : V \times V \rightarrow V$ and $\mu \circ (\iota \times \iota) : V \times V \rightarrow V$ are continuous; that their codomains are V follows from V being a vector subspace of X . Thus V is a topological vector space.

For $i \in I$, define $\phi_i : V \rightarrow V_i$ by $\phi_i = p_i \circ \iota$, which is a morphism. If $j \preceq i$ then $\phi_{ij} \circ \phi_i = \phi_j$, which follows from the definition of V , so $\phi_i : V \rightarrow V_i$ are compatible with the projective system $\{V_i, \phi_{ij}, I\}$. Suppose that Y is a topological vector space and $\psi_i : Y \rightarrow V_i$ are compatible with the projective system $\{V_i, \phi_{ij}, I\}$. Define $\psi : Y \rightarrow V$ by $\psi(y)_i = \psi_i(y)$; that the codomain of ψ is V follows from $\psi_i : Y \rightarrow V_i$ being compatible with the projective system. Since each ψ_i is linear, ψ is linear. Let U_i be open in V_i . We have

$$\psi^{-1}(V \cap p_i^{-1}(U_i)) = \psi_i^{-1}(U_i),$$

hence $\psi^{-1}(U)$ for each subbasic open set U in V , and so ψ is continuous. Let $y \in Y$.

$$\phi_i(\psi(y)) = p_i(\iota(\psi(y))) = \psi(y)_i = \psi_i(y),$$

so $\phi_i \circ \psi = \psi_i$ for each in $i \in I$. Suppose that $\chi : Y \rightarrow V$ is a morphism such that

$$\phi_i \circ \chi = \psi_i$$

for all $i \in I$. On the one hand $\phi_i(\chi(y)) = \chi(y)_i$, and on the other hand $\psi_i(y) = \psi(y)_i$, so $\phi_i \circ \chi = \psi$ for all $i \in I$. Therefore V is a projective limit of the projective system $\{V_i, \phi_{ij}, I\}$.

In any category: Let $\{X_i, \phi_{ij}, I\}$ be a projective system. If (X, ϕ_i) and (Y, ψ_i) are projective limits, then there is a unique isomorphism $\phi : X \rightarrow Y$ such that $\psi_i \circ \phi = \phi_i$ for all i . This can be cleanly proved in a straightforward way; a proof is written down in Ribes and Zalesskii for the category of topological spaces, but it doesn't use anything that is special to that category.³

Thus in the category of topological vector spaces, a projective limit of a projective system exists and is unique up to unique isomorphism. We denote the projective limit of a projective system $\{V_i, \phi_{ij}, I\}$ of topological vector spaces by

$$\varprojlim_{i \in I} V_i.$$

Let $v \in \prod_{i \in I} V_i \setminus \varprojlim_{i \in I} V_i$. Since $v \notin \varprojlim_{i \in I} V_i$, there are some $r, s \in I, r \succeq s$ such that $\phi_{rs}(v_r) \neq v_s$. As $\phi_{rs}(v_r)$ and v_s are distinct points in the Hausdorff space V_s , there are disjoint open sets U_1, U_2 in V_s with $\phi_{rs}(v_r) \in U_1$ and $v_s \in U_2$.

³Ribes and Zalesskii's book is about profinite groups, which are projective limits of finite discrete topological groups; indeed there exist projective limits in the category of topological groups. An example of an infinite profinite group is the p -adic integers \mathbb{Z}_p , for p prime, which is the projective limit of the groups \mathbb{Z}/p^n , where the map $\mathbb{Z}/p^n \rightarrow \mathbb{Z}/p^m, n \geq m$, is taking the remainder modulo p^m .

Let $U'_1 = \phi_{rs}^{-1}(U_1)$; this is an open set in V_r . Define $W_r = U'_1$, $W_s = U_2$, and $W_i = V_i$ if $i \neq r, s$. $\prod_{i \in I} W_i$ is open in $\prod_{i \in I} V_i$, and $w \in \prod_{i \in I} W_i$. Let $w \in \prod_{i \in I} W_i$. $\phi_{rs}(w_r) \in U_1$ and $w_s \in U_2$, and U_1 and U_2 are disjoint, so $\phi_{rs}(w_r) \neq w_s$. Thus $w \notin \varprojlim_{i \in I} V_i$. Therefore, $\varprojlim_{i \in I} V_i$ is a closed subset of $\prod_{i \in I} V_i$.

4 Locally convex topological vector spaces

We say that a topological vector space V is *locally convex* if for every open neighborhood U of 0, there is some convex open $0 \in U' \subseteq U$. Let (V_j, ϕ_{ij}, I) be a projective system of locally convex topological vector spaces and let V be the projective limit in the category of topological vector spaces. We shall show that V is locally convex. Let $J \subset I$ be finite and let U_j be open in V_j for each $j \in J$. Since V_j is locally convex, for each $j \in J$ there is some convex open $U'_j \subseteq U_j$ such that $0 \in U'_j$. Define $W_j = U'_j$ for $j \in J$ and $W_i = V_i$ for $i \in I \setminus J$, and let $W = \prod_{i \in I} W_i$. W is open and convex. This shows that $\prod_{i \in I} V_i$ is locally convex. It is apparent that a vector subspace of a locally convex topological vector space is itself a locally convex topological vector space with the product topology. Thus V is locally convex. It follows that a projective system in the category of locally convex topological vector spaces has a projective limit that is unique up to unique isomorphism. (A morphism of locally convex topological vector spaces is the same as a morphism of topological vector spaces.)

5 Cofinality

Let (I, \preceq) be a directed poset, and suppose that $J \subseteq I$ is also directed with \preceq .⁴ We say that J is *cofinal* in I if for all $i \in I$ there is some $j \in J$ such that $i \preceq j$. Let \mathfrak{C} be a category in which there exists a projective limit of any projective system; we have shown above that the category of topological vector spaces and the category of locally convex topological vector spaces are such categories. This projective limit will be unique up to unique isomorphism.

Let (X_i, ϕ_{ij}, I) be a projective system and let J be cofinal in I . Then (X_i, ϕ_{ij}, J) is a projective system. Let (X_I, ϕ_i, I) be a projective limit of the first system and let (X_J, ϕ'_i, J) be a projective limit of the second system. Let $i \in I$, take $j \in J$ with $j \succeq i$, and define $\psi_i : X_J \rightarrow X_i$ by $\psi_i = \phi_{ji} \circ \phi'_j$; if $j, j' \succeq i$ then $\phi_{j'i} \circ \phi'_{j'} = \phi_{ji} \circ \phi'_j$. Let $i', i \in I$ with $i' \preceq i$, and let $j \in J$ with $i \preceq j$. Because (X_i, ϕ_{ij}, I) is a projective system, $\phi_{ii'} \circ \phi_{ji} = \phi_{ji'}$, and hence

$$\phi_{ii'} \circ \psi_i = \phi_{ii'} \circ \phi_{ji} \circ \phi'_j = \phi_{ji'} \circ \phi'_j = \psi_{i'}$$

showing that the morphisms $\psi_i : X_J \rightarrow X_i$ are compatible with the projective system (X_i, ϕ_{ij}, I) . As (X_I, ϕ_i, I) is a projective limit of (X_i, ϕ_{ij}, I) , it follows that there is a unique morphism $\psi : X_J \rightarrow X_I$ such that $\phi_i \circ \psi = \psi_i$ for all $i \in I$.

⁴This section follows Paul Garrett's notes *Basic categorical constructions*.

Since $\phi_i : X_I \rightarrow X_i$, $i \in I$, are compatible with the projective system (X_i, ϕ_{ij}, I) , it follows that $\phi_j : X_I \rightarrow X_j$, $j \in J$, are compatible with (X_i, ϕ_{ij}, J) . As (X_J, ϕ'_j, J) is a projective limit of (X_i, ϕ_{ij}, I) , there is a unique morphism $\phi : X_I \rightarrow X_J$ such that $\phi'_j \circ \phi = \phi_j$ for all $j \in J$.

On the one hand, $\phi \circ \psi : X_J \rightarrow X_J$ is a morphism such that $\phi'_j \circ (\phi \circ \psi) = \phi'_j$ for all $j \in J$. On the other hand, $\text{id}_{X_J} : X_J \rightarrow X_J$ is a morphism such that $\phi'_j \circ \text{id}_{X_J} = \phi'_j$ for all $j \in J$. By the definition of projective limit, we get $\phi \circ \psi = \text{id}_{X_J}$. Likewise, $\psi \circ \phi = \text{id}_{X_I}$. Thus $\psi : X_J \rightarrow X_I$ and $\phi : X_I \rightarrow X_J$ are isomorphisms. Since X_J and X_I are isomorphic, (X_J, ψ_i, I) is a projective limit of (X_i, ϕ_{ij}, I) . This is similar to how the limit of a convergent sequence is the same as the limit of any infinite subsequence.