Arnold's theorem on analytic circle diffeomorphisms

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We are reorganizing and expanding on the presentation in [6]. Arnold's paper: [1]. Other sources that present the theorem, as I come across them: [2], [3].

Let $S_\sigma = \{z \in \mathbb{C} : |\Im z| < \sigma\}$, let $\|\eta\|_\sigma = \sup_{|\Im z|<\sigma} |\eta(z)|$, and let

$$B_\sigma = \{\eta : \eta \text{ is holomorphic on } S_\sigma,\ \eta(x+1) = \eta(x), \text{ and } \|\eta\|_\sigma < \infty\}.$$

For each $\sigma > 0$, the set B_σ is a Banach space with the norm $\|\cdot\|_\sigma$.

We say that $\rho \in \mathbb{R}$ is of type (K, ν) if $|\rho - \frac{m}{n}| > K|n|^{-\nu}$ for all $(m, n) \in \mathbb{Z}^2$ with $n \neq 0$. If ρ is of type (K, ν) with $K > 1$ then ρ is also of type $(1, \nu)$, and thus we can assume that ρ is of type (K, ν) with $K \leq 1$. Suppose that $\rho \in \mathbb{R} \setminus \mathbb{Q}$. Let $\epsilon > 0$, and let $K > 0$. It follows from Dirichlet's approximation theorem [4, p. 155, Theorem 185] that there is some $(m, n) \in \mathbb{Z}^2$ with $n \geq \frac{1}{K^{1/\epsilon}}$ such that $|\rho - \frac{m}{n}| < \frac{1}{n^2}$. Then

$$|\rho - \frac{m}{n}| < \frac{1}{n^2} = \frac{1}{n^{2-\epsilon}} \frac{1}{n^\epsilon} \leq \frac{1}{n^{2-\epsilon}} K.$$

Hence ρ is not of type $(K, 2-\epsilon)$. Therefore if $\rho \in \mathbb{R} \setminus \mathbb{Q}$ is of type (K, ν) then $\nu \geq 2$.

Let $F : \mathbb{R} \to \mathbb{R}$ be an increasing homeomorphism that satisfies $F(x+1) = F(x) + 1$ for all $x \in \mathbb{R}$. The *rotation number* of F is defined to be

$$\lim_{n\to\infty} \frac{F^n(x) - x}{n}.$$

This limit exists for all $x \in \mathbb{R}$ and is the same for all $x \in \mathbb{R}$ [5, p. 387, Proposition 11.1.1]. If $H : \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism that satisfies $H(x+1) = H(x) + 1$ for all $x \in \mathbb{R}$, then $H^{-1} \circ F \circ H$ has the same rotation number as F [5, p. 388, Proposition 11.1.3].

Proposition 1. *Let $\eta \in B_\sigma$, let $\rho \in \mathbb{R} \setminus \mathbb{Q}$ be of type (K, ν) and define h by $\widehat{h}(k) = \frac{\widehat{\eta}(k)}{e^{2\pi i k\rho} - 1}$ for $k \neq 0$ and $\widehat{h}(0) = 0$. Then for any $0 < \delta < \frac{1}{2\pi}$ we have $h \in B_{\sigma-\delta}$ and*

$$\|h\|_{\sigma-\delta} < \frac{\Gamma(\nu)}{K(2\pi\delta)^\nu} \|\eta\|_\sigma.$$

Proof. We first show that $|\widehat{\eta}(n)| \le \|\eta\|_\sigma e^{-2\pi\sigma|n|}$ for all k. Say $k > 0$. For any $\epsilon > 0$ by the residue theorem we have

$$\widehat{\eta}(k) = \int_0^1 e^{-2\pi i k z} \eta(z) dz = \int_0^{-i(\sigma-\epsilon)} + \int_{-i(\sigma-\epsilon)}^{1-i(\sigma-\epsilon)} + \int_{1-i(\sigma-\epsilon)}^1 .$$

Since $\eta(z+1) = \eta(z)$,

$$\int_{1-i(\sigma-\epsilon)}^1 e^{-2\pi i k z}\eta(z)dz = \int_{-i(\sigma-\epsilon)}^0 e^{-2\pi i k z}\eta(z)dz = -\int_0^{-i(\sigma-\epsilon)} e^{-2\pi i k z}\eta(z)dz,$$

so

$$\widehat{\eta}(k) = \int_{-i(\sigma-\epsilon)}^{1-i(\sigma-\epsilon)} e^{-2\pi i k z}\eta(z)dz = e^{-2\pi k(\sigma-\epsilon)} \int_0^1 e^{-2\pi i k x}\eta(x - i(\sigma-\epsilon))dx,$$

and hence

$$|\widehat{\eta}(k)| \le e^{-2\pi k(\sigma-\epsilon)}\|\eta\|_\sigma.$$

This is true for all $\epsilon > 0$, so we have

$$|\widehat{\eta}(k)| \le e^{-2\pi k\sigma}\|\eta\|_\sigma.$$

For $k < 0$ we use a contour in the upper half-plane rather than a contour in the lower half-plane and get $|\widehat{\eta}(k)| \le e^{2\pi k\sigma}\|\eta\|_\sigma$, proving that $|\widehat{\eta}(k)| \le \|\eta\|_\sigma e^{-2\pi\sigma|k|}$ for all k.

Let $k \neq 0$, and let m be such that $|\rho k - m| \le \frac{1}{2}$. We have

$$|e^{2\pi i\rho k} - 1| = |e^{2\pi i\rho k} - e^{2\pi i m}| = |e^{2\pi i(\rho k - m)} - 1| = 2|\sin\pi(\rho k - m)|.$$

For $|x| \le \frac{\pi}{2}$ we have $|\sin x| \ge \frac{2}{\pi}|x|$, so

$$2|\sin\pi(\rho k - m)| \ge 2\frac{2}{\pi}|\pi(\rho k - m)| = 4|n||\rho - \frac{m}{k}|.$$

But ρ is of type (K, ν) so we get for all $k \neq 0$ that

$$|e^{2\pi i\rho k} - 1| \ge 4K|k|^{-(\nu-1)}.$$

Let $\delta > 0$. If $|\Im z| \le \sigma - \delta$ then

$$\begin{aligned} |h(z)| &= \Big|\sum_{k\neq 0} e^{2\pi i k z}\frac{\widehat{\eta}(k)}{e^{2\pi i k\rho} - 1}\Big| \\ &\le \sum_{k\neq 0} e^{2\pi|k|(\sigma-\delta)}\frac{e^{-2\pi k\sigma}\|\eta\|_\sigma}{4K|k|^{-(\nu-1)}} \\ &= \frac{\|\eta\|_\sigma}{4K}\sum_{k\neq 0} e^{-2\pi|k|\delta}|k|^{\nu-1}. \end{aligned}$$

One can check that $\frac{\Gamma(\nu)}{(2\pi\delta)^\nu} = \int_0^\infty y^{\nu-1}e^{-2\pi\delta y}dy$, and because $2\pi\delta < \nu - 1$ we have that for $y \geq 1$ the integrand is decreasing. Therefore, since $\nu \geq 2$,

$$\sum_{k\geq 1} e^{-2\pi k\delta}k^{\nu-1} \leq e^{-2\pi\delta} + \frac{\Gamma(\nu)}{(2\pi\delta)^\nu} < 2\frac{\Gamma(\nu)}{(2\pi\delta)^\nu},$$

and so

$$\sum_{k\neq 0} e^{-2\pi|k|\delta}|k|^{\nu-1} < 4\frac{\Gamma(\nu)}{(2\pi\delta)^\nu}.$$

□

Proposition 2. *Let $\eta \in B_\sigma$, let $\rho \in \mathbb{R}\setminus\mathbb{Q}$ be of type (K,ν), define h by $\widehat{h}(k) = \frac{\widehat{\eta}(k)}{e^{2\pi ik\rho}-1}$ for $k \neq 0$ and $\widehat{h}(0) = 0$, and let $H(z) = z + h(z)$. If $0 < \delta < \frac{1}{2\pi}$ and $\frac{2\pi\Gamma(\nu)}{K(2\pi\delta)^{\nu+1}}\|\eta\|_\sigma < 1$, then there is a holomorphic $H^{-1}: S_{\sigma-3\delta} \to S_{\sigma-2\delta}$, and $H^{-1}(z) = z - h(z) + g(z)$ with $g \in B_{\sigma-4\delta}$ and*

$$\|g\|_{\sigma-4\delta} < \frac{2\pi\Gamma(\nu)^2}{K^2(2\pi\delta)^{2\nu+1}}\|\eta\|_\sigma^2.$$

Proof. By Proposition 1 we have $h \in B_{\sigma-\delta}$. Using the maximum modulus principle and Cauchy's integral formula we get that $\|h'\|_{\sigma-2\delta} \leq \frac{\|h\|_{\sigma-\delta}}{\delta}$, and by Proposition 1 this is $< \frac{2\pi\Gamma(\nu)}{K(2\pi\delta)^{\nu+1}}\|\eta\|_\sigma$, which by hypothesis is < 1 (and so $\|h\|_{\sigma-\delta} < \delta$). Then for $z \in S_{\sigma-2\delta}$ we have $|H'(z)| = |1 + h'(z)| > 0$, so by the inverse function theorem there is a holomorphic $H^{-1}: H(S_{\sigma-2\delta}) \to S_{\sigma-2\delta}$.

Let $a \in S_{\sigma-3\delta}$, let $K \subset S_{\sigma-2\delta}$ be the circle about a of radius δ, and let $f(z) = z - a$. Then for $z \in K$ we have $|h(z)| < \delta = |f(z)|$, so by Rouché's theorem f and $f + h$ have the same number of zeros in the interior of K. Of course f has one zero in the interior of K so too $f + h = H - a$ has one zero in the interior of K. Thus $a \in H(S_{\sigma-2\delta})$, and we conclude $S_{\sigma-3\delta} \subseteq H(S_{\sigma-2\delta})$. Therefore $H^{-1}: S_{\sigma-3\delta} \to S_{\sigma-2\delta}$.

Now define $g: S_{\sigma-3\delta} \to \mathbb{C}$ by $g(z) = H^{-1}(z) - z + h(z)$. For $z \in B_{\sigma-3\delta}$ we have

$$\begin{aligned} z + 1 &= H(H^{-1}(z)) + 1 \\ &= H^{-1}(z) + 1 + h(H^{-1}(z)) \\ &= H^{-1}(z) + 1 + h(H^{-1}(z) + 1) \\ &= H(H^{-1}(z) + 1), \end{aligned}$$

so $H^{-1}(z+1) = H^{-1}(z) + 1$, from which it follows that $g(z+1) = g(z)$.

Let $a \in S_{\sigma-4\delta}$, and again using Rouché's theorem we get $a \in H(S_{\sigma-3\delta})$. Hence $S_{\sigma-4\delta} \subseteq H(S_{\sigma-3\delta})$. Therefore if $\xi \in S_{\sigma-4\delta}$ then $H^{-1}(\xi) \in S_{\sigma-3\delta}$. Let $\xi \in S_{\sigma-4\delta}$ and let $z = H^{-1}(\xi) \in S_{\sigma-3\delta}$. We have

$$\begin{aligned} z &= H^{-1}(H(z)) \\ &= H(z) - h(H(z)) + g(H(z)) \\ &= z + h(z) - h(z + h(z)) + g(H(z)), \end{aligned}$$

and so

$$g(\xi) = \int_0^1 h'(H^{-1}(\xi) + sh(H^{-1}(\xi)))h(H^{-1}(\xi))ds.$$

Because $\|h\|_{\sigma-\delta} < \delta$, for $0 \le s \le 1$ we have $H^{-1}(\xi) + sh(H^{-1}(\xi)) \in B_{\sigma-2\delta}$. Then since $\|h\|_{\sigma-\delta} < \frac{\Gamma(\nu)}{K(2\pi\delta)^\nu}\|\eta\|_\sigma$ and $\|h'\|_{\sigma-2\delta} < \frac{2\pi\Gamma(\nu)}{K(2\pi\delta)^{\nu+1}}\|\eta\|_\sigma$, we get

$$|g(\xi)| < \frac{2\pi\Gamma(\nu)^2}{K^2(2\pi\delta)^{2\nu+1}}\|\eta\|_\sigma^2,$$

and so

$$\|g\|_{\sigma-4\delta} < \frac{2\pi\Gamma(\nu)^2}{K^2(2\pi\delta)^{2\nu+1}}\|\eta\|_\sigma^2.$$

□

Proposition 3. *Let $\eta \in B_\sigma$, let $\rho \in \mathbb{R} \setminus \mathbb{Q}$ be of type (K, ν), let $\phi(z) = z + \rho + \eta(z)$, define h by $\widehat{h}(k) = \frac{\widehat{\eta}(k)}{e^{2\pi ik\rho}-1}$ for $k \neq 0$ and $\widehat{h}(0) = 0$, define $H(z) = z+h(z)$, define $\psi(z) = H^{-1} \circ \phi \circ H(z)$, and define μ by $\psi(z) = z + \rho + \mu(z)$. If ϕ has rotation number ρ, $0 < \delta < \frac{1}{2\pi}$, and $\frac{2\pi\Gamma(\nu)}{K(2\pi\delta)^{\nu+1}}\|\eta\|_\sigma < 1$, then $\mu \in B_{\sigma-6\delta}$ and*

$$\|\mu\|_{\sigma-6\delta} < \frac{16\pi\Gamma(\nu)^2}{K^2(2\pi\delta)^{2\nu+1}}\|\eta\|_\sigma^2.$$

Proof. For z we have

$$\begin{aligned}
\psi(z) &= H^{-1} \circ \phi(z + h(z)) \\
&= H^{-1}(z + h(z) + \rho + \eta(z + h(z))) \\
&= z + h(z) + \rho + \eta(z + h(z)) - h(z + h(z) + \rho + \eta(z + h(z))) \\
&\quad +g(z + h(z) + \rho + \eta(z + h(z))) \\
&= z + \rho + \Big(h(z) - h(z + \rho) + \eta(z)\Big) + \Big(\eta(z + h(z)) - \eta(z)\Big) \\
&\quad +\Big(h(z + \rho) - h(z + h(z) + \rho + \eta(z + h(z)))\Big) \\
&\quad +g(z + h(z) + \rho + \eta(z + h(z))) \\
&= z + \rho + A(z) + B(z) + C(z) + D(z).
\end{aligned}$$

We have

$$\mu(z) = A(z) + B(z) + C(z) + D(z).$$

First, $h(x + \rho) - h(x) = \eta(x) - \widehat{\eta}(0)$, so $A(z) = \widehat{\eta}(0)$. Second,

$$B(z) = \int_0^1 \eta'(z + sh(z))h(z)ds.$$

Since $\frac{2\pi\Gamma(\nu)}{K(2\pi\delta)^{\nu+1}}\|\eta\|_\sigma < 1$ we have by Proposition 1 that $\|h\|_{\sigma-\delta} < \delta$, so for $z \in S_{\sigma-2\delta}$ and $0 \le s \le 1$ we get $z + sh(z) \in S_{\sigma-\delta}$. Using the maximum modulus principle and Cauchy's integral formula we get that $\|\eta'\|_{\sigma-\delta} \leq \frac{\|\eta\|_\sigma}{\delta}$. Therefore

$$\|B\|_{\sigma-2\delta} < \frac{\|\eta\|_\sigma}{\delta} \frac{\Gamma(\nu)}{K(2\pi\delta)^\nu} \|\eta\|_\sigma = \frac{2\pi\Gamma(\nu)}{K(2\pi\delta)^{\nu+1}} \|\eta\|_\sigma^2.$$

Third,

$$C(z) = \int_0^1 h'(z + \rho + s(h(z) + \eta(z + h(z))))(h(z) + \eta(z + h(z))ds.$$

We have that

$$\|\eta\|_\sigma < \frac{K(2\pi\delta)^{\nu+1}}{2\pi\Gamma(\nu)} \leq \frac{K \cdot \delta}{\Gamma(\nu)} \leq K\delta \leq \delta.$$

For $z \in S_{\sigma-4\delta}$ and $0 \leq s \leq 1$ we have

$$z + \rho + s(h(z) + \eta(z + h(z))) \in S_{\sigma-2\delta}.$$

We have $\|h'\|_{\sigma-2\delta} \leq \frac{\|h\|_{\sigma-\delta}}{\delta}$, and so by Proposition 1 we get $\|h'\|_{\sigma-2\delta} < \frac{2\pi\Gamma(\nu)}{K(2\pi\delta)^{\nu+1}}\|\eta\|_\sigma$. Therefore for $z \in S_{\sigma-4\delta}$ we get

$$|C(z)| < \frac{2\pi\Gamma(\nu)}{K(2\pi\delta)^{\nu+1}} \|\eta\|_\sigma \Big(\frac{\Gamma(\nu)}{K(2\pi\delta)^\nu} \|\eta\|_\sigma + \|\eta\|_\sigma \Big) < \frac{4\pi\Gamma(\nu)^2}{K^2(2\pi\delta)^{2\nu+1}} \|\eta\|_\sigma^2.$$

Fourth, if $z \in S_{\sigma-6\delta}$ then $z + h(z) + \rho + \eta(z + h(z)) \in S_{\sigma-4\delta}$. Thus by Proposition 2 we get

$$\|D\|_{\sigma-6\delta} < \frac{2\pi\Gamma(\nu)^2}{K^2(2\pi\delta)^{2\nu+1}} \|\eta\|_\sigma^2.$$

Since ψ is conjugate to ϕ it has rotation number ρ, so there is some $x_0 \in \mathbb{R}$ such that $\psi(x_0) = x_0 + \rho$. Thus

$$x_0 + \rho = x_0 + \rho + \widehat{\eta}(0) + B(x_0) + C(x_0) + D(x_0),$$

so

$$\widehat{\eta}(0) = -B(x_0) - C(x_0) - D(x_0).$$

Of course $x_0 \in S_{\sigma-6\delta}$, so by what we've done so far in this proof,

$$|\widehat{\eta}(0)| < \frac{2\pi\Gamma(\nu)}{K(2\pi\delta)^{\nu+1}} \|\eta\|_\sigma^2 + \frac{4\pi}{K^2(2\pi\delta)^{2\nu+1}} \Gamma(\nu)^2 \|\eta\|_\sigma^2 + \frac{2\pi\Gamma(\nu)^2}{K^2(2\pi\delta)^{2\nu+1}} \|\eta\|_\sigma^2.$$

Therefore

$$\begin{aligned}\|\mu\|_{\sigma-6\delta} &< 2\cdot\Big(\frac{2\pi\Gamma(\nu)}{K(2\pi\delta)^{\nu+1}}\|\eta\|_\sigma^2+\frac{4\pi\Gamma(\nu)^2}{K^2(2\pi\delta)^{2\nu+1}}\|\eta\|_\sigma^2\\ &\quad+\frac{2\pi\Gamma(\nu)^2}{K^2(2\pi\delta)^{2\nu+1}}\|\eta\|_\sigma^2\Big)\\ &\le 2\cdot\Big(\frac{2\pi\Gamma(\nu)^2}{K^2(2\pi\delta)^{2\nu+1}}\|\eta\|_\sigma^2+\frac{4\pi\Gamma(\nu)^2}{K^2(2\pi\delta)^{2\nu+1}}\|\eta\|_\sigma^2\\ &\quad+\frac{2\pi\Gamma(\nu)^2}{K^2(2\pi\delta)^{2\nu+1}}\|\eta\|_\sigma^2\Big)\\ &= \frac{16\pi\Gamma(\nu)^2}{K^2(2\pi\delta)^{2\nu+1}}\|\eta\|_\sigma^2.\end{aligned}$$

□

Lemma 4. *Let $\rho\in\mathbb{R}\setminus\mathbb{Q}$ be of type (K,ν). Let $\eta_0\in B_{\sigma_0}$ and let $\epsilon_0=\|\eta_0\|_{\sigma_0}$. Suppose that*

$$\epsilon_0<\Big(\frac{K}{16\pi\Gamma(\nu)}\Big(\frac{\sigma_0}{36}\Big)^{\nu+1}\Big)^8, \tag{1}$$

and that $\frac{\sigma_0}{36}<\frac{1}{2\pi}$.

Let $\phi_0(z)=z+\rho+\eta_0(z)$, and suppose that ϕ_0 has rotation number ρ. For $n\ge 0$ let

- $\widehat{h_n}(k)=\frac{\widehat{\eta_n}(k)}{e^{2\pi ik\rho}-1}$ *for $k\neq 0$ and $\widehat{h_n}(0)=0$*
- $H_n(z)=z+h_n(z)$, *and* $g_n(z)=H_n^{-1}(z)-z+h_n(z)$
- $\phi_{n+1}=H_n^{-1}\circ\phi_n\circ H_n$
- $\eta_{n+1}(z)=\phi_{n+1}(z)-z-\rho$
- $\delta_n=\frac{\sigma_0}{36(1+n^2)}$
- $\sigma_{n+1}=\sigma_n-6\delta_n$
- $\epsilon_{n+1}=\epsilon_0^{(3/2)^{n+1}}$

Then for $n\ge 0$ we have that

- $\|\eta_{n+1}\|_{\sigma_{n+1}}\le\epsilon_{n+1}$
- $\|h_n\|_{\sigma_n-\delta_n}<\frac{\Gamma(\nu)\epsilon_n}{K(2\pi\delta_n)^\nu}$
- $\|g_n\|_{\sigma_n-4\delta_n}<\frac{2\pi\Gamma(\nu)^2\epsilon_n^2}{K^2(2\pi\delta_n)^{2\nu+1}}$

Proof. We first verify the claim for $n = 0$. First, $\delta_0 = \frac{\sigma_0}{36} < \frac{1}{2\pi}$, and we have

$$\epsilon_0 < \Big(\frac{K}{16\pi\Gamma(\nu)}\Big(\frac{\sigma_0}{36}\Big)^{\nu+1}\Big)^8 \leq \frac{K}{16\pi\Gamma(\nu)}\Big(\frac{\sigma_0}{36}\Big)^{\nu+1} = \frac{K\delta_0^{\nu+1}}{2\pi\Gamma(\nu)}\frac{1}{8} < \frac{K(2\pi\delta_0)^{\nu+1}}{2\pi\Gamma(\nu)},$$

which gives us that $\frac{2\pi\Gamma(\nu)}{K(2\pi\delta_0)^{\nu+1}}\|\eta_0\|_{\sigma_0} < 1$. Thus by Proposition 3 we have

$$\|\eta_1\|_{\sigma_1} = \|\eta_1\|_{\sigma_0-6\delta_0} < \frac{16\pi\Gamma(\nu)^2}{K^2(2\pi\delta_0)^{2\nu+1}}\|\eta_0\|^2_{\sigma_0} = \frac{16\pi\Gamma(\nu)^2\epsilon_0^2}{K^2(2\pi\delta_0)^{2\nu+1}}.$$

By (1) it follows that $\frac{16\pi\Gamma(\nu)^2\epsilon_0^2}{K^2(2\pi\delta_0)^{2\nu+1}} \leq \epsilon_0^{3/2}$, and so we get $\|\eta_1\|_{\sigma_1} \leq \epsilon_1$.

By Proposition 1 we get

$$\|h_0\|_{\sigma_0-\delta_0} < \frac{\Gamma(\nu)}{K(2\pi\delta_0)^\nu}\|\eta_0\|_{\sigma_0} = \frac{\Gamma(\nu)\epsilon_0}{K(2\pi\delta_0)^\nu},$$

and by Proposition 2 we get

$$\|g\|_{\sigma_0-4\delta_0} < \frac{2\pi\Gamma(\nu)^2}{K^2(2\pi\delta_0)^{2\nu+1}}\|\eta_0\|^2_{\sigma_0} = \frac{2\pi\Gamma(\nu)^2\epsilon_0^2}{K^2(2\pi\delta_0)^{2\nu+1}}.$$

This verifies the claim for $n = 0$. Now we suppose that the claim is true for $n \leq N$, and we shall show that the claim is true for $n = N + 1$.

By assumption we have $\|\eta_{N+1}\|_{\sigma_{N+1}} \leq \epsilon_{N+1}$. Then by Proposition 1 we have

$$\|h_{N+1}\|_{\sigma_{N+1}-\delta_{N+1}} < \frac{\Gamma(\nu)}{K(2\pi\delta_{N+1})^\nu}\|\eta_{N+1}\|_{\sigma_{N+1}} \leq \frac{\Gamma(\nu)\epsilon_{N+1}}{K(2\pi\delta_{N+1})^\nu}.$$

□

One can prove by induction that $\epsilon_n < \frac{K(2\pi\delta_n)^{\nu+1}}{2\pi\Gamma(\nu)}$ for all $n \geq 0$, from which we have $\frac{2\pi\Gamma(\nu)}{K(2\pi\delta_{N+1})^{\nu+1}}\|\eta_{N+1}\|_{\sigma_{N+1}} < 1$. Therefore by Proposition 2 we get

$$\|g_{N+1}\|_{\sigma_{N+1}-4\delta_{N+1}} < \frac{2\pi\Gamma(\nu)^2}{K^2(2\pi\delta_{N+1})^{2\nu+1}}\|\eta_{N+1}\|^2_{\sigma_{N+1}} \leq \frac{2\pi\Gamma(\nu)^2\epsilon_{N+1}^2}{K^2(2\pi\delta_{N+1})^{2\nu+1}}.$$

Finally, we have by Proposition 3 and by assumption that

$$\|\eta_{N+2}\|_{\sigma_{N+1}-6\delta_{N+1}} < \frac{16\pi\Gamma(\nu)^2}{K^2(2\pi\delta_{N+1})^{2\nu+1}}\|\eta_{N+1}\|^2_{\sigma_{N+1}} \leq \frac{16\pi\Gamma(\nu)^2\epsilon_{N+1}^2}{K^2(2\pi\delta_{N+1})^{2\nu+1}}.$$

By (1) it follows that $\frac{16\pi\Gamma(\nu)^2\epsilon_{N+1}^2}{K^2(2\pi\delta_{N+1})^{2\nu+1}} < \epsilon_{N+1}^{3/2}$, and so we get $\|\eta_{N+2}\|_{\sigma_{N+2}} \leq \epsilon_{N+2}$, completing the induction.

Theorem 5. *Arnold's theorem*

Proof. Let $\mathcal{H}_N = H_0 \circ H_1 \circ \cdots H_N$. By Lemma 4, $\mathcal{H}_N$ is holomorphic on $S_{\sigma_N-2\delta_N}$. And for $z \in S_{\sigma_N-2\delta_N}$,

$$|\mathcal{H}_N(z) - z| = |h_N(z) + \cdots + h_0(\cdots)| \leq \sum_{n=0}^N \|h_n\|_{\sigma_n-\delta_n} < \sum_{n=0}^N \frac{\Gamma(\nu)\epsilon_n}{K(2\pi\delta_n)^\nu}.$$

Let $\Delta = \sum_{n=0}^\infty \frac{\Gamma(nu)\epsilon_n}{K(2\pi\delta_n)^\nu}$. □

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