The Bernstein and Nikolsky inequalities for trigonometric polynomials

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January 28, 2015

1 Introduction

Let $T = \mathbb{R}/2\pi\mathbb{Z}$. For a function $f : T \to \mathbb{C}$ and $\tau \in T$, we define $f_\tau : T \to \mathbb{C}$ by $f_\tau(t) = f(t - \tau)$. For measurable $f : T \to \mathbb{C}$ and $0 < r < \infty$, write

$$\|f\|_r = \left( \frac{1}{2\pi} \int_T |f(t)|^r dt \right)^{1/r}.$$

For $f, g \in L^1(T)$, write

$$(f \ast g)(x) = \frac{1}{2\pi} \int_T f(t)g(x - t)dt, \quad x \in T,$$

and for $f \in L^1(T)$, write

$$\hat{f}(k) = \frac{1}{2\pi} \int_T f(t)e^{-ikt}dt, \quad k \in \mathbb{Z}.$$

This note works out proofs of some inequalities involving the support of $\hat{f}$ for $f \in L^1(T)$.

Let $\mathcal{T}_n$ be the set of trigonometric polynomials of degree $\leq n$. We define the \textbf{Dirichlet kernel} $D_n : T \to \mathbb{C}$ by

$$D_n(t) = \sum_{|j| \leq n} e^{ijt}, \quad t \in T.$$

It is straightforward to check that if $T \in \mathcal{T}_n$ then

$$D_n \ast T = T.$$
2 Bernstein’s inequality for trigonometric polynomials

DeVore and Lorentz attribute the following inequality to Szegö.\(^1\)

**Theorem 1.** If \(T \in \mathcal{T}_n\) and \(T\) is real valued, then for all \(x \in \mathbb{T}\),

\[
T'(x)^2 + n^2 T(x)^2 \leq n^2 \|T\|_\infty^2.
\]

**Proof.** If \(T = 0\) the result is immediate. Otherwise, take \(x \in \mathbb{T}\), and for real \(c > 1\) define

\[
P_c(t) = \frac{T(t + x)\text{sgn} T'(x)}{c \|T\|_\infty}, \quad t \in \mathbb{T}.
\]

\(P_c \in \mathcal{T}_n\), and satisfies

\[
P_c'(0) = \frac{T'(x)\text{sgn} T'(x)}{c \|T\|_\infty} \geq 0
\]

and \(\|P_c\|_\infty \leq \frac{1}{c} \leq 1\). Since \(\|P_c\|_\infty < 1\), in particular \(|P_c(0)| < 1\) and so there is some \(\alpha, |\alpha| < \frac{\pi}{2n}\), such that \(\sin n\alpha = P_c(0)\). We define \(S \in \mathcal{T}_n\) by

\[
S(t) = \sin n(t + \alpha) - P_c(t), \quad t \in \mathbb{T},
\]

which satisfies \(S(0) = \sin n\alpha - P_c(0) = 0\). For \(k = -n, \ldots, n\), let \(t_k = -\alpha + \frac{(2k-1)\pi}{2n}\), for which we have

\[
\sin n(t_k + \alpha) = \sin \frac{(2k-1)\pi}{2} = (-1)^{k+1}.
\]

Because \(\|P_c\|_\infty < 1\),

\[
\text{sgn} S(t_k) = (-1)^{k+1},
\]

so by the intermediate value theorem, for each \(k = -n, \ldots, n - 1\) there is some \(c_k \in (t_k, t_k+1)\) such that \(S(c_k) = 0\). Because

\[
t_n - t_{-n} = \frac{(2n-1)\pi}{2n} - \frac{(-2n-1)\pi}{2n} = 2\pi,
\]

it follows that if \(j \neq k\) then \(c_j\) and \(c_k\) are distinct in \(\mathbb{T}\). It is a fact that a trigonometric polynomial of degree \(n\) has \(\leq 2n\) distinct roots in \(\mathbb{T}\), so if \(t \in (t_k, t_k+1)\) and \(S(t) = 0\), then \(t = c_k\). It is the case that \(t_1 = -\alpha + \frac{\pi}{2n} > 0\) and \(t_0 = -\alpha - \frac{\pi}{2n} < 0\), so \(0 \in (t_0, t_1)\). But \(S(0) = 0\), so \(c_0 = 0\). Using \(S(t_1) = 1 > 0\) and the fact that \(S\) has no zeros in \((0, t_1)\) we get a contradiction from \(S'(0) < 0\), so \(S'(0) \geq 0\). This gives

\[
0 \leq P_c'(0) = n \cos n\alpha - S'(0) \leq n \cos n\alpha = n \sqrt{1 - \sin^2 n\alpha} = n \sqrt{1 - P_c(0)^2}.
\]

\(^1\)Ronald A. DeVore and George G. Lorentz, *Constructive Approximation*, p. 97, Theorem 1.1.
Thus \[ P_c'(0) \leq n \sqrt{1 - P_c(0)^2}, \]
or \[ n^2 P_c(0) + P_c'(0)^2 \leq n^2. \]
Because \[ P_c(0)^2 = \frac{T(x)^2}{c^2 \|T\|_\infty^2}, \quad P_c'(0)^2 = \frac{T'(x)^2}{c^2 \|T\|_\infty^2} \]
we get \[ n^2 T(x)^2 + T'(x)^2 \leq c^2 n^2 \|T\|_\infty^2. \]
Because this is true for all \( c > 1 \),
\[ n^2 T(x)^2 + T'(x)^2 \leq n^2 \|T\|_\infty^2, \]
completing the proof.

Using the above we now prove Bernstein’s inequality.\(^3\)

**Theorem 2** (Bernstein’s inequality). If \( T \in \mathcal{T}_n \), then
\[ \|T'\|_\infty \leq n \|T\|_\infty. \]

**Proof.** There is some \( x_0 \in \mathbb{T} \) such that \( |T'(x_0)| = \|T'\|_\infty \). Let \( \alpha \in \mathbb{R} \) be such that \( e^{i\alpha}T'(x_0) = \|T'\|_\infty \). Define \( S(x) = \text{Re} \left( e^{i\alpha}T(x) \right) \) for \( x \in \mathbb{T} \), which satisfies \( S'(x) = \text{Re} \left( e^{i\alpha}T'(x) \right) \) and in particular
\[ S'(x_0) = \text{Re} \left( e^{i\alpha}T'(x_0) \right) = e^{i\alpha}T'(x_0) = \|T'\|_\infty. \]
Because \( S \in \mathcal{T}_n \) and \( S \) is real valued, Theorem 1 yields \[ S'(x_0)^2 + n^2 S(x_0)^2 \leq n^2 \|S\|_\infty^2. \]
A fortiori,
\[ S'(x_0)^2 \leq n^2 \|S\|_\infty^2, \]
giving, because \( S'(x_0) = \|T'\|_\infty \) and \( \|S\|_\infty \leq \|T\|_\infty \),
\[ \|T'\|_\infty^2 \leq n^2 \|T\|_\infty^2, \]
proving the claim. \( \square \)

The following is a version of Bernstein’s inequality.\(^3\)

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Theorem 3. If $T \in \mathcal{T}_n$ and $A \subset \mathbb{T}$ is a Borel set, there is some $x_0 \in \mathbb{T}$ such that
\[
\int_A |T'(t)|dt \leq n \int_{A-x_0} |T(t)|dt.
\]

Proof. Let $A \subset \mathbb{T}$ be a Borel set with indicator function $\chi_A$. Define $Q : \mathbb{T} \to \mathbb{C}$ by
\[
Q(x) = \int_{\mathbb{T}} \chi_A(t)T(t+x)\text{sgn} T'(t)dt, \quad x \in \mathbb{T},
\]
which we can write as
\[
Q(x) = \int_{\mathbb{T}} \chi_A(t) \sum_j \hat{T}(j) e^{ij(t+x)} \text{sgn} T'(t)dt
\]
\[
= \sum_j \hat{T}(j) \left( \int_{\mathbb{T}} \chi_A(t) e^{ijt} \text{sgn} T'(t)dt \right) e^{ijx},
\]
showing that $Q \in \mathcal{T}_n$. Also,
\[
Q'(x) = \int_{\mathbb{T}} \chi_A(t)T'(t+x)\text{sgn} T'(t)dt, \quad x \in \mathbb{T}.
\]
Let $x_0 \in \mathbb{T}$ with $|Q(x_0)| = \|Q\|_\infty$. Applying Theorem 2 we get
\[
\|Q'\|_\infty \leq n \|Q\|_\infty.
\]
Using
\[
Q'(0) = \int_{\mathbb{T}} \chi_A(t)T'(t)\text{sgn} T'(t)dt = \int_{\mathbb{T}} \chi_A(t)|T'(t)|dt,
\]
this gives
\[
\int_{\mathbb{T}} \chi_A(t)|T'(t)|dt \leq n \|Q\|_\infty
\]
\[
= n|Q(t_0)|
\]
\[
= n \left| \int_{\mathbb{T}} \chi_A(t)T(t+x_0)\text{sgn} T'(t)dt \right|
\]
\[
\leq n \int_{\mathbb{T}} \chi_A(t)|T(t+x_0)|dt
\]
\[
= n \int_{\mathbb{T}} \chi_{A-x_0}(t)|T(t)|dt.
\]

Applying the above with $A = \mathbb{T}$ gives the following version of Bernstein’s inequality, for the $L^1$ norm.

Theorem 4 ($L^1$ Bernstein’s inequality). If $T \in \mathcal{T}_n$, then
\[
\|T'\|_1 \leq n \|T\|_1.
\]
3 Nikolsky’s inequality for trigonometric polynomials

DeVore and Lorentz attribute the following inequality to Sergey Nikolsky. 4

**Theorem 5** (Nikolsky’s inequality). If $T \in \mathcal{T}_n$ and $0 < q \leq p \leq \infty$, then for $r \geq \frac{q}{2}$ an integer,

$$\|T\|_p \leq (2nr + 1)^{\frac{1}{q} - \frac{1}{p}} \|T\|_q.$$ 

**Proof.** Let $m = nr$. Then $T^r \in \mathcal{T}_m$, so $T^r * D_m = T^r$, and using this and the Cauchy-Schwarz inequality we have, for $x \in \mathbb{T}$,

$$|T(x)^r| = \left| \frac{1}{2\pi} \int_{\mathbb{T}} T(t)^r D_m(x - t) \right|$$

$$\leq \left| \frac{1}{2\pi} \int_{\mathbb{T}} |T(t)^r| |D_m(x - t)| dt \right|$$

$$\leq \|T\|_r^{\frac{q}{2}} \cdot \frac{1}{2\pi} \int_{\mathbb{T}} |T(t)|^\frac{q}{2} |D_m(x - t)| dt$$

$$\leq \|T\|_r^{\frac{q}{2}} \cdot \|T\|_q^\frac{q}{2} \cdot \|D_m\|_q^\frac{q}{2}$$

$$= \sqrt{2m + 1} \|T\|_r^{\frac{q}{2}} \cdot \|T\|_q^\frac{q}{2}.$$ 

Hence

$$\|T\|_r \leq \sqrt{2m + 1} \|T\|_r^{\frac{q}{2}} \cdot \|T\|_q^\frac{q}{2},$$

thus

$$\|T\|_\infty \leq (2m + 1)^{\frac{1}{q} - \frac{1}{p}} \|T\|_q^\frac{q}{2}.$$ 

Then, using $\|T\|_p \leq \|T\|_\infty^{\frac{p}{q}} \|T\|_q^{\frac{p}{q}}$, we have

$$\|T\|_p \leq (2m + 1)^{\frac{1}{q} - \frac{1}{p}} \|T\|_q^\frac{p}{q} = (2m + 1)^{\frac{1}{q} - \frac{1}{p}} \|T\|_q^\frac{p}{q}.$$ 

$\Box$

4 The complementary Bernstein inequality

We define a **homogeneous Banach space** to be a linear subspace $B$ of $L^1(\mathbb{T})$ with a norm $\|f\|_{L^1(\mathbb{T})} \leq \|f\|_B$ with which $B$ is a Banach space, such that if $f \in B$ and $\tau \in \mathbb{T}$ then $f_\tau \in B$ and $\|f_\tau\|_B = \|f\|_B$, and such that if $f \in B$ then $f_\tau \to f$ in $B$ as $\tau \to 0$.

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Fejér’s kernel is, for \( n \geq 0 \),

\[
K_n(t) = \sum_{|j| \leq n} \left(1 - \frac{|j|}{n + 1}\right) e^{ijt} = \sum_{j \in \mathbb{Z}} \chi_n(j) \left(1 - \frac{|j|}{n + 1}\right) e^{ijt} \quad t \in \mathbb{T}.
\]

One calculates that, for \( t \not\in 4\pi \mathbb{Z} \),

\[
K_n(t) = \frac{1}{n + 1} \left(\frac{\sin \frac{n+1}{2}t}{\sin \frac{1}{2}t}\right)^2.
\]

Bernstein’s inequality is a statement about functions whose Fourier transform is supported only on low frequencies. The following is a statement about functions whose Fourier transform is supported only on high frequencies.\(^5\) In particular, for \( 1 \leq p < \infty \), \( L^p(\mathbb{T}) \) is a homogeneous Banach space, and so is \( C(\mathbb{T}) \) with the supremum norm.

**Theorem 6.** Let \( B \) be a homogeneous Banach space and let \( m \) be a positive integer. Define \( C_m \) as

\[
C_m = \begin{cases} 
  m + 1 & \text{if } m \text{ is even} \\
  \frac{1}{2}m & \text{if } m \text{ is odd}
\end{cases}
\]

If \( f(t) = \sum_{|j| \geq n} a_j e^{ijt}, \quad t \in \mathbb{T}, \)

is \( m \) times differentiable and \( f^{(m)} \in B \), then \( f \in B \) and

\[
\|f\|_B \leq C_m n^{-m} \left\|f^{(m)}\right\|_B.
\]

**Proof.** Suppose that \( m \) is even. It is a fact that if \( a_j, j \in \mathbb{Z}, \) is an even sequence of nonnegative real numbers such that \( a_j \to 0 \) as \( |j| \to \infty \) and such that for each \( j > 0, \)

\[
a_{j-1} + a_{j+1} - 2a_j \geq 0,
\]

then there is a nonnegative function \( f \in L^1(\mathbb{T}) \) such that \( \hat{f}(j) = a_j \) for all \( j \in \mathbb{Z}. \)\(^6\) Define

\[
a_j = \begin{cases} 
  j^{-m} & |j| \geq n \\
  n^{-m} + (n - |j|)(n^{-m} - (n + 1)^{-m}) & |j| \leq n - 1.
\end{cases}
\]

It is apparent that \( a_j \) is even and tends to 0 as \( |j| \to \infty. \) For \( 1 \leq j \leq n - 2, \)

\[
a_{j-1} + a_{j+1} - 2a_j = 0.
\]

For \( j = n - 1, \)

\[
a_{j-1} + a_{j+1} - 2a_j = n^{-m} + (n - (n - 2))(n^{-m} - (n + 1)^{-m}) + n^{-m} - 2 \left(n^{-m} + (n - (n - 1))(n^{-m} - (n + 1)^{-m})\right) = 0.
\]


The function $j \mapsto j^{-m}$ is convex on $\{n, n+1, \ldots\}$, as $m \geq 1$, so for $j \geq n$ we have $a_{j-1} + a_{j+1} - 2a_j \geq 0$. Therefore, there is some nonnegative $\phi_{m,n} \in L^1(T)$ such that

$$\hat{\phi}_{m,n}(j) = a_j, \quad j \in \mathbb{Z}.$$ 

Because $\phi_{m,n}$ is nonnegative, and using $n^{-m} - (n+1)^{-m} < \frac{m}{n} n^{-m}$,

$$||\phi_{m,n}||_1 = \hat{\phi}_{m,n}(0) = n^{-m} + n(n^{-m} - (n+1)^{-m}) < (m+1)n^{-m}.$$ 

Define $d\mu_{m,n}(t) = \frac{1}{2\pi} \phi_{m,n}(t)dt$. For $|j|\geq n$, 

$$\hat{f}^{(m)} * \mu_{m,n}(j) = \hat{f}(j) \hat{\phi}_{m,n}(j) = (ij)^m \hat{f}(j) \cdot |j|^{-m} = i^m \hat{f}(j),$$

so for all $j \in \mathbb{Z}$,

$$f^{(m)} * \mu_{m,n}(j) = i^m \hat{f}(j).$$

This implies that $f^{(m)} * \mu_{m,n} = i^m f$, which in particular tells us that $f \in B$. Then,

$$||f||_B = ||i^m f||_B$$

$$= \left\| f^{(m)} * \mu_{m,n} \right\|_B$$

$$\leq \left\| f^{(m)} \right\|_B ||\mu_{m,n}||_{M(T)}$$

$$= ||\phi_{m,n}||_1 \left\| f^{(m)} \right\|_B$$

$$\leq (m+1)n^{-m} \left\| f^{(m)} \right\|_B.$$ 

This shows what we want in the case that $m$ is even, with $C_m = m+1$.

Suppose that $m$ is odd. For $l$ a positive integer, define $\psi_l : T \to \mathbb{C}$ by

$$\psi_l(t) = \left( e^{2\pi it} + \frac{1}{2} e^{3\pi it} \right) K_{l-1}(t), \quad t \in T.$$ 

There is a unique $l_n$ such that $n \in \{2l_n, 2l_n + 1\}$. For $k \geq 0$ an integer, define $\Psi_{n,k} : T \to \mathbb{C}$ by

$$\Psi_{n,k}(t) = \psi_{l_n, 2^k}(t), \quad t \in T.$$ 

$\Psi_{n,k}$ satisfies

$$||\Psi_{n,k}||_1 \leq \frac{3}{2} ||K_{k-1}||_1 = \frac{3}{2} \frac{1}{2\pi} \int_T |K_{k-1}(t)|dt = \frac{3}{2} \frac{1}{2\pi} \int_T K_{k-1}(t)dt = \frac{3}{2}.$$ 

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On the one hand, for \( j \leq 0 \), from the definition of \( \psi_l \) we have \( \hat{\Psi}_{n,k}(j) = 0 \), hence \( \sum_{k=0}^\infty \hat{\Psi}_{n,k}(j) = 0 \). On the other hand, for \( j \geq n \) we assert that

\[
\sum_{k=0}^\infty \hat{\Psi}_{n,k}(j) = 1.
\]

We define \( \Phi_n : \mathbb{T} \to \mathbb{C} \) by

\[
\Phi_n(t) = \sum_{k=0}^\infty (\hat{\Psi}_{n,k} \ast \phi_{1,n2^k})(t), \quad t \in \mathbb{T}.
\]

We calculate the Fourier coefficients of \( \Phi_n \). For \( j \geq n \),

\[
\hat{\Phi}_n(j) = \sum_{k=0}^\infty \hat{\Phi}_{n,k}(j) \hat{\phi}_{1,n2^k} = \frac{1}{j} \sum_{k=0}^\infty \hat{\Phi}_{n,k}(j) = \frac{1}{j}.
\]

As well,

\[
\|\Phi_n\|_1 \leq \sum_{k=0}^\infty \|\hat{\Psi}_{n,k} \ast \phi_{1,n2^k}\|_1 \leq \sum_{k=0}^\infty \|\hat{\Psi}_{n,k}\|_1 \|\phi_{1,n2^k}\|_1 \leq \frac{3}{2} \sum_{k=0}^\infty 2(n2^k)^{-1} = \frac{6}{n}
\]

We now define

\[
d\mu_{1,n}(t) = \frac{1}{2\pi} (\Phi_n(t) - \Phi_n(-t))dt,
\]

which satisfies for \( |j| \geq n \),

\[
\hat{\mu}_{1,n}(j) = \hat{\Phi}_n(j) - \hat{\Phi}_n(-j) = \frac{1}{j}
\]

and hence

\[
f' \ast \mu_{1,n}(j) = \hat{f}'(j) \hat{\mu}_{1,n}(j) = ij \hat{f}(j) \cdot \frac{1}{j} = i \hat{f}(j).
\]

Because \( \hat{f}(j) = 0 \) for \( |j| < n \), \( f' \ast \mu_{1,n}(j) = 0 \) for \( |j| < n \), it follows that for any \( j \in \mathbb{Z} \),

\[
f' \ast \mu_{1,n}(j) = i \hat{f}(j),
\]

and therefore,

\[
f' \ast \mu_{1,n} = if.
\]

Then

\[
\|f\|_B = \|if\|_B = \|f' \ast \mu_{1,n}\|_B \leq \|\mu_{1,n}\|_{M(\mathbb{T})} \|f'\|_B \leq 2 \|\Phi_n\|_1 \|f'\|_B \leq \frac{12}{n} \|f'\|_B.
\]

That is, with \( C_1 = 12 \) we have

\[
\|f\|_B \leq 12n^{-1} \|f'\|_B.
\]
For $m = 2\nu + 1$, we define

$$\mu_{m,n} = \mu_{1,n} \ast \mu_{2\nu,n},$$

for which we have, for $|j| \geq n$,

$$f^{(m)} \ast \mu_{m,n}(j) = \hat{(ij)^m} \hat{\mu}_{1,n}(j) \hat{\mu}_{2\nu,n}(j) = (ij)^m \hat{f}(j) \cdot \frac{1}{j} \cdot j^{-2\nu} = \hat{i}^m \hat{f}(j).$$

It follows that

$$f^{(m)} \ast \mu_{m,n} = \hat{i}^m f,$$

whence

$$\|f\|_B = \|\hat{i}^m f\|_B$$

$$= \|f^{(m)} \ast \mu_{m,n}\|_B$$

$$\leq \|\mu_{m,n}\|_{M(T)} \|f^{(m)}\|_B$$

$$\leq \|\mu_{1,n}\|_{M(T)} \|\mu_{2\nu,n}\|_{M(T)} \|f^{(m)}\|_B$$

$$\leq \frac{12}{n} \cdot (2\nu + 1)n^{-2\nu} \|f^{(m)}\|_B$$

$$= 12mn^{-m} \|f^{(m)}\|_B.$$ 

That is, with $C_m = 12m$, we have

$$\|f\|_B \leq C_m n^{-m} \|f^{(m)}\|_B,$$

completing the proof. \qed