Proof by bootstrapping

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The *Oxford English Dictionary* defines "to bootstrap" as the following:

> To make use of existing resources or capabilities to raise (oneself) to a new situation or state; to modify or improve by making use of what is already present.

The Picard theorem [17, p. 14, Theorem 1.17]:

Theorem 1. *Let M be a finite dimensional Hilbert space. Let $F : M \to M$ be locally Lipschitz. Let $t_0 \in \mathbb{R}$ and let $u_0 \in M$. Then there exist*

$$-\infty \leq T_- < t_0 < T_+ \leq +\infty$$

such that, for $I = (T_-, T_+)$, there exists a unique $u : I \to M$ satisfying $u(t_0) = u_0$ and

$$\partial_t u(t) = F(u(t)), \qquad t \in I.$$

If T_+ is finite then $\|u(t)\|_M \to \infty$ as $t \to T_+$, and if T_- is finite then $\|u(t)\|_M \to \infty$ as $t \to T_-$.

Taylor's formula:

Theorem 2. *If $f \in C^k(B_r(0))$, then for all $x \in B_r(0)$ we have*

$$f(x) = \sum_{|\alpha| \leq k} \frac{(\partial^\alpha f)(0)}{\alpha!} x^\alpha + R_k(x),$$

where

$$R_k(x) = k \sum_{|\alpha|=k} \frac{x^\alpha}{\alpha!} \int_0^1 (1-t)^{k-1} ((\partial^\alpha f)(tx) - (\partial^\alpha f)(0)) dt.$$

For $x \in B_r(0)$ we have

$$|R_k(x)| \leq \sum_{|\alpha|=k} \frac{|x^\alpha|}{\alpha!} \sup_{0 \leq t \leq 1} |(\partial^\alpha f)(tx) - (\partial^\alpha f)(0)|.$$

For $k = 2$ we can write Taylor's formula as:

Corollary 3. *If $f \in C^2(B_r(0))$, then for all $x \in B_r(0)$ we have*

$$f(x) = f(0) + Df(0)(x) + \frac{1}{2}D^2 f(0)(x,x) + R_2(x),$$

where

$$|R_2(x)| \le \frac{n^2}{2}|x|^2 \sup_{|\alpha|=2, 0\le t\le 1} |(\partial^\alpha f)(tx) - (\partial^\alpha f)(0)|.$$

Thus, for any $\epsilon > 0$ there is some $r > 0$ such that if $x \in B_r(0)$ then $|R_2(x)| \le \epsilon |x|^2$.

1 Potential well example

Theorem 4. *Let M be a finite dimensional Hilbert space and let $V \in C^2_{loc}(M)$ be such that $V(0) = 0$, $DV(0) = 0$, and $D^2V(0)$ is positive definite. Let $N = M \times M$. There is some $\delta > 0$ such that if $\|(m_1, m_2)\|_N < \delta$ then there is a unique $u \in C^1_{loc}(\mathbb{R}, N)$ such that*

$$\partial_t \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} u_2 \\ -V(u_1) \end{pmatrix}, \qquad u(0) = \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}.$$

And u is bounded.

Proof. Define $F : N \to N$ by

$$F(x,y) = \begin{pmatrix} y \\ -V(x) \end{pmatrix}.$$

F is locally Lipschitz, so by Picard's theorem there exist

$$-\infty \le T_- < 0 < T_+ \le +\infty$$

such that, for $I = (T_-, T_+)$, there exists a unique $u : I \to N$ satisfying $u(0) = (m_1, m_2)$ and

$$\partial_t u(t) = F(u(t)), \qquad t \in I.$$

If T_+ is finite then $\|u(t)\|_N \to \infty$ as $t \to T_+$, and if T_- is finite then $\|u(t)\|_N \to \infty$ as $t \to T_-$. We shall show that $\|u(t)\|_N$ is bounded on I, which will show that $T_+ = +\infty$ and $T_- = -\infty$.

Define $E : I \to \mathbb{R}$ by

$$E(t) = \frac{1}{2}\|u_2(t)\|_M^2 + V(u_1(t)).$$

We have

$$\begin{aligned} \frac{dE}{dt}(t) &= \langle u_2(t), \partial_t u_2(t)\rangle + \langle \partial_t u_1(t), DV(u_1(t))\rangle \\ &= \langle u_2(t), -DV(u_1(t))\rangle + \langle u_2(t), DV(u_1(t))\rangle \\ &= 0. \end{aligned}$$

This gives us the following conservation law: for all $t \in I$ we have

$$E(t) = E(0) = \frac{1}{2}\|m_2\|_M^2 + V(m_1).$$

Since $D^2V(0)$ is a symmetric positive definite matrix, there is an orthonormal basis of $\mathbb{R}^n$ whose elements are eigenvectors for $D^2V(0)$ with positive eigenvalues. It follows that $D^2V(0)(v,v) \geq \lambda|v|^2$ for all $v \in \mathbb{R}^n$, where λ is the smallest eigenvalue of $D^2V(0)$.

Let $\epsilon = \frac{\lambda}{4}$ and let $r > 0$ be such that if $\|x\|_M < r$ then $|R_2(x)| \leq \epsilon|x|^2$. For such x we have

$$\begin{aligned} V(x) &= V(0) + DV(0)(x) + \frac{1}{2}D^2V(0)(x,x) + R_2(x) \\ &\geq 0 + 0 + \frac{1}{2}\lambda|x|^2 - \epsilon|x|^2 \\ &= \frac{1}{4}\lambda|x|^2. \end{aligned}$$

Let $\mathbf{H}(t)$ be the statement

$$\|u(t)\|_N \leq \frac{r}{2},$$

and let $\mathbf{C}(t)$ be the statement

$$\|u(t)\|_N \leq \frac{r}{4}.$$

Let $L = \max\{2, \frac{4}{\lambda}\}$, and let $\delta > 0$ be small enough such that both $E(0) \leq \frac{r^2}{16L}$ and $\delta \leq \frac{r}{2}$. We have that $\mathbf{H}(0)$ is true.

If $\mathbf{H}(t)$ is true, then $\|u_1(t)\|_M \leq \frac{r}{2} < r$ and hence

$$\begin{aligned} \|u(t)\|_N^2 &= \|u_1(t)\|_M^2 + \|u_2(t)\|_M^2 \\ &\leq \frac{4}{\lambda}V(u_1(t)) + \|u_2(t)\|_M^2 \\ &\leq L\left(V(u_1(t)) + \frac{1}{2}\|u_2(t)\|_M^2\right) \\ &= LE(t) \\ &= LE(0) \\ &\leq \frac{r^2}{16}, \end{aligned}$$

and hence $\mathbf{C}(t)$ is true.

If $\mathbf{C}(t)$ is true, then for all t' in a neighborhood of t, $\mathbf{H}(t')$ is true. And if $t_k \in I$ converges to $t \in I$ and $\mathbf{C}(t_k)$ is true for each k, then $\mathbf{C}(t)$ is true.

Then by the bootstrap argument, $\mathbf{C}(t)$ is true for all $t \in I$. Thus,

$$\lim_{t \to T_+} \|u(t)\|_N \leq \frac{r}{2} < \infty,$$

and it follows that $T_+ = +\infty$. It likewise follows that $T_- = -\infty$. □

2 Hamiltonian

The following is from [17, p. 32, Exercise 1.29]. Coercive Hamiltonian implies global existence.

Theorem 5. *Let M be a finite dimensional symplectic vector space and let $H \in C^2_{loc}(M)$ be such that $H(0) = 0, DH(0) = 0$, and $D^2H(0)$ is positive definite. There is some $\delta > 0$ such that if $\|u_0\|_M < \delta$ then there is a unique $u \in C^1_{loc}(\mathbb{R}, M)$ such that*

$$\partial_t u = X_H(u), \qquad u(0) = u_0.$$

And u is bounded.

Proof. $X_H : M \to M$ is locally Lipschitz, so by Picard's theorem

$$-\infty \leq T_- < 0 < T_+ \leq +\infty$$

such that, for $I = (T_-, T_+)$, there exists a unique $u : I \to M$ satisfying $u(0) = u_0$ and

$$\partial_t u(t) = X_H(u(t)), \qquad t \in I.$$

If T_+ is finite then $\|u(t)\|_M \to \infty$ as $t \to T_+$, and if T_- is finite then $\|u(t)\|_M \to \infty$ as $t \to T_-$. We shall show that $\|u(t)\|_M$ is bounded on I, which will show that $T_+ = +\infty$ and $T_- = -\infty$.

Since $D^2V(0)$ is a symmetric positive definite matrix, it follows that $D^2V(0)(v, v) \geq \lambda|v|^2$ for all $v \in \mathbb{R}^{2n}$, where λ is the smallest eigenvalue of $D^2V(0)$.

Let $\epsilon = \frac{\lambda}{4}$ and let $r > 0$ be such that if $\|x\|_M < r$ then $|R_2(x)| \leq \epsilon|x|^2$. For such x we have

$$\begin{aligned} H(x) &= H(0) + DH(0)x + \frac{1}{2}D^2H(0)(x, x) + R_2(x) \\ &\geq 0 + 0 + \frac{1}{2}\lambda|x|^2 - \epsilon|x|^2 \\ &= \frac{1}{4}\lambda|x|^2. \end{aligned}$$

Let $\mathbf{H}(t)$ be the statement

$$\|u(t)\|_N \leq \frac{r}{2},$$

and let $\mathbf{C}(t)$ be the statement

$$\|u(t)\|_N \leq \frac{r}{4}.$$

Let $\delta > 0$ be small enough that both $H(u_0) \leq \frac{\lambda r^2}{64}$ and $\delta \leq \frac{r}{2}$. We have that $\mathbf{H}(0)$ is true.

If $\mathbf{H}(t)$ is true, then $\|u(t)\|_M \le \frac{r}{2} < r$ and hence

$$\begin{aligned} \|u(t)\|_M^2 &\le \frac{4}{\lambda} H(u(t)) \\ &= \frac{4}{\lambda} H(u_0) \\ &\le \frac{r^2}{16}, \end{aligned}$$

and hence $\mathbf{C}(t)$ is true.

If $\mathbf{C}(t)$ is true, then for all t' in a neighborhood of t, $\mathbf{H}(t')$ is true. And if $t_k \in I$ converges to $t \in I$ and $\mathbf{C}(t_k)$ is true for each k, then $\mathbf{C}(t)$ is true.

Then by the bootstrap argument, $\mathbf{C}(t)$ is true for all $t \in I$. Thus,

$$\lim_{t \to T_+} \|u(t)\|_M \le \frac{r}{2} < \infty,$$

and it follows that $T_+ = +\infty$. It likewise follows that $T_- = -\infty$. □

Chipot [3, p. 227, §16.4].

Anh.[1]

Grubb [10]

[14, p. 231]

[1]: ellliptic regularity.

Rendall [16, §10.3]. "proof of the stability of Minkowski space by Christodoulou and Klainerman and the theorem on formation of trapped surfaces by Christodoulou"

[11, p. 475]

[18, p. 11, §1.7]

Let $\phi : [0, T] \to [0, \infty)$. If $\phi(0) \le \alpha$ and for t such that $\phi(t) \le \alpha$ we have $\phi(t) \le \alpha/2$, then $\phi(t) \le \alpha/2$ for all $t \in [0, T]$.

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