Convolution semigroups, canonical processes, and Brownian motion

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1 Convolution semigroups, projective families, and canonical processes

Let
\[ E = \mathbb{R}^d \]
and let \( \mathcal{E} = \mathcal{B}_{\mathbb{R}^d} \), the Borel \( \sigma \)-algebra of \( \mathbb{R}^d \), and let \( \mathcal{P}(E) \) be the collection of Borel probability measures on \( \mathbb{R}^d \). With the narrow topology, \( \mathcal{P}(E) \) is a Polish space. For a nonempty set \( J \), we write
\[ \mathcal{E}^J = \bigotimes_{t \in J} \mathcal{E}, \]
the product \( \sigma \)-algebra.

Let \( A : E \times E \to E \) be \( A(x_1, x_2) = x_1 + x_2 \). For \( \nu_1, \nu_2 \in \mathcal{P}(E) \), the convolution of \( \nu_1 \) and \( \nu_2 \) is the pushforward of the product measure \( \nu_1 \times \nu_2 \) by \( A \):
\[ \nu_1 * \nu_2 = A_*(\nu_1 \times \nu_2). \]
The convolution \( \nu_1 * \nu_2 \) is an element of \( \mathcal{P}(E) \).

Let
\[ I = \mathbb{R}_{\geq 0}. \]
A convolution semigroup is a family \( (\nu_t)_{t \in I} \) of elements of \( \mathcal{P}(E) \) such that for \( s, t \in I \),
\[ \nu_{s+t} = \nu_s * \nu_t. \]
From this, it turns out that \( \mu_0 = \delta_0 \)\(^1\). A convolution semigroup is called continuous when the map \( t \mapsto \nu_t \) is continuous \( I \to \mathcal{P}(E) \).

\(^1\)See [http://individual.utoronto.ca/jordanbell/notes/narrow.pdf](http://individual.utoronto.ca/jordanbell/notes/narrow.pdf), Theorem 3.
For \( \nu \in \mathcal{P}(E) \) and \( x \in E \), and for \( B \in \mathcal{E} \),
\[
(\nu \ast \delta_x)(B) = \int_E \left( \int_E 1_B(x_1 + x_2)d\delta_x(x_1) \right) d\nu(x_2) = \nu(B - x),
\]
and we define \( \nu^x \in \mathcal{P}(E) \) by
\[
\nu^x = \nu \ast \delta_x.
\]

For \( \nu \in \mathcal{P}(E) \) and for a Borel measurable function \( f : E \to [0, \infty] \), write
\[
\nu f = \int_E fd\mu.
\]

For \( x \in E \), using the change of variables formula\(^3\) and Fubini’s theorem,
\[
\nu^x f = \int_E \int_E f(x_1 + x_2)d\delta_x(x_2)d\nu(x_1) = \int_E f(x_1 + x)d\nu(x_1).
\]
That is, for \( \nu \in \mathcal{P}(E) \), for \( f : E \to [0, \infty] \) Borel measurable, and for \( x \in E \),
\[
\nu^x f = \int_E fd\nu^x = \int_E f(x + y)d\nu(y). \tag{1}
\]

For nonempty subsets \( J \) and \( K \) of \( I \) with \( J \subset K \), let
\[
\pi_{K,J} : E^K \to E^J
\]
be the projection map. Let \( \mathcal{K} = \mathcal{K}(I) \) be the collection of finite nonempty subsets of \( I \). Let \( (\Omega, \mathcal{F}, P, (X_t)_{t \in I}) \) be a stochastic process with state space \( E \).

For \( J \in \mathcal{K} \), with elements \( t_1 < \ldots < t_n \), we define
\[
X_J = X_{t_1} \otimes \cdots \otimes X_{t_n},
\]
which is measurable \( \mathcal{F} \to \mathcal{E}^J \). The joint distribution \( P_J \) of the family of random variables \( (X_t)_{t \in J} \) is the distribution of \( X_J \), i.e.
\[
P_J = X_{J*}P.
\]

The family of finite-dimensional distributions of \( X \) is the family \( (P_J)_{J \in \mathcal{K}} \).

For \( J, K \in \mathcal{K} \) with \( J \subset K \),
\[
X_J = \pi_{K,J} \circ X_K,
\]
from which
\[(\pi_{K,J})_{*}P_{K} = P_{J}.\] (2)

Forgetting the stochastic process \(X\), a family of probability measures \(P_{J}\) on \(E^{J}\), for \(J \in \mathcal{K}\), is called a projective family when (2) is true. The Kolmogorov extension theorem\[^4\] tells us that if \((P_{J})_{J \in \mathcal{K}}\) is a projective family, then there is a unique probability measure \(P_{I}\) on \(E^{I}\) such that for any \(J \in \mathcal{K}\),
\[(\pi_{I,J})_{*}P_{I} = P_{J}.\] (3)

Then for \(\Omega = E^{I}\) and \(\mathcal{F} = E^{I}\), \((\Omega, \mathcal{F}, P_{I})\) is a probability space, and for \(t \in I\) we define \(X_{t} : \Omega \to E\) by
\[X_{t}(\omega) = \pi_{I,\{t\}}(\omega) = \omega(t),\] (4)
which is measurable \(\mathcal{F} \to \mathcal{E}\), and thus the family \((X_{t})_{t \in I}\) is a stochastic process with state space \(E\). For \(J \in \mathcal{K}\) it is immediate that
\[X_{J} = \pi_{I,J}.\]

For \(B \in \mathcal{E}^{J}\), applying (3) gives
\[(X_{J}, P_{I})(B) = ((\pi_{I,J})_{*}P_{I})(B) = P_{J}(B),\]
which means that \(X_{J} \ast P_{I} = P_{J}\), namely, \((P_{J})_{J \in \mathcal{K}}\) is the family of finite-dimensional distributions of the stochastic process \((X_{t})_{t \in I}\). We call the stochastic process associated with the projective family \((P_{J})_{J \in \mathcal{K}}\).

Let \((\nu_{t})_{t \in I}\) be a convolution semigroup and let \(\mu \in \mathcal{P}(E)\). For \(J \in \mathcal{K}\), with elements \(t_{1} < \ldots < t_{n}\), and for \(B \in \mathcal{E}^{J}\), define
\[P_{J}(B) = \int_{E} \ldots \int_{E} 1_{B}(x_{1}, \ldots, x_{n})d\nu_{x_{n}}^{x_{n-1}}(x_{n}) \ldots d\nu_{x_{1}}^{x_{0}}(x_{1})d\mu(x_{0}).\] (5)

We say that \((P_{J})_{J \in \mathcal{K}}\) is the family of measures induced by the convolution semigroup \((\nu_{t})_{t \in I}\). It is proved that \((P_{J})_{J \in \mathcal{K}}\) is a projective family\[^5\]. Therefore, from the Kolmogorov extension theorem it follows that there is a unique probability measure \(P^{\mu}\) on \(E^{I}\) such that
\[(\pi_{I,J})_{*}P^{\mu} = P_{J}.\] (6)

For \(\Omega = E^{I}\) and \(\mathcal{F} = E^{I}\), \((\Omega, \mathcal{F}, P^{\mu})\) is a probability space. For \(t \in I\) define \(X_{t} : \Omega \to E\) by
\[X_{t}(\omega) = \pi_{I,\{t\}}(\omega) = \omega(t).\]

\[^4\]See [http://individual.utoronto.ca/jordanbell/notes/finitedimdistributions.pdf](http://individual.utoronto.ca/jordanbell/notes/finitedimdistributions.pdf)
\[^5\]See [http://individual.utoronto.ca/jordanbell/notes/markovkernels.pdf](http://individual.utoronto.ca/jordanbell/notes/markovkernels.pdf), Theorem 4.
Applying this with \( \mu \) is, i.e., for \( J \in \mathcal{X} \) with elements \( t_1 < \cdots < t_n \) and for \( B \in \mathcal{E}^J \),

\[
((X_{t_1} \otimes \cdots \otimes X_{t_n})_\ast P^\mu)(B) = \int_E \cdots \int_E 1_B(x_1, \ldots, x_n) d\nu_{t_n-t_{n-1}}(x_n) \cdots d\nu_{t_1}(x_1) d\mu(x_0).
\]

Applying this with \( \mu = \delta_x \) yields

\[
((X_{t_1} \otimes \cdots \otimes X_{t_n})_\ast P^{\delta_x})(B) = \int_E \cdots \int_E 1_B(x_1, \ldots, x_n) d\nu_{t_n-t_{n-1}}(x_n) \cdots d\nu_{t_1}(x_1),
\]

and thus, for any \( \mu \in \mathcal{P}(E) \),

\[
\int_E \cdots \int_E 1_B(x_1, \ldots, x_n) d\nu_{t_n-t_{n-1}}(x_n) \cdots d\nu_{t_1}(x_1) d\mu(x) = \int_E ((X_{t_1} \otimes \cdots \otimes X_{t_n})_\ast P^{\delta_x})(B) d\mu(x).
\]

That is, for \( \mu \in \mathcal{P}(E) \), for \( J \in \mathcal{X} \), and \( B \in \mathcal{E}^J \),

\[
(X_{J\ast} P^\mu)(B) = \int_E (X_{J\ast} P^{\delta_x})(B) d\mu(x). \tag{7}
\]

For \( J \in \mathcal{X} \), \( A_t \in \mathcal{E} \) for \( t \in J \), and \( A = \prod_{t \in J} A_t \times \prod_{t \notin J} E \in \mathcal{F} = \mathcal{E}^J \), namely \( A \) is a cylinder set\(^6\), let \( B = \pi_{I,J}(A) = \prod_{t \in J} A_t \in \mathcal{E}^J \),

\[
X_J^{-1}(B) = \pi_{I,J}^{-1}(B) = A,
\]

so by (7),

\[
P^\mu(A) = \int_E P^{\delta_x}(A) d\mu(x). \tag{8}
\]

Because this is true for all cylinder sets in the product \( \sigma \)-algebra \( \mathcal{E}^J \) and \( \mathcal{E}^J \) is generated by the collection of cylinder sets, \( \text{[8]} \) is true for all \( A \in \mathcal{F} \).

Let \( J \in \mathcal{X} \), with elements \( t_1 < \cdots < t_n \), and let \( \sigma_n : E^{n+1} \to E^n \) be \( \sigma_n(x_0, x_1, \ldots, x_n) = (x_0 + x_1, x_0 + x_1 + x_2, \ldots, x_0 + x_1 + x_2 + \cdots + x_n) \).

For \( B \in \mathcal{E}^n \) using \( \text{\textbullet} \) we obtain by induction

\[
\int_E \cdots \int_E 1_B(x_1, \ldots, x_{n-1}, x_n) d\nu_{t_{n-1}-t_{n-2}}(x_n) \cdots d\nu_{t_1}(x_1) d\mu(x_0) = \int_E \cdots \int_E 1_B(x_1, \ldots, x_{n-1}, x_n + x_1) d\nu_{t_{n-1}-t_{n-2}}(x_n) \cdots d\nu_{t_1}(x_1) d\mu(x_0) = \cdots = \int_E \cdots \int_E 1_B \circ \sigma_n d\nu_{t_{n-1}-t_{n-2}}(x_n) \cdots d\nu_{t_1}(x_1) d\mu(x_0).
\]

\(^6\)See http://individual.utoronto.ca/jordanbell/notes/productmeasure.pdf

\(^7\)See http://individual.utoronto.ca/jordanbell/notes/productmeasure.pdf
Thus, with $P_J$ the probability measure on $\mathcal{E}^J$ defined in [3],

$$
\int_{E^J} 1_B dP_J = P_J(B) = \int_E \cdots \int_E 1_B \circ \sigma_n d\nu_{t_n} \cdots d\nu_1(x_1) d\mu(x_0).
$$

For $f : E^n \to [0, \infty]$ a Borel measurable function, there is a sequence of measurable simple functions pointwise increasing to $f$, and applying the monotone convergence theorem yields

$$
\int_{E^n} f dP_J = \int_E \cdots \int_E f \circ \sigma_n d\nu_{t_n} \cdots d\nu_1(x_1) d\mu(x_0).
$$

### 2 Increments

Let $(\Omega, \mathcal{F}, P, (X_t)_{t \in I})$ be a stochastic process with state space $E$. $X$ is said to have **stationary increments** if there is a family $(\nu_t)_{t \in I}$ of probability measures on $\mathcal{E}$ such that for all $s, t \in I$ with $s \leq t$,

$$
P_*(X_t - X_s) = \nu_{t-s}.
$$

In particular, for $s = t$ this implies that $P_*(0) = \nu_0$, hence $\nu_0 = \delta_0$.

A stochastic process is said to have **independent increments** if for any $J \in \mathcal{K}$, with elements $0 = t_0 < t_1 < \cdots < t_n$, the random variables

$$
X_{t_0}, X_{t_1} - X_{t_0}, \ldots, X_{t_n} - X_{t_{n-1}}
$$

are independent.

We now prove that the canonical process associated with the projective family of probability measures induced by a convolution semigroup and any initial distribution has stationary and independent increments.

**Theorem 1.** Let $(\nu_t)_{t \in I}$ be a convolution semigroup, let $(P_J)_{J \in \mathcal{K}}$ be the family of measures induced by this convolution semigroup, let $\mu \in \mathcal{P}(E)$, and let $(\Omega, \mathcal{F}, P^\mu, (X_t)_{t \in I}), \Omega = E^I$ and $\mathcal{F} = \mathcal{E}^I$, be the associated canonical process. $X$ has stationary increments,

$$
(X_t - X_s)_s P^\mu = \nu_{t-s}, \quad s \leq t,
$$

and has independent increments.

**Proof.** $\nu_0 = \delta_0$, so (10) is immediate when $s = t$. When $s < t$, let

$$
Y = X_s \otimes X_t = X_{(s,t)} = \pi_{I,(s,t)},
$$

which is measurable $\mathcal{F} \to \mathcal{E} \otimes \mathcal{E}$, and let $q : E \times E \to E$ be $(x_1, x_2) \mapsto x_2 - x_1$, which is continuous and hence Borel measurable. Then $q \circ Y$ is measurable.

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\[ \mathcal{F} \rightarrow \mathcal{F}, \text{ and for } B \in \mathcal{F}, \]
\[ (q \circ Y)^{-1}(B) = \{ \omega \in \Omega : (q \circ Y)(\omega) \in B \} \]
\[ = \{ \omega \in \Omega : X_t(\omega) - X_s(\omega) \in B \} \]
\[ = (X_t - X_s)^{-1}(B), \]

and thus
\[ (q \circ Y)_* P^\mu = (X_t - X_s)_* P^\mu. \tag{11} \]

Now, according to (6),
\[ Y_* P^\mu = (\pi_{I,\{s,t\}})_* P^\mu = P_{\{s,t\}}. \]

Therefore, using that \( x_2 - x_1 \in B \) if and only if \( x_2 \in x_1 + B \) and also using \( \nu_{t-s}^E(x_1 + B) = \nu_{t-s}(B) \),
\[ (X_t - X_s)_* P^\mu(B) = (q \circ Y)_* P^\mu(B) \]
\[ = Y_* P^\mu(q^{-1}(B)) \]
\[ = P_{\{s,t\}}(q^{-1}(B)) \]
\[ = \int_{E} \int_{E} 1_{q^{-1}(B)}(x_1, x_2) d\nu_{t-s}^E(x_2) d\nu^E_s(x_1) d\mu(x) \]
\[ = \int_{E} \int_{E} 1_{x_1 + B}(x_2) d\nu_{t-s}^E(x_2) d\nu^E_s(x_1) d\mu(x) \]
\[ = \nu_{t-s}(B) \int_{E} d\nu^E_s(x_1) d\mu(x) \]
\[ = \nu_{t-s}(B) \int_{E} d\nu^E_s(d\mu(x)) \]
\[ = \nu_{t-s}(B) \int_{E} d\nu^E_s(d\mu(x)) \]
\[ = \nu_{t-s}(B) \int_{E} d\mu(x) \]
\[ = \nu_{t-s}(B), \]

which shows that
\[ (X_t - X_s)_* P^\mu = \nu_{t-s}, \]

and thus that \( X \) has stationary increments.

Let \( 0 = t_0 < t_1 < \cdots < t_n \), let \( J = \{t_0, t_1, \ldots, t_n\} \in \mathcal{F} \), write \( X_{t-1} = 0 \), and let
\[ Y_0 = X_{t_0} - X_{t-1}, \quad Y_1 = X_{t_1} - X_{t_0}, \quad \ldots, \quad Y_n = X_{t_n} - X_{t_{n-1}}. \]

For the random variables \( Y_0, \ldots, Y_n \) to be independent means for their joint distribution to be equal to the product of the distributions of each, i.e. to prove that \( X \) has independent increments, writing
\[ Z = Y_0 \otimes \cdots \otimes Y_n = \tau_n \circ (X_{t_0} \otimes \cdots \otimes X_{t_n}) = \tau_n \circ X_J = \tau_n \circ \pi_{I,J}, \]
with $\tau_n : E^{n+1} \to E^n$ defined by

$$\tau_n(x_0, x_1, \ldots, x_n) = (x_0, x_1 - x_0, \ldots, x_n - x_{n-1}),$$

we have to prove that

$$Z_* P^\mu = \prod_{j=0}^n Y_j^* P^\mu.$$

To prove this, it suffices (because the collection of cylinder sets generates the product $\sigma$-algebra) to prove that for any $A_0, \ldots, A_n \in \mathcal{E}$ and for $A = \prod_{j=0}^n A_j \in \mathcal{E}^{n+1},$

$$(Z_* P^\mu)(A) = \left( \prod_{j=0}^n Y_j^* P^\mu \right)(A),$$

i.e. that

$$(Z_* P^\mu)(A) = \prod_{j=0}^n (Y_j^* P^\mu)(A_j).$$

We now prove this. Using the change of variables theorem and (6),

$$\begin{align*}
(Z_* P^\mu)(A) &= \int_{E^{n+1}} 1_A d(Z_* P^\mu) \\
&= \int_{\Omega} 1_A \circ Z dP^\mu \\
&= \int_{\Omega} 1_A \circ \tau_n \circ (X_{t_0} \otimes \cdots \otimes X_{t_n}) dP^\mu \\
&= \int_{E^J} 1_A \circ \tau_n d(X_* P^\mu) \\
&= \int_{E^J} 1_A \circ \tau_n dP_J.
\end{align*}$$

Then applying (9) with $f = 1_A \circ \tau_n,$

$$\begin{align*}
\int_{E^J} 1_A \circ \tau_n dP_J &= \int_{E^J} \cdots \int_E 1_A \circ \tau_n \circ \sigma_{n+1} d\nu_{t_n - t_{n-1}}(x_n) \cdots d\nu_{t_0}(x_0) d\mu(x-1) \\
&= \int_{E^J} \cdots \int_E 1_A(x_{-1} + x_0, x_1, \ldots, x_n) d\nu_{n - t_{n-1}}(x_n) \cdots d\nu_{t_0}(x_0) d\mu(x-1) \\
&= \int_{E^J} \cdots \int_E 1_A(x_{-1} + x_0) 1_A(x_1) \cdots 1_A(x_n) d\nu_{n - t_{n-1}}(x_n) \cdots d\nu_{t_0}(x_0) d\mu(x-1) \\
&= \prod_{j=1}^n \nu_{j - t_{j-1}}(A_j) \int_{E^J} 1_A(x_{-1} + x_0) d\nu_{t_0}(x_0) d\mu(x-1),
\end{align*}$$

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and because $t_0 = 0$ and $\nu_0 = \delta_0$,

$$\int_E \int_E 1_{A_0}(x - 1 + x_0) d\nu_0(x_0) d\mu(x - 1) = \int_E \int_E 1_{A_0}(x_0 + 1 + x_0) d\delta_0(x_0) d\mu(x - 1)$$

$$= \int_E 1_{A_0}(x_0) d\mu(x - 1)$$

$$= \mu(A_0),$$

and therefore

$$(Z_* P^\mu)(A) = \mu(A_0) \cdot \prod_{j=1}^n \nu_{t_j - t_{j-1}}(A_j).$$

But we have already proved that (10), which tells us that for each $j$,

$$Y_j^* P^\mu = (X_{t_j} - X_{t_{j-1}})^* P^\mu = \nu_{t_j - t_{j-1}},$$

and thus

$$(Z_* P^\mu)(A) = \mu(A_0) \cdot \prod_{j=1}^n (Y_j^* P^\mu)(A_j).$$

But $Y_0^* P^\mu = X_0^* P^\mu$ and from (7) we have

$$(X_0^* P^\mu)(A_0) = \int_E (X_0^* P^\delta_x)(A_0) d\mu(x) = \int_E (\pi_0^* P^\delta_x)(A_0) d\mu(x),$$

and, from (5),

$$(\pi_0^* P^\delta_x)(A_0) = \int_E \int_E 1_{A_0}(x_0) d\nu_0^\mu(x_0) d\delta_x(y)$$

$$= \int_E \int_E 1_{A_0}(x_0) d\delta_0(x_0) d\delta_x(y)$$

$$= \int_E 1_{A_0}(y) d\delta_x(y)$$

$$= 1_{A_0}(x),$$

thus

$$(X_0^* P^\mu)(A_0) = \int_E 1_{A_0}(x) d\mu(x) = \mu(A_0).$$

Therefore

$$(Z_* P^\mu)(A) = (X_0^* P^\mu)(A_0) \cdot \prod_{j=1}^n (Y_j^* P^\mu)(A_j) = \prod_{j=0}^n (Y_j^* P^\mu)(A_j),$$

which completes the proof that $X$ has independent increments. 

\[\square\]
3 The Brownian convolution semigroup and Brownian motion

For $a \in \mathbb{R}$ and $\sigma > 0$, let $\gamma_{a,\sigma^2}$ be the Gaussian measure on $\mathbb{R}$, the probability measure on $\mathbb{R}$ whose density with respect to Lebesgue measure is

$$p(x, a, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-a)^2}{2\sigma^2}\right).$$

For $\sigma = 0$, let $\gamma_{a,0} = \delta_a$.

Define for $t \in I$,

$$\nu_t = \prod_{k=1}^d \gamma_{0,t},$$

which is an element of $\mathcal{P}(E)$. For $s, t \in I$, we calculate

$$\nu_s * \nu_t = \left( \prod_{k=1}^d \gamma_{0,s} \right) * \left( \prod_{k=1}^d \gamma_{0,t} \right) = \prod_{k=1}^d (\gamma_{0,s} * \gamma_{0,t}) = \prod_{k=1}^d \gamma_{0,s+t} = \nu_{s+t},$$

showing that $(\nu_t)_{t \in I}$ is a convolution semigroup. It is proved using Lévy’s continuity theorem that $t \mapsto \nu_t$ is continuous $I \to \mathcal{P}(E)$, showing that $(\nu_t)_{t \in I}$ is a continuous convolution semigroup.

We first prove a lemma (which is made explicit in Isserlis’s theorem) about the moments of random variables with Gaussian distributions.

**Lemma 2.** If $Z : \Omega \to E$ is a random variable with Gaussian distribution $\nu_{\tau}$, $\tau > 0$, then for each $n$ there is some $C_n > 0$ such that

$$E(|Z|^{2n}) = C_n \tau^n.$$

In particular, $C_2 = d$ and $C_4 = d(d+2)$.

**Proof.** That $Z$ has distribution $\nu_{\tau}$ means that

$$Z_*P = \nu_{\tau} = \prod_{j=1}^d \gamma_{0,\tau}.$$

Write $Z = Z_1 \otimes \cdots \otimes Z_d$, each of which has distribution $\gamma_{0,\tau}$, and $Z_*P = \prod_{j=1}^d Z_j$, which means that $Z_1, \ldots, Z_d$ independent. Let $U_j = \tau^{-1/2}Z_j$ for $j = 1, \ldots, d$, and then $U_1, \ldots, U_d$ are independent random variables each with

9 [http://individual.utoronto.ca/jordanbell/notes/markovkernels.pdf §6.](http://individual.utoronto.ca/jordanbell/notes/markovkernels.pdf)

distribution $\gamma_{0,1}$. Then using the multinomial formula,

$$
E(|Z|^2n) = E((Z_1^2 + \cdots + Z_d^2)^n) \\
= \tau^n \cdot E((U_1^2 + \cdots + U_d^2)^n) \\
= \tau^n \cdot E\left(\sum_{k_1+\cdots+k_d=n} \frac{n!}{k_1! \cdots k_d!} \prod_{1 \leq i \leq d} U_{j_i}^{2k_i}\right) \\
= \tau^n \cdot \sum_{k_1+\cdots+k_d=n} \frac{n!}{k_1! \cdots k_d!} E\left(\prod_{1 \leq i \leq d} U_{j_i}^{2k_i}\right).
$$

For $n = 2$, since $E(U_i U_j) = E(U_i) E(U_j) = 0$ for $i \neq j$,

$$
\tau^2 \cdot \sum_{k_1+\cdots+k_d=2} \frac{2}{k_1! \cdots k_d!} E\left(\prod_{1 \leq i \leq d} U_i^{2k_i}\right) = \tau^2 \cdot \sum_{j=1}^d E(U_j^2) = \tau^2 \cdot \sum_{j=1}^d 1 = d \tau^2,
$$

showing that $C_2 = d$. \hfill \Box

A stochastic process $(\Omega, \mathcal{F}, P, (X_t)_{t \in I})$ with state space $E$ is called a d-dimensional Brownian motion when:

1. For $s \leq t$,

$$
(X_t - X_s)_* P = \nu_{t-s},
$$

and thus $X$ has stationary increments.

2. $X$ has independent increments.

3. For almost all $\omega \in \Omega$, the path $t \mapsto X_t(\omega)$ is continuous $I \to E$.

We call $X_0_* P$ the initial distribution of the Brownian motion. When $X_0_* P = \delta_x$ for some $x \in E$, we say that $x$ is the starting point of the Brownian motion. We now prove that for any Borel probability measure on $\mathcal{E}$, in particular $\delta_x$, there is a $d$-dimensional Brownian motion which has this as its initial distribution.\textsuperscript{11}

**Theorem 3** (Brownian motion). For any $\mu \in \mathcal{P}(E)$, there is a $d$-dimensional Brownian motion with initial distribution $\mu$.

**Proof.** Let $(P_j)_{j \in \mathcal{E}}$ be the family of measures induced by the Brownian convolution semigroup

$$
\nu_t = \prod_{k=1}^d \gamma_{0,t}, \quad t \in I,
$$

and let $(\Omega, \mathcal{F}, P^\mu, (X_t)_{t \in I})$, $\Omega = E^I$ and $\mathcal{F} = \mathcal{E}^I$, be the associated canonical process. Theorem\textsuperscript{11} tells us that $X$ has stationary increments,

$$
(X_t - X_s)_* P^\mu = \nu_{t-s}, \quad s \leq t, \quad (12)
$$

\textsuperscript{11}Heinz Bauer, *Probability Theory*, p. 342, Theorem 40.3.
and has independent increments. For \( \tau = t - s > 0 \), by \( \text{[12]} \) and Lemma \( \text{[2]} \),

\[
E(|X_t - X_s|^4) = d(d + 2)r^2 = d(d + 2)|t - s|^2. 
\]

Because \( E(|X_t - X_s|^4) = E(0) = 0 \), we have that for any \( s, t \in I \),

\[
E(|X_t - X_s|^4) = d(d + 2)|t - s|^2. 
\]

The initial distribution of \( X \) is \( X_0, P^\mu = \mu \). For \( \alpha = 4, \beta = 1, c = d(d + 2) \), the Kolmogorov continuity theorem\( \text{[13]} \) tells us that there is a continuous modification \( B \) of \( X \). That is, there is a stochastic process \( (B_t)_{t \in I} \) such that for each \( \omega \in \Omega \), the path \( t \mapsto B_t(\omega) \) is continuous \( I \rightarrow E \), namely, \( B \) is a continuous stochastic process, and for each \( t \in I \),

\[
P(X_t = B_t) = 1,
\]

namely, \( B \) is a modification of \( X \). Because \( B \) is a modification of \( X \), \( B \) has the same finite-dimensional distributions as \( X \)\( \text{[13]} \) from which it follows that \( B \) satisfies \( \text{[12]} \) and has independent increments. For \( A \in \mathcal{E} \), because \( B \) is a modification of \( X \),

\[
(B_0, P^\mu)(A) = P^\mu(B_0 \in A) = P^\mu(X_0 \in A) = (X_0, P^\mu)(A),
\]

thus \( B_0, P^\mu = X_0, P^\mu = \mu \), namely, \( B \) has initial distribution \( \mu \). Therefore, \( B \) is a Brownian motion (indeed, all the paths of \( B \) are continuous, not merely almost all of them) that has initial distribution \( \mu \), proving the claim. \( \square \)

For \( \mu \in \mathcal{P}(E) \), let \((\Omega, \mathcal{F}, P^\mu, (B_t)_{t \in I})\) be the \( d \)-dimensional Brownian motion with initial distribution \( \mu \) constructed in Theorem \( \text{[3]} \) we are not merely speaking about some \( d \)-dimensional Brownian motion but about this construction, for which \( \Omega = E^I \), all whose paths are continuous rather than merely almost all whose paths are continuous. For a measurable space \((A, \mathcal{A})\) and topological spaces \( X \) and \( Y \), a function \( f : X \times A \rightarrow Y \) is called a Carathéodory function if for each \( x \in X \), the map \( a \mapsto f(x, a) \) is measurable \( \mathcal{A} \rightarrow \mathcal{B}_Y \), and for each \( a \in A \), the map \( x \mapsto f(x, a) \) is continuous \( X \rightarrow Y \). It is a fact\( \text{[13]} \) that if \( X \) is a separable metrizable space and \( Y \) is a metrizable space, then any Carathéodory function \( f : X \times A \rightarrow Y \) is measurable \( \mathcal{B}_X \otimes \mathcal{A} \rightarrow \mathcal{B}_Y \), namely it is jointly measurable. \( B : I \times \Omega \rightarrow E \) is a Carathéodory function. \( I = \mathbb{R}_{\geq 0} \), with the subspace topology inherited from \( \mathbb{R} \), is a separable metrizable space, and \( E = \mathbb{R}^d \) is a metrizable space, and therefore the \( d \)-dimensional Brownian motion \( B \) is jointly measurable.


The Kolmogorov-Chentsov theorem\textsuperscript{17} says that if a stochastic process $(X_t)_{t \in I}$ with state space $E$ satisfies, for $\alpha, \beta, c > 0$,

$$E(|X_s - X_t|^\alpha) \leq c|t - s|^{1+\beta}, \quad s, t \in I,$$

and almost every path of $X$ is continuous, then for almost every $\omega \in \Omega$, for every $0 < \gamma < \frac{n}{2}$ the map $t \mapsto X_t(\omega)$ is locally $\gamma$-Hölder continuous: for each $t_0 \in I$ there is some $0 < \epsilon_{t_0} < 1$ and some $C_{t_0}$ such that

$$|X_t(\omega) - X_s(\omega)| \leq C_{t_0}|t - s|^{\gamma}, \quad |s - t_0| < \epsilon_{t_0}, |t - t_0| < \epsilon_{t_0}.$$

For $\mu \in \mathcal{P}(E)$, let $(\Omega, \mathcal{F}, P^\mu, (B_t)_{t \in I})$ be the $d$-dimensional Brownian motion with initial distribution $\mu$ formed in Theorem\textsuperscript{3}. For $s \leq t$, $(B_t - B_s)_n = \mu_{t-s}$, and thus Lemma\textsuperscript{2} tells us that for each $n \geq 1$ there is some $C_n$ with which $E(|B_t - B_s|^{2n}) = C_n(t - s)^n$ for all $s < t$. Then $E(|B_t - B_s|^{2n}) \leq C_n|t - s|^n$ for all $s, t \in I$. For $n > 1$ and for $\alpha_n = 2n$ and $\beta_n = n - 1$,

$$\frac{\beta_n}{\alpha_n} = \frac{n - 1}{2n} = \frac{1}{2} - \frac{1}{2n},$$

and for $n > 2$, take some $\frac{\beta_n - 1}{\alpha_n - 1} < \gamma_n < \frac{\beta_n}{\alpha_n}$. Let $N_n$ be the set of those $\omega \in \Omega$ for which $t \mapsto B_t(\omega)$ is not locally $\gamma_n$-Hölder continuous. Then the Kolmogorov-Chentsov theorem yields $P^\mu(N_n) = 0$. Let $N = \bigcup_{n \geq 2} N_n$, which is a $P^\mu$-null set. For $\omega \in \Omega \setminus N$ and for any $0 < \gamma < \frac{1}{2}$, there is some $\gamma_n$ satisfying $\gamma \leq \gamma_n < \frac{1}{2}$, and hence the map $t \mapsto B_t(\omega)$ is locally $\gamma_n$-Hölder continuous, which implies that this map is locally $\gamma$-Hölder continuous. We summarize what we have just said in the following theorem.

**Theorem 4.** Let $\mu \in \mathcal{P}(E)$ and let $(\Omega, \mathcal{F}, P^\mu, (B_t)_{t \in I})$ be the $d$-dimensional Brownian motion with initial distribution $\mu$ formed in Theorem\textsuperscript{3}. For almost all $\omega \in \Omega$, for all $0 < \gamma < \frac{1}{2}$, the map $t \mapsto B_t(\omega)$ is locally $\gamma$-Hölder continuous.

### 4 Lévy processes

A stochastic process $(X_t)_{t \in I}$ with state space $E$ is called a Lévy process\textsuperscript{17} if (i) $X_0 = 0$ almost surely, (ii) $X$ has stationary and independent increments, and (iii) for any $a > 0$,

$$\lim_{t \uparrow 0} P(|X_t| \geq \epsilon) = 0.$$

Because $X_0 = 0$ almost surely and $X$ has stationary increments, (iii) yields for any $t \in I$,

$$\lim_{s \rightarrow t} P(|X_s - X_s| \geq \epsilon) = 0. \quad (13)$$

\textsuperscript{17}See David Applebaum, *Lévy Processes and Stochastic Calculus*, p. 39, §1.3.
In any case, (13) is sufficient for (iii) to be true. Moreover, (iii) means that $X_s \to X_t$ in the **topology of convergence in probability** as $s \to t$, and if $X_s \to X_t$ almost surely then $X_s \to X_t$ in the topology of convergence in probability; this is proved using Egorov’s theorem.\(^{18}\) Thus, a $d$-dimensional Brownian motion with starting point 0 is a Lévy process; we do not merely assert that the Brownian motion formed in Theorem 3 is a Lévy process. There is much that can be said generally about Lévy processes, and thus the fact that any $d$-dimensional Brownian motion with starting point 0 is a Lévy process lets us work in a more general setting in which some results may be more naturally proved: if we work merely with a Lévy process we know less about the process and thus have less open moves.

\(^{18}\)http://individual.utoronto.ca/jordanbell/notes/L0.pdf Theorem 3.