The Cameron-Martin theorem

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1 Gaussian vectors in a Hilbert space

Lemma 1. Let \((\Omega, \mathcal{F})\) be a measurable space and let \((Y, d)\) be a metric space. Suppose that \((f_n)\) is a sequence of measurable functions \((\Omega, \mathcal{F}) \to (Y, \mathcal{B}_Y)\), \(A \in \mathcal{F}\), \(y_0 \in Y\), and \(f_n(\omega)\) converges in \(Y\) for all \(\omega \in A\). Then \(f : \Omega \to Y\) defined by

\[
f(\omega) = \begin{cases} 
\lim_{n \to \infty} f_n(\omega) & \omega \in A \\
y_0 & \omega \not\in A
\end{cases}
\]

is measurable.

Proof. Because the Borel \(\sigma\)-algebra \(\mathcal{B}_Y\) is generated by the collection of closed sets in \(Y\), it suffices to prove that \(f^{-1}(F) \in \mathcal{F}\) when \(F\) is a closed set in \(Y\). Let

\[G_n = \left\{ y \in Y : d(y, F) < \frac{1}{n} \right\}.
\]

Because \(y \to d(y, F)\) is continuous, each \(G_n\) is open. Because \(F\) is closed, \(F = \bigcap_{n=1}^{\infty} G_n\).

If \(\omega \in A \cap f^{-1}(F)\) and \(k \geq 1\), then because \(G_k\) is an open neighborhood of \(f(\omega)\) and \(f_n(\omega) \to f(\omega) \in G_k\), there is some \(m_k\) such that for \(n \geq m_k\) the point \(f_n(\omega)\) belongs to \(G_k\). Thus

\[A \cap f^{-1}(F) \subset A \cap \bigcap_{k=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} f_n^{-1}(G_k).
\]

On the other hand, if \(\omega\) belongs to the right-hand side then for each \(k\) there is some \(m_k\) such that for \(n \geq m_k\), \(f_n(\omega) \in G_k\). Because \(f_n(\omega) \to f(\omega)\), this means that \(f(\omega) \in \overline{G_k}\). This is true for all \(k\), so \(f(\omega) \in \bigcap_{k=1}^{\infty} \overline{G_k}\), and because \(\overline{G_k} \subset G_k\), it is the case that \(f(\omega) \subset \bigcap_{k=1}^{\infty} G_k = F\). Therefore,

\[A \cap f^{-1}(F) = A \cap \bigcap_{k=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} f_n^{-1}(G_k),
\]
which shows that $A \cap f^{-1}(F) \in \mathfrak{F}$. If $y_0 \in F$, then $f^{-1}(F) = A^c \cup (A \cap f^{-1}(F)) \in \mathfrak{F}$, and if $y_0 \not\in F$, then $f^{-1}(F) = A \cap f^{-1}(F) \in \mathfrak{F}$. Therefore $f$ is measurable.

Let $\mathcal{H}$ be a separable real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and let $(e_j)$ be an orthonormal basis for $\mathcal{H}$. Let $(\xi_j)$ be a sequence of independent random variables $(\Omega, \mathcal{F}, \mathbb{P}) \to (\mathbb{R}, \mathcal{B}_\mathbb{R})$ with distribution $(\xi_j)_* \mathbb{P} = \gamma_1$, where $\gamma_2$ is the Gaussian measure on $\mathbb{R}$ with variance $\sigma^2$. Let $(\sigma_j)$ be a sequence of nonnegative real numbers satisfying $\sum_{j=1}^{\infty} \sigma_j^2 < \infty$. Define $X_n : \Omega \to \mathcal{H}$ by

$$X_n(\omega) = \sum_{j=1}^{n} \sigma_j \xi_j(\omega) e_j,$$

which is measurable $(\Omega, \mathcal{F}) \to (\mathcal{H}, \mathcal{B}_\mathcal{H})$. For $X_n(\omega)$ to be a Cauchy sequence in $\mathcal{H}$, it is necessary and sufficient that $\sum_{j=1}^{\infty} |\sigma_j \xi_j(\omega)|^2 < \infty$. But

$$\sum_{j=1}^{\infty} E|\sigma_j \xi_j|^2 = \sum_{j=1}^{\infty} \sigma_j^2 E|\xi_j|^2 = \sum_{j=1}^{\infty} \sigma_j^2 < \infty$$

implies that the series $\sum_{j=1}^{\infty} |\sigma_j \xi_j|^2$ is convergent almost surely: for some $A \in \mathfrak{F}$ with $\mathbb{P}(A) = 1$ the series $\sum_{j=1}^{\infty} |\sigma_j \xi_j(\omega)|^2$ converges for $\omega \in A$. For $\omega \in A$ we define $X(\omega) \in \mathcal{H}$ to be the limit of the Cauchy sequence $X_n(\omega)$,

$$X(\omega) = \sum_{j=1}^{\infty} \sigma_j \xi_j(\omega) e_j,$$

and otherwise we define $X(\omega) = 0$. By Lemma 1 $X$ is measurable $(\Omega, \mathcal{F}) \to (\mathcal{H}, \mathcal{B}_\mathcal{H})$.

For $X$ defined in (1) and for $f \in \mathcal{H}$ with

$$f = \sum_{j} \langle f, e_j \rangle e_j = \sum_{j} f_j e_j,$$

we have for $\omega \in A$,

$$\langle f, X \rangle = \sum_{j=1}^{\infty} f_j \sigma_j \xi_j(\omega).$$

This satisfies

$$E \langle X, f \rangle = 0,$$

http://individual.utoronto.ca/jordanbell/notes/gaussian.pdf

http://individual.utoronto.ca/jordanbell/notes/parseval.pdf

Mikhail Lifshits, *Lectures on Gaussian Processes*, p. 7, Example 2.2; Lifshits calls this a Karhunen-Loève expansion of $X$. 

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and for \( f, g \in \mathcal{H} \),
\[
\text{Cov}(\langle X, f \rangle, \langle X, g \rangle) = E(\langle X, f \rangle \cdot \langle X, g \rangle)
\]
\[
= E \left( \sum_{j=1}^{\infty} \sigma_j f_j \xi_j \cdot \sum_{k=1}^{\infty} \sigma_k g_k \xi_k \right)
\]
\[
= \sum_{j=1}^{\infty} \sigma_j^2 f_j g_j.
\]

Define \( K : \mathcal{H} \to \mathcal{H} \) by
\[
K e_j = \sigma_j^2 e_j,
\]
which is a Hilbert-Schmidt operator\(^4\) It satisfies
\[
\langle K f, g \rangle = \text{Cov}(\langle X, f \rangle, \langle X, g \rangle).
\]

\section{Wiener measure}

Let \( \mathcal{X} = C[0,1] \), which is a separable Banach space with the supremum norm, whose dual space \( \mathcal{X}^* \) is the signed measures of bounded variation on \([0,1]\)\(^5\)

For \( \mu \in \mathcal{X}^* \) and \( f \in \mathcal{X} \), write
\[
\langle f, \mu \rangle = \int_{[0,1]} f d\mu.
\]

Let \( W \in \mathcal{X}^* \) be Wiener measure on \( \mathcal{X} \), define \( B_t f = f(t) \), and define \( B : \mathcal{X} \to \mathcal{X}^* \) by \( B f = f' \)\(^6\) The stochastic process \((B_t)_{t \in [0,1]}\) is a Brownian motion. For \( s, t \in [0,1] \),
\[
\mathbb{E} B_t = 0, \quad \text{Cov}(B_s, B_t) = \mathbb{E}(B_s B_t) = \min(s,t).
\]

\( B : (\mathcal{X}, \mathcal{B}_\mathcal{X}) \to (\mathcal{X}^*, \mathcal{B}_\mathcal{X}^*) \) is measurable, and \( B_* W = W \), i.e. the distribution of \( B \) is Wiener measure. For \( \mu \in \mathcal{X}^* \),
\[
\mathbb{E} \langle B, \mu \rangle = \mathbb{E} \int_{[0,1]} B_t d\mu(t) = \int_{[0,1]} \mathbb{E} B_t d\mu = 0
\]

and for \( \mu, \nu \in \mathcal{X}^{*-} \),
\[
\text{Cov}(\langle B, \mu \rangle, \langle B, \nu \rangle) = \mathbb{E} \left( \int_{[0,1]} B_s d\mu(s) \cdot \int_{[0,1]} B_t d\nu(t) \right)
\]
\[
= \int_{[0,1] \times [0,1]} \mathbb{E}(B_s B_t) d\mu(s) d\nu(t)
\]
\[
= \int_{[0,1] \times [0,1]} \min(s,t) d\mu(s) d\nu(t).
\]

\(^4\)http://individual.utoronto.ca/jordanbell/notes/traceclass.pdf
\(^5\)http://individual.utoronto.ca/jordanbell/notes/CK.pdf
\(^6\)http://individual.utoronto.ca/jordanbell/notes/donsker.pdf
Define $K : \mathcal{X}^* \to \mathcal{X}$ by

$$(K\mu)(t) = \int_{[0,1]} \min(s,t) d\mu(s),$$

which satisfies

$$\text{Cov}(\langle B, \mu \rangle, \langle B, \nu \rangle) = \langle K\mu, \nu \rangle.$$ 

3 Measurable linear functionals

Let $\mathcal{X}$ be a Fréchet space with dual space $\mathcal{X}^*$, and for $f \in \mathcal{X}$ and $\mu \in \mathcal{X}^*$ denote the dual pairing by

$$\langle f, \mu \rangle,$$

and we also use this notation when $\mu$ is a function $\mathcal{X} \to \mathbb{R}$ that need not belong to $\mathcal{X}^*$. Suppose that $X : (\Omega, \mathcal{F}, \mathbb{P}) \to (\mathcal{X}, \mathcal{B}_X)$ is measurable, and that $X$ is Gaussian with $\mathbb{E}X = 0 \in \mathcal{X}$ and covariance $K : \mathcal{X}^* \to \mathcal{X}$. That is,

$$\mathbb{E}\langle X, \mu \rangle = \langle 0, \mu \rangle = 0$$

for all $\mu \in \mathcal{X}^*$, and $K : \mathcal{X}^* \to \mathcal{X}$ is a continuous linear operator satisfying

$$\mathbb{E}\langle X, \mu \rangle \cdot \langle X, \nu \rangle = \text{Cov}(\langle X, \mu \rangle, \langle X, \nu \rangle) = \langle K\mu, \nu \rangle$$

for all $\mu, \nu \in \mathcal{X}^*$. Let $P = X_*\mathbb{P}$ be the distribution of $X$; $P$ is a Borel probability measure on $\mathcal{X}$.

For $\mu \in \mathcal{X}^*$, by the change of variables formula,

$$\mathbb{E}|\langle X, \mu \rangle|^2 = \int_\Omega |\langle X(\omega), \mu \rangle|^2 d\mathbb{P}(\omega) = \int_{\mathcal{X}} |\langle f, \mu \rangle|^2 d\mathbb{P}(f) = \int_{\mathcal{X}} |\mu|^2 d\mathbb{P}.$$

Let $J : \mathcal{X}^* \to L^2(\mathcal{X}, P)$ be the embedding, and let $\mathcal{X}^*_P$ be the closure of $J(\mathcal{X}^*)$ in $L^2(\mathcal{X}, P)$. Thus $\mathcal{X}^*_P$ is a Hilbert space with the inner product

$$\langle \phi, \psi \rangle_{\mathcal{X}^*_P} = \int_{\mathcal{X}} \phi \cdot \psi dP = \mathbb{E}(\langle X, \phi \rangle \cdot \langle X, \psi \rangle).$$

Elements of $\mathcal{X}^*_P$ are called measurable linear functionals; elements of $\mathcal{X}^*$ are continuous linear functionals.

$J : \mathcal{X}^* \to \mathcal{X}^*_P$ is a continuous linear map, and there is a unique continuous linear map $I : (\mathcal{X}^*_P)^* \to \mathcal{X}$ satisfying

$$\langle I\phi, \mu \rangle = \langle \phi, J\mu \rangle_{\mathcal{X}^*_P} = \langle \phi, \mu \rangle_{\mathcal{X}^*_P} = \mathbb{E}(\langle X, \phi \rangle \cdot \langle X, \mu \rangle)$$

for $\phi \in (\mathcal{X}^*_P)^* = \mathcal{X}^*_P$ and $\mu \in \mathcal{X}^*$. If $I\phi = 0$ then

$$\langle \phi, J\mu \rangle_{\mathcal{X}^*_P} = \langle I\phi, \mu \rangle = \langle 0, \mu \rangle = 0.$$
for all $\mu \in \mathcal{X}^*$. Let $\mu_n \in \mathcal{X}^*$ with $J\mu_n \to \phi$ in $L^2(\mathcal{X}, P)$. Then $\langle \phi, J\mu_n \rangle_{\mathcal{X}^*_P} \to \langle \phi, \phi \rangle_{\mathcal{X}^*_P}$, and because each $\langle \phi, J\mu_n \rangle_{\mathcal{X}^*_P} = 0$ we get that $\langle \phi, \phi \rangle_{\mathcal{X}^*_P} = 0$, which means that $\phi = 0$. Therefore $I$ is injective.

We have assumed that $X$ has covariance $K : \mathcal{X}^* \to \mathcal{X}$, which means that for $\mu, \nu \in \mathcal{X}^*$,

$$\langle K\mu, \nu \rangle = \mathbb{E}(\langle X, \mu \rangle \cdot \langle X, \nu \rangle).$$

For $\mu, \nu \in \mathcal{X}^*$,

$$\langle I\mu, \nu \rangle = \langle J\mu, J\nu \rangle_{\mathcal{X}^*_P} = \mathbb{E}(\langle X, \mu \rangle \cdot \langle X, \nu \rangle) = \langle K\mu, \nu \rangle,$$

which implies that $K = IJ$.

Let

$$H_P = I\mathcal{X}^*_P,$$

which is a linear subspace of $\mathcal{X}$. For $f, g \in H_P$, let

$$\langle f, g \rangle_{H_P} = \langle I^{-1}f, I^{-1}g \rangle_{\mathcal{X}^*_P}.$$

### 4 Examples of $H_P$

Take $X : \Omega \to \mathcal{H}$ from §1 with $EX = 0$ and with covariance $K : \mathcal{H} \to \mathcal{H}$ defined by

$$K e_j = \sigma^2_j e_j,$$

which is a Hilbert-Schmidt operator satisfying

$$\langle K f, g \rangle = \text{Cov}(\langle X, f \rangle, \langle X, g \rangle).$$

For $f, g \in \mathcal{H} = \mathcal{H}^*$,

$$\langle J f, J g \rangle_{\mathcal{X}^*_P} = \langle K f, g \rangle = \sum_{j=1}^{\infty} \sigma^2_j f_j g_j,$$

Check that $\mathcal{X}^*_P$ is the set of those $\phi : \mathcal{H} \to \mathbb{R}$ such that

$$\sum_{j=1}^{\infty} |\langle e_j, \phi \rangle|^2 \sigma^2_j < \infty.$$

Writing $\phi_j = \langle \phi, e_j \rangle$,

$$\langle I \phi, e_k \rangle = \langle \phi, Je_k \rangle_{\mathcal{X}^*_P},$$

$$= \langle \phi, e_k \rangle_{\mathcal{X}^*_P},$$

$$= \mathbb{E}(\langle X, \phi \rangle \cdot \langle X, e_k \rangle)$$

$$= \mathbb{E} \left( \sum_{j=1}^{\infty} \phi_j \sigma_j \xi_j \cdot \sigma_k \xi_k \right),$$

$$= \sigma^2_k \phi_j.$$

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For $\phi \in X_P^*$, write $h = I\phi$, and then
\[ \|h\|_{H_P}^2 = \|\phi\|_{X_P^*}^2 = \sum_{j=1}^{\infty} \sigma_j^2 \phi_j^2. \]

But
\[ h_k = \langle h, e_k \rangle = \langle I\phi, e_k \rangle = \langle \phi, Je_k \rangle = \langle X, e_k \rangle \]
\[ = \mathbb{E} \left( \sum_{j=1}^{\infty} \phi_j \sigma_j \xi_j \cdot \sigma_k \xi_k \right) = \sigma_k^2 \phi_k, \]

so
\[ \|h\|_{H_P}^2 = \sum_{j=1}^{\infty} \frac{h_j^2}{\sigma_j^2}. \]

We then check that
\[ H_P = \left\{ h \in \mathcal{H} : \sum_{j=1}^{\infty} \frac{h_j^2}{\sigma_j^2} < \infty \right\}. \]

Now take $\mathcal{H} = \mathbb{R}^d$ and let $X : (\Omega, \mathcal{F}, \mathbb{P}) \to (\mathbb{R}^d, \mathcal{B}_{\mathbb{R}^d})$ be a random vector that is Gaussian with $\mathbb{E}X = 0$ and positive-definite covariance $K : \mathbb{R}^d \to \mathbb{R}^d$, and let $P = X^*P$, a Borel probability measure on $\mathbb{R}^d$. Because $K$ is a positive-definite symmetric matrix, by the spectral theorem there is an orthonormal basis $e_1, \ldots, e_d$ for $\mathbb{R}^d$ and positive real numbers $\sigma_1, \ldots, \sigma_d$ such that $Ke_j = \sigma_j^2 e_j$. For almost all $\omega \in \Omega$,
\[ X(\omega) = \sum_{j=1}^{d} \sigma_j \xi_j(\omega) e_j. \]

From our work before, $H_P = \mathbb{R}^d$, and
\[ \langle f, g \rangle_{H_P} = \sum_{j=1}^{\infty} \frac{f_j g_j}{\sigma_j^2} = \langle K^{-1} f, g \rangle. \]

### 5 The factorization theorem

The following is proved in Lifshits, and there is called the factorization theorem.

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7Mikhail Lifshits, Lectures on Gaussian Processes, p. 26, Theorem 4.1.
Theorem 2 (Factorization theorem). If $\mathcal{X}$ is a Fréchet space, $\mathcal{H}$ is a Hilbert space, and $L : \mathcal{H} \to \mathcal{X}$ is an injective linear map such that

$$K = LL^*,$$

then

$$H_P = L\mathcal{H}$$

and

$$\langle f, g \rangle_{H_P} = \langle L^{-1}f, L^{-1}g \rangle_{\mathcal{X}}$$

for all $f, g \in H_P$.

Let $\mathcal{X} = C[0, 1]$, from §2. Here, $B : (\mathcal{X}, \mathcal{B}_{\mathcal{X}}, W) \to (\mathcal{X}, \mathcal{B}_{\mathcal{X}})$ is $Bf = f$, $P = B,W = W$, and the covariance of $B$ is $K : \mathcal{X}^* \to \mathcal{X}$,

$$(K\mu)(t) = \int_{[0,1]} \min(s,t)d\mu(s),$$

satisfying

$$\langle K\mu, \nu \rangle = \mathbb{E}(\langle B, \mu \rangle, \langle B, \nu \rangle).$$

Take $\mathcal{H} = L^2[0, 1]$, with Lebesgue measure. Define $L : \mathcal{H} \to \mathcal{X}$ by

$$(Lf)(t) = \int_0^t f(s)ds.$$ 

Indeed, $Lf$ is continuous, and $L$ is linear and injective. $\mathcal{X}^*$ is the signed measures of bounded variation on $[0, 1]$. For $\mu \in \mathcal{X}^*$, Fubini’s theorem yields

$$\langle f, L^*\mu \rangle_{\mathcal{X}} = \langle Lf, \mu \rangle$$

$$\hspace{1cm} = \int_{[0,1]} (Lf)(t)d\mu(t)$$

$$\hspace{1cm} = \int_{[0,1]} \left( \int_0^t f(s)ds \right) d\mu(t)$$

$$\hspace{1cm} = \int_0^1 \left( \int_{[s,1]} d\mu(t) \right) f(s)ds$$

$$\hspace{1cm} = \int_0^1 \mu[s, 1]f(s)ds$$

$$\hspace{1cm} = \langle s \mapsto \mu[s, 1], f \rangle_{\mathcal{X}}.$$

This shows that $L^* : \mathcal{X}^* \to \mathcal{H}$ is

$$(L^*\mu)(s) = \mu[s, 1].$$
For $\mu \in \mathcal{X}^*$,

\[ (LL^* \mu)(t) = \int_0^t (L^* \mu)(s) ds \]

\[ = \int_0^t \mu[s, 1] ds \]

\[ = \int_0^1 1_{[0, t]}(s) \left( \int_{[s, 1]} d\mu(r) \right) ds \]

\[ = \int_{[0, 1]} \left( \int_0^t 1_{[0, t]}(s) ds \right) d\mu(r) \]

\[ = \int_{[0, 1]} \min(r, t) d\mu(r), \]

showing that $LL^* = K$. Then by Theorem 2

\[ H_W = L\mathcal{X} \]

and

\[ (F, G)_{H_W} = \langle L^{-1} F, L^{-1} G \rangle_{\mathcal{X}}, \]

for $F, G \in H_W$. This means that if $F \in H_W$ if and only if there is $f \in L^2[0, 1]$ such that

\[ F(t) = (Lf)(t) = \int_0^t f(s) ds. \]

This is equivalent to $F$ being absolutely continuous, with $F(0) = 0$ and for almost all $t \in [0, 1], F$ is differentiable at $t$, and $F' \in L^2[0, 1]$. Thus, $H_W$ is the collection of absolutely continuous functions $F : [0, 1] \to \mathbb{R}$ satisfying $F(0) = 0$ and $F' \in L^2[0, 1]$. Furthermore,

\[ (F, G)_{H_W} = \langle L^{-1} F, L^{-1} G \rangle_{L^2[0, 1]} = \langle F', G' \rangle_{L^2[0, 1]} \]

6 The Cameron-Martin theorem

Let $\mathcal{X}$ be a Fréchet space and let $X : (\Omega, \mathcal{F}, \mathbb{P}) \to (\mathcal{X}, \mathcal{B}_\mathcal{X})$ be a random vector with distribution $P = X_\# \mathbb{P}$. For $h \in \mathcal{X}$, $X + h$ is a random vector, and we write $P_h = (X + h)_\# \mathbb{P}$. For $A \in \mathcal{B}_\mathcal{X}$,

\[ P_h(A) = \mathbb{P}(X + h \in A) = \mathbb{P}(X \in A - h) = P(A - h). \]

If $P_h$ is absolutely continuous with respect to $P$, written $P_h \ll P$, we say that $h$ is an admissible shift.

For $\mathcal{X} = \mathbb{R}^d$, let $X$ be a random vector with state space $\mathcal{X}$ and Gaussian distribution with $\mathbb{E}X = 0$ and covariance $I_d$, namely a random vector on $\mathcal{X}$.

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http://indvidual.utoronto.ca/jordanbell/notes/totalvariation.pdf
with the standard Gaussian distribution. Let $\lambda_d$ be Lebesgue measure on $\mathbb{R}^d$. For $P = X_*\mathbb{P}$, which is a standard Gaussian measure on $\mathbb{R}^d$,

$$dP(x) = (2\pi)^{-d/2}e^{-(x,x)/2}d\lambda_d(x).$$

That is, the density of $P$ with respect to $\lambda_d$ is

$$\frac{dP}{d\lambda_d}(x) = (2\pi)^{-d/2}e^{-(x,x)/2}.$$

For $h \in \mathbb{R}^d$ and $A$ a Borel set in $\mathbb{R}^d$,

$$P_h(A) = P(A - h) = \int_{A-h}(2\pi)^{-d/2}e^{-(x,x)/2}dx = \int_{A}(2\pi)^{-d/2}e^{-(y-h,y-h)/2}dy,$$

which shows that

$$\frac{dP_h}{d\lambda_d}(x) = (2\pi)^{-d/2}e^{-(x-h,x-h)/2}.$$

Because $\lambda_d \ll P$, with

$$\frac{d\lambda_d}{dP}(x) = (2\pi)^{d/2}e^{(x,x)/2},$$

the chain rule for the Radon-Nikodym derivative yields

$$\frac{dP_h}{dP}(x) = \frac{dP_h}{d\lambda_d}(x) \cdot \frac{d\lambda_d}{dP}(x) = (2\pi)^{-d/2}e^{-(x-h,x-h)/2} \cdot (2\pi)^{d/2}e^{(x,x)/2},$$

which is

$$\frac{dP_h}{dP}(x) = e^{(h,x)-(h,h)/2}.$$

We now get to the Cameron-Martin theorem\textsuperscript{9}

**Theorem 3** (Cameron-Martin theorem). Let $\mathcal{X}$ be a Fréchet space, let $X : (\Omega, \mathcal{F}, \mathbb{P}) \rightarrow (\mathcal{X}, \mathcal{B}_\mathcal{X})$ be a random vector that is Gaussian with $\mathbb{E}X = 0$ and covariance $K : \mathcal{X}^* \rightarrow \mathcal{X}$, and with distribution $P = X_*\mathbb{P}$. In this case, $P_h \ll P$ if and only if $h \in H_P$.

If $h \in H_P$, then there is some $\phi \in \mathcal{X}_P^*$ such that $L\phi = h$ and

$$\frac{dP_h}{dP}(f) = e^{\langle f, \phi \rangle - (h,h)/2}, \quad f \in \mathcal{X}.$$

We have established that

$$H_W = \{h \in AC[0,1] : h(0) = 0, h' \in L^2[0,1]\}$$

and

$$\|h\|_{H_W}^2 = \int_0^1 |h'(s)|^2 ds, \quad h \in H_W.$$

For $h = H_W$ let $\phi \in L^2[0,1]$ such that $L\phi = h$, i.e. $\phi = L^{-1}h$ which means $\phi = h'$ in $L^2[0,1]$.

\textsuperscript{9}Mikhail Lifshits, *Lectures on Gaussian Processes*, p. 34, Theorem 5.1.