The left shift map and expanding endomorphisms of the circle

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1 Sequences

Let $m \geq 2$ and let $N_m = \{0, \dots, m-1\}$, which is a discrete topological space. Let $I = \mathbb{Z}_{\geq 1}$, for $i \in I$ write $A_i = N_m$, and let ν_i be the probability measure on A_i defined by $\nu_i(\{a\}) = \frac{1}{m}$ for $a \in A_i$. Let

$$\Sigma_m = \prod_{i \in I} A_i.$$

Define $\pi_i : \Sigma_m \to A_i$ by $\pi_i(x) = x(i)$. A cylinder set is a subset of Σ_m of the form

$$\prod_{i \in I} B_i,$$

where $B_i \subset A_i$ and $\{i \in I : B_i \neq A_i\}$ is finite. In other words, a cylinder set is an intersection of finitely many sets of the form $\pi_i^{-1}(B_i)$ with $B_i \subset A_i$. Let $\mathscr{C}$ be the collection of cylinder sets. The product σ-algebra is the σ-algebra generated by $\mathscr{C}$.

Assign Σ_m the product topology, the initial topology for $\{\pi_i : i \in I\}$. Because A_i is finite, with the discrete topology it is compact and so Σ_m is compact. The discrete topology on A_i is induced by the metric $d_i(a, b) = |a - b|$. For $x, y \in \Sigma_m$ let

$$d(x, y) = \sum_{i \in \mathbb{Z}_{\geq 1}} \frac{d_i(x(i), y(i))}{m^i} = \sum_{i \in \mathbb{Z}_{\geq 1}} \frac{|x(i) - y(i)|}{m^i}.$$

It is a fact that d is a metric on Σ_m that induces the product topology.[1]

It is a fact that the Borel σ-algebra of Σ_m is equal to the product σ-algebra. (This is true for any countable product of second-countable topological spaces.[2])

[1] cf. http://individual.utoronto.ca/jordanbell/notes/uniformmetric.pdf
[2] http://individual.utoronto.ca/jordanbell/notes/kolmogorov.pdf

Let $\mu_m = \bigotimes_{i \in I} \nu_i$, the product measure:[3] for $\prod_{i \in I} B_i \in \mathscr{C}$,

$$\mu_m \left(\prod_{i \in I} B_i \right) = \prod_{i \in I} \nu_i(B_i).$$

2 The left shift

Define $\sigma : \Sigma_m \to \Sigma_m$ by

$$(\sigma x)(i) = x(i+1), \qquad i \in I.$$

For $\prod_{i \in I} B_i \in \mathscr{C}$, let $C_1 = N_m$ and otherwise let $C_i = B_{i-1}$. Then

$$\sigma^{-1} \left(\prod_{i \in I} B_i \right) = \prod_{i \in I} C_i.$$

This shows that σ is continuous. Moreover, this shows that for $C \in \mathscr{C}$,

$$(\sigma_* \mu_m)(C) = \mu_m(\sigma^{-1}(C)) = \mu_m(C).$$

It follows that

$$\sigma_* \mu_m = \mu_m. \tag{1}$$

That is, σ is measure-preserving.

For $k \geq 1$, $\mathrm{Fix}\,(\sigma^k)$ is the set of those $x \in \Sigma_m$ such that for each $i \in I$, $x(i+k) = x(i)$. Check that $|\mathrm{Fix}\,(\sigma^k)| = m^k$.

3 The circle

Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, which is a compact abelian group using addition, and let μ be the Haar measure with $\mu(\mathbb{T}) = 1$. For $m \in \mathbb{Z}_{\geq 1}$ let $E_m : \mathbb{T} \to \mathbb{T}$ be

$$E_m t = mt,$$

which is an endomorphism of the topological group $\mathbb{T}$: E_m is continuous, and for $s, t \in \mathbb{T}$, $E_m(s+t) = E_m s + E_m t$.

Define $\phi : \Sigma_m \to \mathbb{T}$ by

$$\phi(x) = \sum_{i \geq 1} \frac{x(i)}{m^i} + \mathbb{Z}.$$

[3] `http://individual.utoronto.ca/jordanbell/notes/productmeasure.pdf`

ϕ is continuous and surjective. For $x \in \Sigma_m$,

$$\begin{aligned}(E_m \circ \phi)(x) &= \sum_{i\geq 1} m \cdot \frac{x(i)}{m^i} + \mathbb{Z} \\ &= \sum_{i\geq 2} \frac{x(i)}{m^{i-1}} + \mathbb{Z} \\ &= \sum_{i\geq 1} \frac{x(i+1)}{m^i} + \mathbb{Z} \\ &= (\phi \circ \sigma)(x),\end{aligned}$$

which means that

$$E_m \circ \phi = \phi \circ \sigma. \tag{2}$$

Thus $E_m : \mathbb{T} \to \mathbb{T}$ and $\sigma : \Sigma_m \to \Sigma_m$ are **topologically semiconjugate**.

Check that

$$\phi_* \mu_m = \mu. \tag{3}$$

Using (1), (2), and (3),

$$\begin{aligned}E_{m*}\mu &= E_{m*}(\phi_* \mu_m) \\ &= (E_m \circ \phi)_* \mu_m \\ &= (\phi \circ \sigma)_* \mu_m \\ &= \phi_*(\sigma_* \mu_m) \\ &= \phi_* \mu_m \\ &= \mu.\end{aligned}$$

This means that $E_m : \mathbb{T} \to \mathbb{T}$ is measure-preserving.

For $k \geq 1$,

$$\phi \circ \sigma^k = (E_m \circ \phi) \circ \sigma^{k-1} = \cdots = E_m^k \circ \phi.$$

If $x \in \mathrm{Fix}\,(\sigma^k)$, then

$$\phi(x) = (\phi \circ \sigma^k)(x) = (E_m^k \circ \phi)(x),$$

hence $\phi(x) \in \mathrm{Fix}\,(E_m^k)$. Now, let $z_0(i) = 0$ for all i and let $z_1(i) = m - 1$ for all i; $z_0, z_1 \in \mathrm{Fix}\,(\sigma^k)$. $\phi(z_0) = 0 + \mathbb{Z}$ and $\phi(z_1) = \sum_{i\geq 1} \frac{m-1}{m^i} + \mathbb{Z} = 1 + \mathbb{Z} = 0 + \mathbb{Z}$, so $\phi(z_0) = \phi(z_1)$. Check that if $x, y \in \mathrm{Fix}\,(\sigma^k)$ are distinct and $\{x, y\} \neq \{z_0, z_1\}$ then $\phi(x) \neq \phi(y)$. It follows that[4]

$$|\mathrm{Fix}\,(E_m^k)| = |\mathrm{Fix}\,(\sigma^k)| - 1 = m^k - 1.$$

[4]Michael Brin and Garrett Stuck, *Introduction to Dynamical Systems*, p. 6, §1.3.