1 Definitions

Let $H$ be a complex Hilbert space. If $\lambda \in \mathbb{C}$, we also write $\lambda$ to denote $\lambda \cdot \text{id}_H \in B(H)$.

For $T \in B(H)$, the spectrum $\sigma(T)$ of $T$ is the set of those $\lambda \in \mathbb{C}$ such that the map $T - \lambda$ is not bijective.\footnote{\(\sigma(T)\) is defined to be the set of $\lambda \in \mathbb{C}$ such that $v \mapsto T v - \lambda v$ is not a bijection. It is a fact that if $T \in B(H)$ and $v \mapsto T v - \lambda v$ is a bijection then it is an element of $B(H)$. That it is linear can be proved quickly. The fact that it is bounded is proved using the open-mapping theorem, which states that a surjective bounded linear map from one Banach space to another is an open map, from which it follows that a bijective bounded linear map from one Banach space to another has a bounded inverse.} It happens that it is useful for some purposes to write $\sigma(T)$ as a union of three particular disjoint subsets of itself.

- The point spectrum $\sigma_{\text{point}}(T)$ is the set of those $\lambda \in \mathbb{C}$ such that $T - \lambda$ is not injective. Equivalently, $\lambda \in \sigma_{\text{point}}(T)$ if $\lambda$ is an eigenvalue of $T$.\footnote{The point spectrum is often called the discrete spectrum. From the definition by itself it is not apparent what $\sigma_{\text{point}}(T)$ has to do either with points or discreteness.}

- The continuous spectrum $\sigma_{\text{cont}}(T)$ is the set of those $\lambda \in \mathbb{C}$ such that $T - \lambda$ is injective, has dense image, and is not surjective.

- The residual spectrum $\sigma_{\text{res}}(T)$ is the set of those $\lambda \in \mathbb{C}$ such that $T - \lambda$ is injective and does not have dense image.

It is apparent that the sets $\sigma_{\text{point}}(T)$, $\sigma_{\text{cont}}(T)$, and $\sigma_{\text{res}}(T)$ are disjoint, and that

$$\sigma(T) = \sigma_{\text{point}}(T) \cup \sigma_{\text{cont}}(T) \cup \sigma_{\text{res}}(T).$$

If $T \in B(H)$ then $\sigma(T) \neq \emptyset$, but any of the above three sets may be empty; they merely can’t all be empty for a given operator.
2 Properties

2.1 Residual spectrum

If \( T \in B(H) \) is a normal operator then \( \sigma_{\text{res}}(T) = \emptyset \).\(^3\) We prove this. Suppose that \( T - \lambda \) is injective. We have to show that \( \text{im} (T - \lambda) \) is dense in \( H \), and thus that \( \lambda \not\in \sigma_{\text{res}}(T) \). (\( \lambda \) might be in \( \sigma_{\text{cont}}(T) \) or might not be in \( \sigma(T) \); we merely want to show that it is not in \( \sigma_{\text{res}}(T) \).) We have

\[
H = \overline{\text{im} (T - \lambda)} \oplus (\text{im} (T - \lambda))^\perp.
\]

Let \( w \in (\text{im} (T - \lambda))^\perp \); we have to show that \( w = 0 \). For all \( v \in H \),

\[
\langle (T - \lambda)v, w \rangle = 0,
\]

so for all \( v \in H \) we have \( \langle v, (T - \lambda)^*w \rangle = 0 \) and therefore \( (T - \lambda)^*w = 0 \), so \( w \in \ker(T - \lambda)^* = \ker(T - \lambda) \).\(^4\) As \( T - \lambda \) is injective, \( w = 0 \), completing the proof.

2.2 Point spectrum

If \( A \in B(H) \) is normal then it is straightforward to show that \( \ker A = \ker A^* \). Also, if \( T \in B(H) \) is normal then for any \( z \in \mathbb{C} \), \( T - z \) is normal. Thus, \( \ker(T - z) = \{0\} \) if and only if \( \ker((T - z)^*)_z = \{0\} \). That is, \( \lambda \in \sigma_{\text{point}}(T) \) if and only if \( \lambda \in \sigma_{\text{point}}(T^*) \). For \( X \subseteq \mathbb{C} \) we define \( X^* = \{ \overline{z} : z \in X \} \). We have shown that if \( T \in B(H) \) is normal then

\[
\sigma_{\text{point}}(T)^* = \sigma_{\text{point}}(T^*).
\]

2.3 Continuous spectrum

If \( \lambda \in \sigma_{\text{cont}}(T) \), then \( \text{im} (T - \lambda) \) is dense in \( H \). Also,

\[
(T - \lambda)^{-1} : \text{im} (T - \lambda) \to H
\]

is a surjective linear map (the inverse of a linear map is itself a linear map) that is not continuous. For \( \text{im} (T - \lambda) \) is dense in \( H \), so if \( (T - \lambda)^{-1} \) were continuous then it would have a unique extension to a continuous, hence bounded, map \( H \to H \). Using this and the fact that \( T - \lambda \) is not surjective will give a contradiction.

\(^3\)\( T \in B(H) \) is normal if \( T^*T = TT^* \); in particular, a self-adjoint operator is normal. Equivalently, \( T \in B(H) \) is normal if and only if \( \|Tv\| = \|T^*v\| \) for all \( v \in H \).

\(^4\)If \( S \) is normal then \( \ker S = \ker S^* \). Proof: If \( v \in \ker S \) then \( \langle Sv, Sv \rangle = 0 \), hence \( \langle S^*Sv, v \rangle = 0 \), hence \( \langle SS^*v, v \rangle = 0 \), hence \( \langle S^*v, S^*v \rangle = 0 \), hence \( S^*v = 0 \), hence \( v \in \ker S^* \).
2.4 Approximate point spectrum

Let \( \lambda \in \mathbb{C} \setminus (\sigma_{\text{point}}(T) \cup \sigma_{\text{res}}(T)) \). I claim that \( \lambda \in \sigma_{\text{cont}}(T) \) if and only if \( \lambda \) is in the approximate point spectrum of \( T \), the set of those \( \lambda \in \mathbb{C} \) such that there is a sequence \( v_n \in H \) with \( \|v_n\| = 1 \) and \( \|(T - \lambda)v_n\| \to 0 \) as \( n \to \infty \). It is a fact that for \( T \in B(H) \), if \( T \) is invertible if and only if \( T(H) \) is dense in \( H \) and there is some \( \alpha > 0 \) such that \( \|Tv\| \geq \alpha \|v\| \) for all \( v \in H \).\(^5\) If \( \lambda \in \sigma_{\text{cont}}(T) \), then \( T - \lambda \) is not invertible but the image of \( T - \lambda \) is dense in \( H \), then it must therefore be that \( T - \lambda \) is not bounded below. That is, there is no \( \alpha > 0 \) such that for every \( w \in H \) we have \( \|(T - \lambda)w\| \geq \alpha \|w\| \). Then for each \( n \) there is some \( w \in H \) such that \( \|(T - \lambda)w_n\| < \frac{1}{n} \|w_n\| \). Let \( v_n = \frac{w_n}{\|w_n\|} \). We have \( \|v_n\| = 1 \) and \( \|(T - \lambda)v_n\| < \frac{1}{n} \), showing that \( \lambda \in \sigma_{\text{ap}}(T) \).

On the other hand, if \( \lambda \in \sigma_{\text{ap}}(T) \) then there is a sequence \( v_n \in H \) such that \( \|(T - \lambda)v_n\| \to 0 \) as \( n \to \infty \). Then for any \( \alpha > 0 \), there is some \( v_n \) such that \( \|(T - \lambda)v_n\| < \alpha \|v_n\| \), so \( T - \lambda \) is not invertible and hence \( \lambda \in \sigma(T) \). Since we assumed that \( \lambda \in \mathbb{C} \setminus (\sigma_{\text{point}}(T) \cup \sigma_{\text{res}}(T)) \), we then have \( \lambda \in \sigma_{\text{cont}}(T) \).

Halmos shows in Problem 62 of his Hilbert Space Problem Book that \( \sigma_{\text{ap}}(T) \) is a closed subset of \( \mathbb{C} \), and proves in Problem 63 that \( \partial\sigma(T) \subseteq \sigma_{\text{ap}}(T) \): the boundary of the spectrum of \( T \) is contained in the approximate point spectrum of \( T \).

3 Normal operators

We showed earlier that if \( T \in B(H) \) is normal then \( \sigma_{\text{point}}(T)^* = \sigma_{\text{point}}(T^*) \), where, for \( X \subseteq \mathbb{C} \), \( X^* = \{ \tau : z \in X \} \). This is one reason why it can be helpful to know that an operator is normal. Using this we can show something more about normal operators. Let \( T \in B(H) \) be normal and suppose that \( \lambda, \mu \in \sigma_{\text{point}}(T) \) are distinct. Then there are nonzero \( v, w \in H \) with \( Tv = \lambda v \) and \( Tw = \mu w \). Using \( T^*w = \overline{\mu}w \) we get

\[
\lambda \langle v, w \rangle = \langle \lambda v, w \rangle = \langle Tv, w \rangle = \langle v, T^*w \rangle = \langle v, \overline{\mu}w \rangle = \mu \langle v, w \rangle.
\]

As \( \lambda \neq \mu \), this means that \( \langle v, w \rangle = 0 \). In words, if \( T \in B(H) \) is normal, then its eigenspaces are mutually orthogonal.

\(^5\)In words, \( T \in B(H) \) is invertible if and only if it has dense image and is bounded below. This result is proved in Paul Halmos, Introduction to Hilbert Space and the Theory of Spectral Multiplicity, p. 38, §21, Theorem 3.
4 Compact operators

Let $K(H)$ be the closure in $B(H)$ of the set of finite-rank operators. We call the elements of $K(H)$ compact operators. The following are equivalent ways to state that an operator is compact.

- $T \in B(H)$ is compact if and only if for every bounded subset $S$ of $H$, the closure of the image $T(S)$ is compact.

- $T \in B(H)$ is compact if and only if for every sequence $v_n \in H$ with $\|v_n\| = 1$, $T(v_n)$ has a convergent subsequence.

- Let $B$ be the closed unit ball in $H$, and let $B$ be a topological space with the weak topology: a net $v_\alpha \in B$ converges weakly to $v \in B$ if for all $w \in H$ we have $\langle v_\alpha, w \rangle \to \langle v, w \rangle$. If $H$ is separable, then the weak topology on $B$ is metrizable and thus can be characterized using merely sequences instead of nets. $T \in B(H)$ is compact if and only if the restriction of $T$ to $B$ is continuous $B \to H$, where $B$ has the weak topology and $H$ has the norm topology.

If $K \in K(H)$ and $V$ is a closed subspace of $V$ such that $K(V) \subseteq V$, then the restriction of $K$ to $V$ is an element of $K(V)$.

$B(H)$ is a $C^*$-algebra, and $K(H)$ is a $C^*$-subalgebra of $B(H)$. If $T \in K(H)$ and $S \in B(H)$, then $ST, TS \in K(H)$.

Hence $K(H)$ is an ideal of the $C^*$-algebra $B(H)$.

Useful facts about compact operators are proved in Yuri A. Abramovich and Charalambos D. Aliprantis, *An Invitation to Operator Theory*, p. 272, §7.1.

5 Fredholm alternative

The Fredholm alternative states that if $K \in K(H)$, $\lambda \neq 0$, and $\ker(K - \lambda) = \{0\}$ then $(K - \lambda)^{-1} \in B(H)$. Equivalently, if $K \in K(H)$, $\lambda \neq 0$, and $\lambda \notin \sigma_{\text{point}}(K)$ then $\lambda \notin \sigma(K)$. Equivalently, if $K \in K(H)$, then $\sigma(K) \subseteq \sigma_{\text{point}}(K) \cup \{0\}$.

The above forms are the ones that we want to use. The following is the one that we want to prove, which is equivalent because a nonzero multiple of a

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6This is proved in Paul Halmos, *Hilbert Space Problem Book*, Problem 18.

7The $C^*$-algebra $B(H)/K(H)$ is called the Calkin algebra of $H$. $T \in B(H)$ is called a Fredholm operator if $T + K(H)$ is an invertible element of the Calkin algebra. In particular, if $T \in K(H)$ then $\text{id}_H - T$ is a Fredholm operator.

8We are following Paul Halmos, *Hilbert Space Problem Book*, p. 293, Problem 140.
compact operator is compact: If \( K \in K(H) \) and \( \ker(id_H - K) = \{0\} \) then \((id_H - K)^{-1} \in B(H)\).

We prove two standalone lemmas that we then use to prove the Fredholm alternative.

- Let \( K \in K(H) \) let \( A = id_H - K \in B(H) \), and suppose that \( A(H) = H \).
  Define \( K_n = \ker(A^n) \). We have \( K_1 \subseteq K_2 \subseteq \cdots \). Assume by contradiction that \( K_1 \neq \{0\} \). Then there is some nonzero \( f_1 \in K_1 \). As \( A(H) = H \), there is some \( f_2 \in H \) with \( Af_2 = f_1 \); but \( A^2 f_2 = Af_1 = 0 \) and \( Af_2 = f_1 \neq 0 \), so \( f_2 \in K_2 \setminus K_1 \). Let \( f_{n+1} \in K_{n+1} \setminus K_n \) with \( Af_{n+1} = f_n \). Therefore \( K_1, K_2, \ldots \) are a strictly increasing sequence of subspaces of \( H \). Using Gram-Schmidt, there is an orthonormal sequence \( e_1, e_2, \ldots \) with \( e_n \in K_n \) for all \( n \); we caution that we do not necessarily have \( Ae_{n+1} = e_n \). As \( Ae_{n+1} \in K_n \), \( (Ae_{n+1}, e_{n+1}) \), giving
  \[
  \|Ke_{n+1}\|^2 = \|e_{n+1} - Ae_{n+1}\|^2 = \|e_{n+1}\|^2 + \|Ae_{n+1}\|^2 \geq 1 \|e_{n+1}\|^2 = 1.
  \]
  Each \( e_n \) is an element of the closed unit ball \( B \), and \( e_n \to 0 \) weakly (this is the case for any orthonormal sequence in \( H \), basis or not, and is proved using Bessel’s inequality). Since \( K \) is compact, it is continuous \( B \to H \) where \( B \) has the weak topology and \( H \) has the norm topology; but \( e_n \to 0 \), \( K(0) = 0 \), and \( \|Ke_n\| \geq 1 \), so \( Ke_n \) does not converge to 0 in \( H \), a contradiction. Therefore \( K_1 = \{0\} \), that is, \( \ker A = \{0\} \).

- Let \( K \in K(H) \) and let \( A = id_H - K \in B(H) \). Suppose by contradiction that \( A \) is not bounded below on \((\ker A)^\perp \). So for every \( \alpha > 0 \) there is some \( w \in (\ker A)^\perp \) such that \( \|Aw\| \geq \alpha \|w\| \). Then for all \( n \) there is some \( w_n \in (\ker A)^\perp \) with \( \|Aw_n\| < \frac{1}{n} \|w_n\| \). Let \( v_n = \frac{w_n}{\|w_n\|} \in (\ker A)^\perp \). Then
  \[
  \|Av_n\| < \frac{1}{n}.
  \]
  As \( K \) is compact, there is some subsequence \( v_{n(n)} \) such that \( Kv_{n(n)} \) converges to some \( v \in H \). \( Av_{n(n)} \to Av \) and \( Av_n \to 0 \), so \( v \in \ker A \). On the other hand, because \( v_n = Av_n + Kv_n \to v \) and \((\ker A)^\perp \) is closed, we have \( v \in (\ker A)^\perp \), so \( v \in \ker A \cap (\ker A)^\perp = \{0\} \). But as \( \|v_n\| \geq 1 \) for each \( n \), we have \( \|v\| = 1 \), a contradiction. Therefore, \( A \) is bounded below on \((\ker A)^\perp \).

If \( K \in K(H) \) and \( A = id_H - K \), then by the second of the two lemmas, we have that \( A \) is bounded below on \((\ker A)^\perp \): there is some \( \alpha > 0 \) such that \( \|Av\| > 0 \) for all \( v \in (\ker A)^\perp \). If
  \[
  w_n \in A(H) = A((\ker A) \oplus (\ker A)^\perp) = A((\ker A)^\perp)
  \]
  with \( w_n \to w \in H \), then there are \( v_n \in (\ker A)^\perp \) such that \( Av_n = w_n \). As \( w_n \)
  converges, for all \( \epsilon > 0 \) there is some \( N \) such that if \( n, m \geq N \) then \( \|w_n - w_m\| \leq \epsilon \). But
  \[
  \|A(v_n - v_m)\| \geq \alpha \|v_n - v_m\|,
  \]
so $v_n$ is a Cauchy sequence and hence converges, say to $v$. $v_n \in (\ker A) \dual$, which is closed, so $v \in (\ker A) \dual$. Then $w_n = Av_n \to Av \in A(H)$. Therefore, if $K \in K(H)$ and $A = \id_H - K$, then $A(H)$ is closed in $H$.

Let $K \in K(H)$ and $A = \id_H - K$, and suppose that $\ker A = \{0\}$. By the above paragraph, $A(H)$ is closed in $H$. $K^* \in K(H)$ and $A^* = \id_H - K^*$, so by the above paragraph we also get that $A^*(H)$ is closed in $H$. It is a fact that if $T \in B(H)$ then $\ker T^* = (T(H))^\dual$, so using this with $T = A^*$ we get

$$\ker A = (A^*(H))^\dual;$$

taking orthogonal complements and using the fact that the double orthogonal complement of a subspace is its closure and that $A^*(H)$ is closed, we obtain

$$(\ker A)^\dual = A^*(H).$$

Since $\ker A = \{0\}$, we have $A^*(H) = H$. Then we can apply the first of the two lemmas: as $A^* = \id_H - K^*$, $K^* \in K(H)$, and $A^*(H) = H$, we have $\ker A^* = \{0\}$. We now apply the second of the two lemmas: $A^*$ is bounded below on $(\ker A^*)^\dual = H$. Using the fact that if $T \in B(H)$ is bounded below and has dense image then $T^{-1} \in B(H)$, we get $(A^*)^{-1} \in B(H)$ ($A^*(H) = H$ so $A^*$ certainly has dense image). Taking adjoints commutes with taking inverses, so $A^{-1} \in B(H)$. This completes the proof of the Fredholm alternative.

## 6 Compact self-adjoint operators

It is a fact that if $T \in B(H)$ is self-adjoint then

$$\|T\| = \sup_{\|v\| \leq 1} |\langle Tv, v \rangle| = \sup_{\|v\| = 1} |\langle Tv, v \rangle|.$$

Let $T \in B(H)$ be compact and self-adjoint and $T \neq 0$. Since $T$ is self-adjoint, $\langle Tv, v \rangle \in \mathbb{R}$, so either $\|T\| = \sup_{\|v\| = 1} \langle Tv, v \rangle$ or $\|T\| = -\inf_{\|v\| = 1} \langle Tv, v \rangle$. Say the first is the case. Let $\|v_n\| = 1$ and $\langle Tv_n, v_n \rangle \to \|T\|$ as $n \to \infty$. Then, as $T = T^*$,

$$\langle Tv_n - \|T\| v_n, v_n - \|T\| v_n \rangle = \langle Tv_n, Tv_n \rangle - \langle Tv_n, \|T\| v_n \rangle - (\|T\| v_n, Tv_n) + (\|T\| v_n, \|T\| v_n)$$

$$= \|Tv_n\|^2 - 2\|T\| \langle Tv_n, v_n \rangle + \|T\|^2 \|v_n\|^2$$

$$\leq \|T\|^2 \|v_n\|^2 - 2\|T\| \langle Tv_n, v_n \rangle + \|T\|^2 \|v_n\|^2$$

$$= 2\|T\|^2 - 2\|T\| \langle Tv_n, v_n \rangle.$$

---

9The adjoint of a compact operator is itself a compact operator. This is true even for Banach spaces, and a proof of this is given by Paul Garrett in his note *Compact operators on Banach spaces: Fredholm-Riesz*. A bounded linear map $K : X \to Y$, where $X$ and $Y$ are Banach spaces, is said to be compact if for every bounded subset $S$ of $X$, the closure of $K(S)$ in $Y$ is compact.

10This is proved in Anthony W. Knapp, *Advanced Real Analysis*, p. 37, Proposition 2.2.
Thus \( \|Tv_n - T\| v_n \| \to 0 \) as \( n \to \infty \), that is, \( Tv_n - T\| v_n \to 0 \). On the other hand, as \( \|v_n\| = 1 \) for each \( n \), there is some subsequence \( v_{a(n)} \) such that \( Tv_{a(n)} \) converges, say to \( v \). Together with \( Tv_n - \|T\| v_n \to 0 \) this gives \( \|T\| v_{a(n)} \to v \) as \( n \to \infty \), from which we get \( \|v\| = \|Tv\| > 0 \). Thus \( v \neq 0 \). And

\[
(T - \|T\|)v = (T - \|T\|) \lim_{n \to \infty} v_{a(n)} = \lim_{n \to \infty} (T - \|T\|)v_{a(n)} = 0,
\]

which means that \( \|T\| \in \sigma_{\text{point}}(T) \). Likewise, in the case \( \|T\| = -\inf_{\|v\|=1} \langle Tv, v \rangle \) we get \( -\|T\| \in \sigma_{\text{point}}(T) \).

7 Multiplication operators

7.1 The Hilbert space \( L^2(X) \)

Let \((X, \Sigma, \mu)\) be a \( \sigma \)-finite measure space. \( L^2(X) \) is a Hilbert space\(^{11}\) with inner product

\[
\langle f, g \rangle = \int fg^* d\mu,
\]

where \( g^*(x) = g(x) \).

7.2 The \( C^* \)-algebra homomorphism \( \phi \mapsto M_\phi \)

A multiplication operator on \( L^2(X) \) is an operator \( M_\phi : L^2(X) \to L^2(X) \), \( \phi \in L^\infty(X) \), of the form

\[
(M_\phi f)(x) = \phi(x)f(x).
\]

As

\[
\|M_\phi f\|^2 = \int_X \phi(x)f(x)\overline{\phi(x)}f(x)d\mu(x) \leq \|\phi\|_\infty^2 \|f\|^2,
\]

where \( \|\phi\|_\infty \) is the essential supremum of \( \phi(x) \) for \( x \in X \), we have \( \|M_\phi\| \leq \|\phi\| \) and so \( M_\phi \in B(H) \). \( L^\infty(X) \) is a \( C^* \)-algebra, and so is \( B(L^2(X)) \). If \( X \) is \( \sigma \)-finite then I claim that

\[
\phi \mapsto M_\phi, \quad L^\infty(X) \to B(L^2(X))
\]

is an injective homomorphism of \( C^* \)-algebras. It is straightforward to show that this map is a homomorphism of \( C^* \)-algebras, and this does not use the

\(^{11}\)Whether \( L^2(X) \) is separable depends on the measure space \((X, \mu)\). Let \( \mathcal{A} \) be the set of all measurable subsets of \( X \) with finite measure, and let \( \rho(A, B) = \mu(A \cup B \setminus A \cap B) \), the measure of the symmetric difference of \( A \) and \( B \). One shows that \( \mathcal{A} \) is a pseudometric space with pseudometric \( \rho \). It is a fact that \( L^2(X) \) is separable if and only if \( \mathcal{A} \) is separable; cf. Paul Halmos, Measure Theory, p. 177, §42. For this to be the case, it suffices that \( X \) is \( \sigma \)-finite and that its \( \sigma \)-algebra is countably generated.
assumption that $X$ is $\sigma$-finite. For our benefit, we shall show that $\phi \mapsto M_\phi$ is injective. If $\phi \neq 0$, then $\|\phi\|_\infty > 0$, so for

$$E = \{ x \in X : |\phi(x)| \geq \frac{1}{2} \|\phi\|_\infty \}$$

we have $0 < \mu(E) \leq \infty$. Because $(X, \mu)$ is $\sigma$-finite, there is some subset $F$ of $E$ with $0 < \mu(F) < \infty$. As $f = \phi^* \cdot \chi_F \in L^2(X)$, we have

$$M_\phi f = \int_F \phi(x) \phi(x) d\mu(x)$$

$$\geq \int_F \frac{1}{4} \|\phi\|_\infty^2 d\mu(x)$$

$$= \frac{1}{4} \|\phi\|_\infty^2 \cdot \mu(F)$$

$$> 0,$$

so $M_\phi \neq 0$. Generally, an injective homomorphism of $C^*$-algebras is an isometry, so $\|M_\phi\| = \|\phi\|_\infty$.

As $M_{\phi^*} = M_\phi M_{\phi^*} = M_\phi \phi^*$ and $M_{\phi^*} = M_{\phi^*} M_\phi$, we have $M_\phi M_{\phi^*} = M_{\phi^*} M_\phi$, namely, a multiplication operator is a normal operator. Since residual spectrum of a normal operator is empty, the residual spectrum of a multiplication operator is empty.

### 7.3 Essential range

For $\phi \in L^\infty(X)$, we define the essential range of $\phi$ to be the set

$$\{ z \in \mathbb{C} : \text{if } \epsilon > 0 \text{ then } \mu(\{ x \in X : |f(x) - z| < \epsilon \}) > 0 \}.$$

Equivalently, the essential range of $\phi$ is the set of those $z \in \mathbb{C}$ such that for all $\epsilon > 0$,

$$\mu(\phi^{-1}(D_\epsilon(z))) > 0,$$

in words, those $z \in \mathbb{C}$ such that the inverse image of every $\epsilon$-disc about $z$ has positive measure. Equivalently, the essential range of $\phi$ is the intersection of all closed subsets $K$ of $\mathbb{C}$ such that for almost all $x \in X$, $\phi(x) \in K$. It is a fact that if $\phi \in L^\infty(X)$ then the essential range of $\phi$ is a compact subset of $\mathbb{C}$.

Let $\phi \in L^\infty(X)$. If $\lambda$ is not in the essential range of $\phi$, then there is some $\epsilon > 0$ such that $\mu(\phi^{-1}(D_\epsilon(\lambda))) = 0$, which means that for almost all $x \in X$ we have $|\phi(x) - \lambda| \geq \epsilon$. Define $\psi(x) = \frac{1}{\phi(x) - \lambda}$. For almost all $x \in X$,

$$|\psi(x)| = \frac{1}{|\phi(x) - \lambda|} \leq \frac{1}{\epsilon},$$

hence $\psi \in L^\infty(X)$. Then

$$M_\psi M_{\phi - \lambda} = M_{\phi - \lambda} M_\psi = M_{(\phi - \lambda) \cdot \psi} = M_1 = \text{id}_{L^2(X)},$$
so $M_{\phi-\lambda}$ is invertible. But $M_{\phi-\lambda} = M_{\phi} - \lambda$, so $M_{\phi} - \lambda$ is invertible and hence $\lambda \notin \sigma(M_{\phi})$.

If $\lambda$ is in the essential range of $\phi$, then for each $n$ we have $0 < \mu(\phi^{-1}(D_{1/n}(\lambda))) \leq \infty$; since $\Sigma$ is $\sigma$-finite, for each $n$ there is a subset $E_n$ of $\phi^{-1}(D_{1/n}(\lambda))$ with $0 < \mu(E_n) < \infty$, and so $\chi_{E_n} \in L^2(X)$. We have, since $|\phi(x) - \lambda| < \frac{1}{n}$ for $x \in E_n$,

\[
\| (M_{\phi} - \lambda) \chi_{E_n} \|^2 = \int_X |(\phi(x) - \lambda)\chi_{E_n}(x)|^2 d\mu(x) \\
= \int_{E_n} |\phi(x) - \lambda|^2 d\mu(x) \\
\leq \frac{1}{n^2} \int_{E_n} d\mu(x) \\
= \frac{1}{n^2} \int_X \chi_{E_n} d\mu(x) \\
= \frac{1}{n^2} \| \chi_{E_n} \|^2,
\]

so for each $n$,

\[
\| (M_{\phi} - \lambda) \chi_{E_n} \| \leq \frac{1}{n} \| \chi_{E_n} \|.
\]

It follows that $M_{\phi} - \lambda$ is not invertible, as it is not bounded below. Therefore $\lambda \in \sigma(M_{\phi})$. Therefore the essential range of $\phi \in L^\infty(X)$ is equal to the spectrum of $M_{\phi} \in B(L^2(X))$.

We say that $\phi \in L^\infty(X)$ is invertible if there is some $\psi \in L^\infty(X)$ such that $\phi(x)\psi(x) = 1$ for almost all $x \in X$. (It would not make sense to demand that $\phi(x)\psi(x) = 1$ for all $x \in X$.) For $\phi \in L^\infty(X)$ to be invertible, it is necessary and sufficient that there is some $\alpha > 0$ such that $|\phi(x)| \geq \alpha$ for almost all $x \in X$ (lest its inverse not have an essential supremum).

### 7.4 Isolated elements of the essential range

If $\lambda$ is not just an element of the essential range $\phi$ but is an isolated element of the essential range, then we can say more than just that $\lambda \in \sigma(M_{\phi})$. In this case, there is some $\epsilon > 0$ such that the intersection of $D_\epsilon(\lambda)$ and the essential range of $\phi$ is equal to the singleton $\{\lambda\}$. Let $E$ be a subset of $\phi^{-1}(D_\epsilon(\lambda))$ with $0 < \mu(E) < \infty$. For almost all $x \in X$, $\phi(x)$ is an element of the essential range of $\phi$, hence for almost all $x \in E$ we have $\phi(x) = \lambda$. Therefore, for almost all $x \in X$ we have

\[
(M_{\phi} \chi_E)(x) = \phi(x) \chi_E(x) = \lambda \chi_E(x).
\]

Hence $(M_{\phi} - \lambda) \chi_E = 0$, and as $\mu(E) > 0$ we have $\chi_E \neq 0$. Therefore, if $\lambda$ is an isolated element of the essential range of $\phi$ then $\lambda \in \sigma_{\text{point}}(M_{\phi})$.

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12If $\lambda$ is an isolated element of the essential range of $\phi$ then one finds that the inverse image of the singleton $\{\lambda\}$ has positive measure. I would be surprised if this were not the origin of the term point spectrum. Being isolated corresponds to being discrete.
8 Functional calculus

Let $T \in B(H)$ be self-adjoint. The spectrum $\sigma(T)$ is a compact subset of $\mathbb{R}$, and one checks that the set $C(\sigma(T))$ of continuous functions $\sigma(T) \to \mathbb{C}$ is a $C^*$-algebra, with norm $\|f\| = \sup_{\lambda \in \sigma(T)} |f(\lambda)|$.

Let $\mathbb{C}[x]$ be the set of polynomials with complex coefficients. For $T \in B(H)$ self-adjoint and $p(x) = \sum_{k=0}^{n} a_k x^k \in \mathbb{C}[x]$, we define $\sigma(T) 

\begin{align*}
p(T) \in B(H) 
\end{align*}

by

\begin{align*}
p(T) = \sum_{k=0}^{n} a_k T^k.
\end{align*}

$T \mapsto p(T)$ is a homomorphism of $C^*$-algebras, where

\begin{align*}
\left( \sum_{k=0}^{n} a_k x^k \right)^* = \sum_{k=0}^{n} \overline{a_k} x^k.
\end{align*}

It is a fact that

\begin{align*}
\sigma(p(T)) = p(\sigma(T)),
\end{align*}

where for $M \subseteq \mathbb{C}$ we define $p(M) = \{p(z) : z \in M\}$. It is also a fact that

\begin{align*}
\|p(T)\| = \|p\| = \sup_{\lambda \in \sigma(T)} |p(\lambda)|;
\end{align*}

this is proved using the result that the norm of a normal operator $T$ is equal to its spectral radius, which is given by the two following expressions that one proves are equal:

\begin{align*}
r(T) &= \lim_{n \to \infty} \|T^n\|^{1/n} = \max_{\lambda \in \sigma(T)} |\lambda|.
\end{align*}

The above is used to define $f(T)$ for any continuous function $f : \sigma(T) \to \mathbb{C}$. This map $C(\sigma(T)) \to B(H)$ is called the continuous functional calculus. It is an isometric homomorphism of $C^*$-algebras. The continuous functional calculus can be used to prove things about the spectrum of self-adjoint operators that do not obviously have to do with making sense of continuous functions applied to these operators.

Let $T \in B(H)$ be self-adjoint and let $\lambda \in \sigma(T)$ be an isolated point in $\sigma(T)$. I will show that that $\lambda \in \sigma_{\text{point}}(T)$. Since $\lambda$ is isolated in $\sigma(T)$, the function $f : \sigma(T) \to \mathbb{C}$ defined by

\begin{align*}
f(z) = \begin{cases} 1 & z = \lambda, \\
0 & z \neq \lambda,
\end{cases}
\end{align*}

is continuous at $\lambda$. Then $f(T) = \lambda$ is an eigenvalue of $T$.

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13 If we had not stipulated that $T$ be self-adjoint then we would have to define the conjugation of polynomials as conjugation of polynomial functions: for $p$ defined by $p(z) = \sum_{k=0}^{n} a_k z^k$, then $p^*$ is defined by $p^*(z) = \sum_{k=0}^{n} \overline{a_k} z^k$.

is continuous. Since \( f \) is continuous, \( f(T) \in B(H) \), and because \( f = f^* \), \( f(T) \) is self-adjoint. Let \( P = f(T) \). As \( \|P\| = \|f\| = 1 \), \( P \neq 0 \). Define \( g \in C(\sigma(T)) \) by \( g(x) = (x - \lambda)f(x) \). Then \( g(x) = 0 \) for all \( x \in \sigma(T) \), so \( g(T) = 0 \), i.e. \( (T - \lambda)P = 0 \).

Hence \( \text{im} \ P \subseteq \ker(T - \lambda) \). As \( P \neq 0 \), there is some \( v \in H \) with \( Pv \neq 0 \). Then \( (T - \lambda)Pv = 0 \), \( Pv \neq 0 \), so \( \lambda \in \sigma_{\text{point}}(T) \).

### 9 Spectral measures

It is a fact that if \( f \geq 0 \) then \( f(T) \geq 0 \), where, for a self-adjoint operator \( T \), \( T \geq 0 \) means \( \langle Tv, v \rangle \geq 0 \) for all \( v \in H \). For \( T \in B(H) \) self-adjoint and \( v \in H \), using the continuous functional calculus talked about in the previous section we define \( \phi : C(\sigma(T)) \to \mathbb{C} \) by

\[
\phi(f) = \langle v, f(T)v \rangle.
\]

\( \phi \) is a positive linear functional: if \( f \) is real-valued and \( f(x) \geq 0 \) for all \( x \in \sigma(T) \), then \( \phi(f) \geq 0 \). \( \sigma(T) \) is indeed a locally compact Hausdorff space and since \( \sigma(T) \) is compact the continuous functions of compact support on \( \sigma(T) \) are precisely the continuous functions on \( \sigma(T) \), so we satisfy the conditions of the Riesz-Markov theorem. Thus there exists a unique regular Borel measure \( \mu \) on the Borel \( \sigma \)-algebra of \( \sigma(T) \) such that, for all \( f \in C(\sigma(T)) \),

\[
\langle v, f(T)v \rangle = \phi(f) = \int_{\sigma(T)} f(x) d\mu(x).
\]

Lebesgue’s decomposition theorem states that

\[
\mu = \mu_{\text{ac}} + \mu_{\text{sing}} + \mu_{\text{pp}},
\]

where

- \( \mu_{\text{ac}} \) is absolutely continuous with respect to Lebesuge measure: if \( A \) is a measurable subset of \( \sigma(T) \) and its Lebesgue measure is 0, then \( \mu_{\text{ac}}(A) = 0 \).
- \( \mu_{\text{sing}} \) and Lebesgue measure are mutually singular,\(^{15}\) and if \( \lambda \in \sigma(T) \) then \( \mu_{\text{sing}}(\{\lambda\}) = 0 \).
- There is a countable subset \( J \) of \( \sigma(T) \) such that

\[
\mu_{\text{pp}} = \sum_{\lambda \in J} a_\lambda \delta_\lambda, \quad a_\lambda \in \mathbb{C}.
\]

\(^{15}\)There are disjoint measurable sets \( A \) and \( B \) with \( A \cup B = \sigma(T) \) such that \( \mu_{\text{sing}}(A) = 0 \) and the Lebesgue measure of \( B \) is 0.
We define $H_{ac}$ to be the set of those $v \in H$ such that $\mu$ is equal to the absolutely continuous part of its Lebesgue decomposition, i.e. the other two parts are 0, and we define $H_{sing}$ and $H_{pp}$ likewise. (Note that we first took $v \in H$ and then defined $\mu$ using $v$.) One proves that $H_{ac}$, $H_{sing}$ and $H_{pp}$ are closed subspaces of $H$ and that they are invariant under $T$, and defines the absolutely continuous spectrum of $T$ to be the spectrum of the restriction of $T$ to $H_{ac}$; the singular spectrum of $T$ to be the spectrum of the restriction of $T$ to $H_{sing}$; and the pure point spectrum of $T$ to be the spectrum of the restriction of $T$ to $H_{pp}$. It is a fact that\(^{16}\)

$$\sigma(T) = \sigma_{ac}(T) \cup \sigma_{sing}(T) \cup \sigma_{pp}(T),$$

but these three sets might not be disjoint.