Denjoy’s theorem on circle diffeomorphisms

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In this note I’m just presenting the proof of Denjoy’s theorem in Michael Brin and Garrett Stuck’s Introduction to dynamical systems, Cambridge University Press, 2002.

Let $S^1 = \mathbb{R}/\mathbb{Z}$. For $\alpha \in \mathbb{R}$, define $R_\alpha : S^1 \to S^1$ by $R_\alpha(x) = x + \alpha + \mathbb{Z}$.

We say that a homeomorphism $f : S^1 \to S^1$ is orientation preserving if it lifts to an increasing homeomorphism $F : \mathbb{R} \to \mathbb{R}$: $\pi \circ F = f \circ \pi$.

The rotation number of an orientation preserving homeomorphism $f$ is defined by

$$\rho(f) = \lim_{n \to \infty} \frac{F^n(x) - x}{n}.$$  

One proves that this is independent both of the lift $F$ of $f$ and the point $x \in \mathbb{R}$. Some facts about the rotation number: it is an invariant of topological conjugacy, and $\rho(f)$ is rational if and only if $f$ has a periodic point. A periodic point is $x \in S^1$ such that $f^n(x) = x$ for some $n \geq 1$.

There are some lemmas in Chapter 7 that I don’t want to write out. The important theorem that we’re going to use without proof is that if $f : S^1 \to S^1$ is an orientation preserving homeomorphism that is topologically transitive with irrational rotation number $\rho(f)$, then $f$ is topologically conjugate to $R_{\rho(f)}$. This reduces our problem to showing that a map is topologically transitive.

We will use the following lemma in the proof of Denjoy’s theorem.

**Lemma 1.** Let $f : S^1 \to S^1$ be a $C^1$ diffeomorphism and let $J$ be an interval in $S^1$. Let $g = \log f'$. If the interiors of $J, f(J), \ldots, f^{n-1}(J)$ are pairwise disjoint, then for any $x, y \in J$ and any $n \in \mathbb{Z}$ we have

$$\text{Var}(g) \geq |\log((f^n)'(x)) - \log((f^n)'(y))|.$$  

**Proof.** The intervals $[x, y], [f(x), f(y)], \ldots, [f^{n-1}(x), f^{n-1}(y)]$ are pairwise disjoint, so they are part of a partition of $[0, 1]$. The total variation of $g$ is defined as a supremum over all partitions, so in particular it will be $\geq$ the sum coming from any particular partition or a subset of that partition.
\[
\text{Var}(g) \geq \sum_{k=0}^{n-1} |g(f^k(y)) - g(f^k(x))| \\
\geq \left| \sum_{k=0}^{n-1} g(f^k(y)) - g(f^k(x)) \right| \\
= \left| \log \prod_{k=0}^{n-1} f'(f^k(y)) - \log \prod_{k=0}^{n-1} f'(f^k(x)) \right| \\
= \left| \log((f^n)'(x)) - \log((f^n)'(y)) \right|.
\]

Now we can prove Denjoy’s theorem.

**Theorem 2.** If \( f : S^1 \to S^1 \) is a \( C^1 \) diffeomorphism that is orientation preserving, that has irrational rotation number \( \rho(f) \), and whose derivative \( f' : S^1 \to \mathbb{R} \) has bounded variation, then \( f \) is topologically conjugate to \( R_{\rho(f)} \).

*Proof.* Suppose by contradiction that \( f \) is not topologically transitive. It’s a fact proved in Chapter 7 of Brin and Stuck that this implies that \( \omega(x) \) is perfect and nowhere dense, and is independent of the point \( x \). (Recall that \( \omega(x) = \bigcap_{n \geq 1} \bigcup_{i \geq n} f^i(x) \).) It follows that there is an interval \( I = (a, b) \) in its complement.

The intervals \( f^n(I), n \in \mathbb{Z} \), are pairwise disjoint, for otherwise \( f \) would have a periodic point. Let \( \mu \) be Haar measure on \( S^1 \). Then

\[
\sum_{n \in \mathbb{Z}} \mu(f^n(I)) \leq 1.
\]

Let \( x \in S^1 \). Suppose for the moment that there are infinitely \( n \geq 1 \) such that the intervals \( (x, f^{-n}(x)), (f(x), f^{1-n}(x)), \ldots, (f^n(x), x) \) are pairwise disjoint; we shall prove that this is true later. By applying the lemma we proved with \( y = f^{-n}(x) \) we get

\[
\text{Var}(g) \geq \left| \log \left( \frac{(f^n)'(x)}{(f^n)'(y)} \right) \right| = \left| \log((f^n)'(x)(f^{-n})'(x)) \right|.
\]

To see the equality in the above line it helps to write out what \( (f^{-n})'(x) \) is.
Then for infinitely many $n$ we have
\[
\mu(f^n(I)) + \mu(f^{-n}(I)) = \int_I (f^n)'(x)dx + \int_I (f^{-n})'(x)dx
\]
\[
= \int_I ((f^n)'(x) + (f^{-n})'(x))dx
\]
\[
\geq \int_I \sqrt{(f^n)'(x)(f^{-n})'(x)}dx
\]
\[
= \int_I \sqrt{\exp(\log((f^n)'(x)(f^{-n})'(x)))}dx
\]
\[
\geq \int_I \sqrt{\exp(-|\log((f^n)'(x)(f^{-n})'(x))|)}dx
\]
\[
\geq \int_I \sqrt{\exp(-\Var(g))}dx
\]
\[
= \exp\left(-\frac{1}{2}\Var(g)\right)\mu(I).
\]
Since $\mu(I) > 0$ this implies that $\sum_{n \in \mathbb{Z}} \mu(f^n(I)) = \infty$, a contradiction. Therefore $f$ is topologically transitive, and so it is topologically conjugate to the $R_{\rho(f)}$.

It is indeed necessary that $f'$ has bounded variation. Brin and Stuck give an example on p. 161 that they attribute to Denjoy: for any irrational number $\rho \in (0, 1)$, there is a nontransitive orientation preserving $C^1$ diffeomorphism of $S^1$ with rotation number $\rho$. The only condition of Denjoy’s theorem that isn’t satisfied here is that $f'$ have bounded variation.