Estimating a product of sines using Diophantine approximation

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For $\sigma > 0$ and $A > 0$, let

$$D(A, \sigma) = \left\{ \alpha \in [0, 1] : \text{if } p, q \in \mathbb{Z} \text{ and } q \neq 0 \text{ then } \left| \alpha - \frac{p}{q} \right| \geq A|q|^{-\sigma} \right\}.$$

Let $D_\sigma = \bigcup_{A>0} D(A, \sigma)$. We can check that α has bounded partial quotients if and only if $\alpha \in D_2$. Elements of D_2 are also called *badly approximable numbers*. $\mu(D_2) = 0$. Dodson and Kristensen [1, §§3–4] survey results on the measure and Hausdorff dimension of the sets D_σ. A result of Jarník [1, Theorem 4.3] shows that D_2 has Hausdorff dimension 1, and a result of Jarník and Besicovitch [1, Theorem 4.4] shows that if $\sigma > 2$ then $[0, 1] \setminus D_\sigma$ has Hausdorff dimension < 1. And while $\mu(D_2) = 0$, it is not difficult to show that if $\sigma > 2$ then $\mu(D_\sigma) = 1$.

Theorem 1. *If $\alpha \in D(A, \sigma)$ then*

$$\prod_{k=1}^{n} |\sin \pi k \alpha| \geq (2A)^n (n!)^{-\sigma+1}.$$

Proof. If $\alpha \in D(A, \sigma)$ then for all $q \neq 0$ we have

$$|e^{2\pi i q \alpha} - 1| \geq 4A|q|^{-\sigma+1}.$$

Therefore if $\alpha \in D(A, \sigma)$ then

$$\begin{aligned} \prod_{k=1}^{n} |\sin \pi k \alpha| &= \prod_{k=1}^{n} \frac{1}{2} \left| 1 - e^{2\pi i k \alpha} \right| \\ &\geq \prod_{k=1}^{n} \frac{1}{2} 4Ak^{-\sigma+1} \\ &= (2A)^n (n!)^{-\sigma+1}. \end{aligned}$$

□

The measure theoretic notion of a small set is a set with measure 0, and the topological notion of a small set is *meager set*, also called a set *of first category*, defined as follows. (The set theoretic notion of a small set is a set in bijection with a subset of the integers, namely, a finite or countable set.) A set $E \subset [0,1]$ is *nowhere dense* if for all a and b with $0 \leq a < b \leq 1$ there exist c and d with $0 \leq a \leq c < d \leq b \leq 1$ such that $E \cap (c,d) = \emptyset$. A meager set is a countable union of nowhere dense sets. We have

$$[0,1] \setminus D(A,\sigma) = \bigcup_{q=2}^{\infty} \bigcup_{p=-\infty}^{\infty} \left(\frac{p}{q} - \frac{A}{q^\sigma}, \frac{p}{q} + \frac{A}{q^\sigma} \right).$$

Since $\frac{p}{q} \in [0,1] \setminus D(A,\sigma)$, it follows that $[0,1] \setminus D(A,\sigma)$ is dense in $[0,1]$. But $[0,1] \setminus D(A,\sigma)$ is a union of open sets so it is an open set and the complement of an open dense set is nowhere dense. Hence, each set $D(A,\sigma)$ is nowhere dense, and so D_σ, which can be written as a countable union of the sets $D(A,\sigma)$, is a meager set. For $\sigma > 2$, this gives us an example of sets that are topologically small (they are meager) which have measure 1; cf. Oxtoby [4, Chapter 2].

Let

$$\mathscr{D} = \bigcup_{\sigma \geq 2} D_\sigma.$$

The elements of $\mathscr{D}$ are called *Diophantine numbers*. Since each D_σ is meager, it follows that $\mathscr{D}$ is meager.

A theorem of Liouville states that if α is an algebraic number of degree $n \geq 1$, then there is some A such that $\alpha \in D(A,n)$. Therefore, the irrational algebraic numbers are a subset of the Diophantine numbers. The complement of $\mathscr{D} \cup \mathbb{Q}$ in $\mathbb{R}$ is called the *Liouville numbers*, and since the irrational algebraic numbers are a subset of the Diophantine numbers, the Liouville numbers are a subset of the transcendental numbers.

Mahler proves that $\pi \in D(1,42)$. Feldman and Nesterenko show relations of Diophantine numbers to transcendence theory.

Diophantine numbers in dynamics: Ghys [2], Milnor [3]

References

[1] M. Maurice Dodson and Simon Kristensen, *Hausdorff dimension and Diophantine approximation*, Fractal Geometry and Applications: A Jubilee of Benoît Mandelbrot, Part 1 (Michel L. Lapidus and Machiel van Frankenhuijsen, eds.), Proceedings of Symposia in Pure Mathematics, vol. 72, American Mathematical Society, Providence, RI, 2004, pp. 305–347.

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