The Dirac delta distribution and Green’s functions

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1 $u_s(x) = |x|^s$

If $s \in \mathbb{C}$ and $\Re s \geq 2$, then $u_s(x) = |x|^s$ is in $C^2_{\text{loc}}(\mathbb{R}^n)$. \footnote{This is all an expansion and gloss on Paul Garrett’s note Meromorphic continuations of distributions, which is on his homepage.} \[ \Delta : C^2_{\text{loc}}(\mathbb{R}^n) \to C^0_{\text{loc}}(\mathbb{R}^n) \]
and, \[ \Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}, \]
and $|x|^2 = x_1^2 + \cdots + x_n^2$.

\[ (\Delta u_s)(x) = \Delta |x|^s \]
\[ = \sum_{i=1}^n \frac{\partial}{\partial x_i} \left( \frac{s}{2} \cdot 2x_i \cdot (|x|^2)^{\frac{s}{2} - 1} \right) \]
\[ = \sum_{i=1}^n \left( \frac{s}{2} \cdot 2 \cdot (|x|^2)^{\frac{s}{2} - 1} + \frac{s}{2} \left( \frac{s}{2} - 1 \right) \cdot (2x_i)^2 \cdot (|x|^2)^{\frac{s}{2} - 2} \right) \]
\[ = ns \cdot |x|^{s-2} + s(s-2)|x|^{s-2} \]
\[ = s(s+n-2) \cdot |x|^{s-2} \]
\[ = s(s+n-2) \cdot u_{s-2}. \]

We take $n > 2$ in the following.

Typically we talk about functions $\mathbb{C} \to \mathbb{C}$ that are holomorphic (or meromorphic if they are defined on a subset of $\mathbb{C}$). But we can also talk about functions $\mathbb{C} \to V$ that are holomorphic/meromorphic for certain types of topological vector spaces over $\mathbb{C}$. In particular, we can talk about holomorphic/meromorphic functions that take values in the tempered distributions on $\mathbb{R}^n$. If $\Re s > -n$, then $u_s$ is locally integrable (for any point in $\mathbb{R}^n$, there is a neighborhood of the point on which $u_s$ is $L^1$), and hence it is a tempered distribution for $\Re s > -n$. Thus for $\Re s > -n$, $\Delta u_s$ is a tempered distribution.

For $\Re s \geq 2$ we have

\[ u_{s-2} = \frac{\Delta u_s}{s(s+n-2)}, \]
and hence for $\Re s \geq 0$ we have

$$u_s = \frac{\Delta u_{s+2}}{(s+2)(s+n)}.$$ 

As $u_0$ is a constant, $\Delta u_0 = 0$, and so $s - 2$ is a removable singularity of the right-hand side. It follows that $u_s$ is meromorphic and that its only possible pole is at $s = -n$. One iterates this argument and obtains that $u_s$ is meromorphic on $\mathbb{C}$, with at most simple poles at $s = -n, -n - 2, -n - 4, \ldots$.

Let $\gamma = e^{-|x|^2}$ and let $f$ be a Schwartz function on $\mathbb{R}^n$. For $\Re s > -n - 1$, we have $u_s \cdot (f - f(0)\gamma) \in L^1(\mathbb{R}^n)$ (the term $f - f(0)\gamma$ is certainly integrable at infinity and will still be integrable at infinity after being multiplied by $|x|^s$, and while $|x|^s$ might not be integrable at 0, the term $f - f(0)\gamma$ goes to 0 like $|x|^2$). The tempered distribution $u_s$ maps the Schwartz function $f - f(0)\gamma$ to

$$u_s(f - f(0)\gamma) = \int_{\mathbb{R}^n} |x|^s \cdot (f(x) - f(0)\gamma(x)) dx.$$ 

In the above equation (for fixed $f$), the right-hand side is holomorphic for $\Re s > -n - 1$, thus so is the left. Hence the residue of the left side at $s = -n$ is 0:

$$\text{Res}_{s=-n} u_s(f - f(0)\gamma) = 0.$$ 

Thus

$$\text{Res}_{s=-n} u_s(f) = \text{Res}_{s=-n} u_s(f - f(0)\gamma) + \text{Res}_{s=-n} u_s(f(0)\gamma)$$

$$= \text{Res}_{s=-n} u_s(f(0)\gamma)$$

$$= f(0) \text{Res}_{s=-n} u_s(\gamma)$$

$$= \delta(f) \text{Res}_{s=-n} u_s(\gamma).$$

Using polar coordinates, with $\sigma(S^{n-1}) = \frac{2\pi^{n/2}}{\Gamma(n/2)}$,

$$\text{Res}_{s=-n} u_s(\gamma) = \text{Res}_{s=-n} \int_{\mathbb{R}^n} |x|^s e^{-|x|^2} dx$$

$$= \text{Res}_{s=-n} \int_0^\infty \int_{S^{n-1}} |rx|^s e^{-|rx|^2} r^{n-1} d\sigma(x') dr$$

$$= \sigma(S^{n-1}) \text{Res}_{s=-n} \int_0^\infty r^{s+n-1} e^{-r^2} dr$$

$$= \sigma(S^{n-1}) \frac{2}{\Gamma(s+n)} \text{Res}_{s=-n} \int_0^\infty e^{-t} t^{s+n-2} dt$$

$$= \sigma(S^{n-1}) \frac{2}{\Gamma(s+n)} \text{Res}_{s=-n} \Gamma(s+n) - s.$$ 

As $\Gamma(z + 1) = z \Gamma(z)$,

$$\text{Res}_{s=-n} u_s(\gamma) = \frac{\sigma(S^{n-1})}{2} \text{Res}_{s=-n} \frac{2}{s+n}$$

$$= \frac{\sigma(S^{n-1})}{2} \cdot 2$$

$$= \sigma(S^{n-1}).$$

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Therefore for any Schwartz function $f$, we have

$$\text{Res}_{s=-n} u_s(f) = \sigma(S^{n-1}) \cdot \delta(f),$$

hence

$$\text{Res}_{s=-n} u_s = \sigma(S^{n-1}) \cdot \delta.$$ 

We know that $u_s$ has poles at most at $s = -n, -n - 2, -n - 4, \ldots$, and we have just explicitly found its residue at $s = -n$.

This fact has an important consequence. As $u_s$ has a simple pole at $s = -n$, the value of $(s + n)u_s$ at $s = -n$ is $\text{Res}_{s=-n} u_s$. But

$$u_s = \frac{\Delta u_{s+2}}{(s + 2)(s + n)},$$

so

$$\Delta u_{-n+2} = (-n + 2) \cdot \sigma(S^{n-1}) \cdot \delta,$$

i.e.

$$\Delta \frac{1}{|x|^{n-2}} = (-n + 2) \cdot \sigma(S^{n-1}) \cdot \delta,$$

with $\sigma(S^{n-1}) = \frac{2\pi^{n/2}}{\Gamma(n/2)}$. Recall that we have assumed $n > 2$. In other words, we have just determined the Green's function of the Laplace operator on $\mathbb{R}^n$, $n > 2$. 
\( w_s(x) = |x|^s \cdot \log |x| \)

If \( \Re s > 2 \), then \( w_s(x) = |x|^s \cdot \log |x| \in C^2_{\log}(\mathbb{R}^2) \). Let \( u_s(x) = |x|^s \). We have

\[
(\Delta w_s)(x) = \frac{1}{2} \sum_{i=1}^{2} \frac{\partial^2}{\partial x_i^2} \left( (x_1^2 + x_2^2)^{s/2} \log(x_1^2 + x_2^2) \right)
= \frac{1}{2} \sum_{i=1}^{2} \frac{\partial}{\partial x_i} \left( \frac{s}{2}(x_1^2 + x_2^2)^{s/2 - 1} \cdot 2x_i \cdot \log(x_1^2 + x_2^2) \right)
+ (x_1^2 + x_2^2)^{s/2} \cdot \frac{2x_i}{x_1^2 + x_2^2} \cdot \frac{2x_i}{x_1^2 + x_2^2} 
= \sum_{i=1}^{2} \frac{\partial}{\partial x_i} \left( \frac{s}{2}(x_1^2 + x_2^2)^{s/2 - 1} \cdot x_i \cdot \log(x_1^2 + x_2^2) + (x_1^2 + x_2^2)^{s/2 - 1} \cdot x_i \right)
= \sum_{i=1}^{2} \frac{s}{2} \left( \frac{s}{2} - 1 \right) (x_1^2 + x_2^2)^{s/2 - 2} \cdot 2x_i^2 \cdot \log(x_1^2 + x_2^2)
+ \frac{s}{2}(x_1^2 + x_2^2)^{s/2 - 1} \cdot \log(x_1^2 + x_2^2)
+ \frac{s}{2}(x_1^2 + x_2^2)^{s/2 - 1} \cdot \frac{2x_i^2}{x_1^2 + x_2^2}
+ \left( \frac{s}{2} - 1 \right) (x_1^2 + x_2^2)^{s/2 - 2} \cdot 2x_i^2 + (x_1^2 + x_2^2)^{s/2 - 1}
= \sum_{i=1}^{2} s(s - 2)|x|^{s-4} \cdot x_i^2 \cdot \log |x| + s|x|^{s-2} \cdot \log |x| + s|x|^{s-4} \cdot x_i^2
+ (s - 2)|x|^{s-4} \cdot x_i^2 + |x|^{s-2}
= s(s - 2)|x|^{s-2} \log |x| + 2s|x|^{s-2} \log |x| + s|x|^{s-2} + (s - 2)|x|^{s-2} + 2|x|^{s-2}
= s^2w_{s-2}(x) + 2su_{s-2}(x).
\]

Hence
\[ \Delta w_s = s^2w_{s-2} + 2su_{s-2}, \]
and so
\[ (s + 2)^2w_s = -2(s + 2)u_s + \Delta w_{s+2}. \]

We calculate
\[
\int_{\mathbb{R}^2} |x|^s \log |x| e^{-|x|^2} dx = \int_0^\infty \int_{S^1} |r x'|^s \log |r x'| e^{-|r x'|^2} r d\sigma(x') dr 
= 2\pi \int_0^{\infty} r^{s+1} \cdot \log r \cdot e^{-r^2} dr 
= 2\pi \cdot \frac{1}{4} \Gamma \left( 1 + \frac{s}{2} \right) \psi \left( 1 + \frac{s}{2} \right),
\]
where \( \psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} \), namely the digamma function. Using \( \Gamma(z + 1) = z\Gamma(z) \) and
\[ \psi(z + 1) = \psi(z) + \frac{1}{2}, \text{ with } \gamma(x) = e^{-x^2}. \]

\[ \text{Res}_{s=-2}(s + 2)w_s(\gamma) = \frac{\pi}{2} \cdot \text{Res}_{s=-2}(s + 2) \frac{1}{1 + \frac{s}{2}} \left( \psi \left( 1 + \frac{s}{2} + 1 \right) - \frac{1}{1 + \frac{s}{2}} \right) \]

\[ = \pi \cdot \text{Res}_{s=-2} \left( -C - \frac{2}{s + 2} \right) \]

\[ = -2\pi, \]

where \( C \) is Euler’s constant; it is a fact that \( \psi(1) = -C. \)

Thus like in the previous section, if \( f \) is a Schwartz function then

\[ \text{Res}_{s=-2}(s + 2)w_s(f) = \delta(f) \text{Res}_{s=-2}(s + 2)w_s(\gamma) = -2\pi \cdot \delta(f). \]

Because

\[ (s + 2)^2w_s = -2(s + 2)u_s + \Delta w_{s+2}, \]

the value of \((s + 2)^2w_s\) at \( s = -2 \) is \( \Delta w_0 - 2 \cdot \text{Res}_{s=-2}u_s \). On the other hand, the value of \((s + 2)^2w_s\) at \( s = -2 \) is \( \text{Res}_{s=-2}(s + 2)w_s = -2\pi \cdot \delta, \) hence

\[ \Delta w_0 = -2\pi \cdot \delta + 2 \cdot \text{Res}_{s=-2}u_s. \]

We can calculate \( \text{Res}_{s=-2}u_s \) just like in the previous section. If \( f \) is a Schwartz function and \( \gamma = e^{-|x|^2} \), then

\[ \text{Res}_{s=-2}u_s(f) = \delta(f) \text{Res}_{s=-2}u_s(\gamma) \]

\[ = \delta(f) \text{Res}_{s=-2} \frac{1}{2} \Gamma \left( 1 + \frac{s}{2} \right) \]

\[ = 2\pi \cdot \delta(f). \]

Therefore

\[ \Delta w_0 = 2\pi \cdot \delta, \]

i.e.,

\[ \Delta \log |x| = 2\pi \cdot \delta. \]

Recall that here \( n = 2 \). In other words, we have just determined the Green’s function of the Laplace operator on \( \mathbb{R}^2 \).

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Let \( T = \mathbb{R}/\mathbb{Z} \). On \( \mathbb{R}^n \), tempered distributions integrate against a larger class of functions than do distributions, so it’s stronger to be a tempered distribution. But on \( T \), any Schwartz function has compact support, and moreover, any \( C^\infty \)

\footnote{Historical note: In the papers of Euler’s that I’ve seen where he mentions the Euler constant, the notation he uses is either \( C \) or \( O \), not once the modern \( \gamma \).}
function on $\mathbb{T}$ is a Schwartz function. Thus distributions on $\mathbb{T}$ integrate smooth functions on $\mathbb{T}$. For $\mathbb{R}s > 2$, define the following distribution on $\mathbb{T}$:

$$u_s = \sum_{0 < \frac{p}{q} \leq 1 \atop \gcd(p,q) = 1} \frac{1}{q^s} \cdot \delta_{p/q}.$$  

Why is this in fact a distribution? If $f \in C^\infty(\mathbb{T})$ then $f$ is certainly bounded (indeed, $u_s$ can take any continuous function on $\mathbb{T}$ as an argument, not just smooth functions). Let $|f(t)| \leq K$ for all $t \in \mathbb{T}$. Then,

$$|u_s(f)| \leq K \sum_{0 < \frac{p}{q} \leq 1 \atop \gcd(p,q) = 1} \frac{1}{q^{\mathbb{R}s}} < K \sum_{q = 1}^{\infty} \frac{q}{q^{\mathbb{R}s}}.$$  

Since $\mathbb{R}s > 2$, this series converges.

Doing some series manipulations we get (probably the hardest step to see is that summing over the products of $d$ and $q$ is the same as summing over $q$ and then over those $d$ that divide it)

$$\zeta(s) \cdot u_s = \sum_{d \geq 1} \frac{1}{d^s} \sum_{q \geq 1} \frac{1}{q^s} \sum_{0 < p \leq q \atop \gcd(p,q) = 1} \delta_{p/q}$$

$$= \sum_{d \geq 1} \sum_{q \geq 1} \frac{1}{(qd)^s} \sum_{0 < p \leq q \atop \gcd(pd,qd) = d} \delta_{p/q}$$

$$= \sum_{q = 1}^{\infty} \frac{1}{q^s} \sum_{d \mid q \atop d \geq 1 \atop \gcd(p,qd) = d} \delta_{p/q}$$

$$= \sum_{q = 1}^{\infty} \frac{1}{q^s} \sum_{0 < p \leq q} \delta_{p/q}$$

$$= v_s.$$  

(The last equality is a definition.) To summarize: $\zeta(s) \cdot u_s = v_s$.

Supposing we are interested in $u_s$, using the above formula we can instead investigate $v_s$, which for some purposes is more analytically tractable. We shall determine the Fourier series of $v_s$. For $\mathbb{R}s > 2$ and for $n \in \mathbb{Z}$ (recalling that $v_s$ is a distribution, i.e. it integrates functions)

$$\hat{v}_s(n) = v_s(e^{-2\pi inx})$$

$$= \sum_{q = 1}^{\infty} \frac{1}{q^s} \sum_{0 < p \leq q} \delta_{p/q}(e^{-2\pi inx})$$

$$= \sum_{q = 1}^{\infty} \frac{1}{q^s} \sum_{0 < p \leq q} e^{-2\pi inp/q}.$$
$p \mapsto e^{-2\pi i np/q}$, $\mathbb{Z}/q \to \mathbb{C}$, is a character, and, unless it is the trivial character, the sum over $\mathbb{Z}/q$ is equal to 0. So if $q \nmid n$ then the inner sum is 0, and if $q|n$ then the inner sum is equal to $q$. (If the language of characters of $\mathbb{Z}/q$ isn’t familiar, you can check this fact directly; to show the inner sum is 0, you show that the inner sum is equal to itself times something that is nonzero.) Thus

$$\hat{v}_s(n) = \sum_{q|n} \frac{1}{q^{s-1}}.$$ 

For $n = 0$, we get

$$\hat{v}_s(0) = \zeta(s-1).$$

Otherwise, the above can be written using a standard arithmetic function, the sum of powers of positive divisors. Let $\sigma_\alpha(n)$ denote the sum of the $\alpha$th powers of the positive divisors of $n$. Thus for $n \neq 0$ we have

$$\hat{v}_s(n) = \sigma_{1-s}(n).$$

Using $\zeta(s) \cdot u_s = v_s$ we get

$$\hat{u}_s(n) = \begin{cases} \frac{\zeta(s-1)}{\zeta(s)} & n = 0, \\ \frac{\sigma_{1-s}(n)}{\zeta(s)} & n \neq 0. \end{cases}$$

The expression on the right-hand side has poles at $s = 2$ and at the zeros of the Riemann zeta function. Otherwise, for a fixed $s$, the right-hand side has at most polynomial growth in $n$, and therefore it is the Fourier series of a distribution on $\mathbb{T}$ (see Katznelson, p. 48, Chapter 1, Exercise 7.5), and for $\Re s \leq 2$ we shall define $u_s$ to be this distribution. In summary: $u_s$ is originally defined as a distribution for $\Re s > 2$, and now we have defined it to be a distribution for $s \neq 2$ and $\zeta(s) \neq 0$. Thus $u_s$ is a meromorphic distribution valued functions on $\mathbb{C}$ with poles at $s = 2$ and at the zeros of the Riemann zeta function.

Since $\zeta(1) = \infty$, if $n \neq 0$ then $\hat{u}_1(n) = 0$. The only pole of the Riemann zeta function is at $s = 1$, hence $\zeta(0)/\zeta(1) = 0$. Thus $\hat{u}_1(0) = 0$, and it follows that as a distribution on $\mathbb{T}$,

$$u_1 = 0$$

(although the distribution $u_1$ is 0, this doesn’t mean that we can put $s = 1$ into the original definition of $u_s$ and assert that this is 0, as the original definition of $u_s$ was only for $\Re s > 2$, and we have analytically continued $u_s$ as a meromorphic distribution valued function on $\mathbb{C}$). Likewise, although $\zeta(0) = -\frac{1}{2}$, it is incorrect to conclude that $1 + 1 + 1 + \cdots = -\frac{1}{2}$, although for certain formal arguments this may be a correct interpretation.)