The Dunford-Pettis theorem

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April 18, 2015

1 Weak topology and weak-* topology

If \((E, \tau)\) is a topological vector space, we denote by \(E^*\) the set of continuous linear maps \(E \to \mathbb{C}\), the dual space of \(E\). The weak topology on \(E\), denoted \(\sigma(E, E^*)\), is the coarsest topology on \(E\) with which each function \(x \mapsto \lambda x, \lambda \in E^*\) is continuous \(E \to \mathbb{C}\). Thus, \(\sigma(E, E^*) \subset \tau\). If \((E, \tau)\) is a locally convex space, it follows by the Hahn-Banach separation theorem that \(E^*\) separates \(X\), and hence \(|\lambda|, \lambda \in E^*\), is a separating family of seminorms on \(E\) that induce the topology \(\sigma(E, E^*)\). Therefore, if \((E, \tau)\) is a locally convex space, then \((E, \sigma(E, E^*))\) is a locally convex space.

If \((E, \tau)\) is a topological vector space, the weak-* topology on \(E^*\), denoted \(\sigma(E^*, E)\), is the coarsest topology on \(E^*\) with which each function \(\lambda \mapsto \lambda x, x \in E\) is continuous \(E^* \to \mathbb{C}\). It is a fact that \(E^*\) with this topology is a locally convex space.

If \(E\) is a normed space, then \(\|\lambda\|_{op} = \sup_{\|x\| \leq 1} |\lambda x|\) is a norm on the dual space \(E^*\), and that \(E^*\) with this norm is a Banach space. The Banach-Alaoglu theorem states that \(\{\lambda \in E^*: \|\lambda\|_{op} \leq 1\}\) is a compact subset of \((E^*, \sigma(E^*, E))\).

If \((X, \Sigma, \mu)\) is a \(\sigma\)-finite measure space, for \(g \in L^\infty(\mu)\) define \(\phi_g \in (L^1(\mu))^*\) by \(\phi_g(f) = \int_X fgd\mu\). The map \(g \mapsto \phi_g\) is an isometric isomorphism \(L^\infty(\mu) \to (L^1(\mu))^*\).

Let \((X, \Sigma, \mu)\) be a probability space. If \(\Psi \in (L^\infty(\mu))^*\) and \(A \mapsto \Psi(\chi_A)\) is countably additive on \(\Sigma\), then there is some \(f \in L^1(\mu)\) such that

\[
\Psi(g) = \int_X gfd\mu, \quad g \in L^\infty(\mu),
\]

and \(\|\Psi\|_{op} = \|f\|_{1}\). Also, an additive function \(F\) on an algebra of sets \(\mathcal{A}\) is countably additive if and only if whenever \(A_n\) is a decreasing sequence of
elements of $\mathcal{A}$ with $\bigcap_{n=1}^{\infty} A_n = \emptyset$, we have \( \lim_{n \to \infty} F(A_n) = 0 \). Using that \( \mu \) is countably additive we get the following.

**Theorem 1.** Suppose that \((X, \Sigma, \mu)\) be a probability space and that \( \Psi \in (L^{\infty}(\mu))^* \), and suppose that for each \( \epsilon > 0 \) there is some \( \delta > 0 \) such that \( E \in \Sigma \) and \( \mu(E) \leq \delta \) imply that \( |\Psi(\chi_A)| \leq \epsilon \). Then there is some \( f \in L^1(\mu) \) such that

\[
\Psi(g) = \int_X g \, d\mu, \quad g \in L^{\infty}(\mu).
\]

## 2 Normed spaces

If \( E \) is a normed space, its dual space \( E^* \) with the operator norm is a Banach space, and \( E^{**} = (X^*)^* \) with the operator norm is a Banach space. Define \( i : E \to E^{**} \) by

\[
i(x)(\lambda) = \lambda(x), \quad x \in E, \quad \lambda \in E^*.
\]

It follows from the Hahn-Banach extension theorem that \( i : E \to E^{**} \) is an isometric linear map.

If \( E \) and \( F \) are normed spaces and \( T : E \to F \) is a bounded linear map, we define the **transpose** \( T^* : F^* \to E^* \) by \( T^* \lambda = \lambda \circ T \) for \( \lambda \in F^* \). If \( T \) is an isometric isomorphism, then \( T^* : F^* \to E^* \) is an isometric isomorphism, where \( E^* \) and \( F^* \) are each Banach spaces with the operator norm. In particular, we have said that when \((X, \Sigma, \mu)\) is a \( \sigma \)-finite measure space, then the map \( \phi : L^{\infty}(\mu) \to (L^1(\mu))^* \) defined for \( g \in L^{\infty}(\mu) \) by

\[
\phi_g(f) = \int_X f \, g \, d\mu, \quad f \in L^1(\mu),
\]

is an isometric isomorphism, and hence \( \phi^* : (L^1(\mu))^{**} \to (L^{\infty}(\mu))^* \) is an isometric isomorphism. Therefore, for \( E = L^1(\mu) \) we have that

\[
\phi^* \circ i : L^1(\mu) \to (L^{\infty}(\mu))^*
\]

is an isometric linear map. For \( f \in L^1(\mu) \) and \( g \in L^{\infty}(\mu) \),

\[
(\phi^* \circ i)(f)(g) = (\phi^*(i(f)))(g)
= (i(f) \circ \phi)(g)
= i(f)(\phi_g)
= \phi_g(f).
\]

The **Eberlein-Smulian theorem** states that if \( E \) is a normed space and \( A \) is a subset of \( E \), then \( A \) is weakly compact if and only if \( A \) is weakly sequentially compact.\(^3\)

\(^3\)V. I. Bogachev, *Measure Theory*, volume I, p. 9, Proposition 1.3.3.

3 Equi-integrability

Let \((X, \Sigma, \mu)\) be a probability space and let \(\mathcal{F}\) be a subset of \(L^1(\mu)\). We say that \(\mathcal{F}\) is **equi-integrable** if for every \(\epsilon > 0\) there is some \(\delta > 0\) such that for any \(A \in \Sigma\) with \(\mu(A) \leq \delta\) and for all \(f \in \mathcal{F}\),

\[
\int_A |f| d\mu \leq \epsilon.
\]

If \(\mathcal{F}\) is a bounded subset of \(L^1(\mu)\), it is a fact that \(\mathcal{F}\) being equi-integrable is equivalent to

\[
\lim_{C \to \infty} \sup_{f \in \mathcal{F}} \int_{\{|f| > C\}} |f| d\mu = 0. \tag{2}
\]

The following theorem gives a condition under which a sequence of integrable functions is bounded and equi-integrable.5

**Theorem 2.** Let \((X, \Sigma, \mu)\) be a probability space and let \(f_n\) be a sequence in \(L^1(\mu)\). If for each \(A \in \Sigma\) the sequence \(\int_A f_n d\mu\) has a finite limit, then \(\{f_n\}\) is bounded in \(L^1(\mu)\) and is equi-integrable.

4 The Dunford-Pettis theorem

A subset \(A\) of a topological space \(X\) is said to be **relatively compact** if \(A\) is contained in some compact subset of \(X\). When \(X\) is a Hausdorff space, this is equivalent to the closure of \(A\) being a compact subset of \(X\).

The following is the **Dunford-Pettis theorem**.6

**Theorem 3** (Dunford-Pettis theorem). Suppose that \((X, \Sigma, \mu)\) is a probability space and that \(\mathcal{F}\) is a bounded subset of \(L^1(\mu)\). \(\mathcal{F}\) is equi-integrable if and only if \(\mathcal{F}\) is a relatively compact subset of \(L^1(\mu)\) with the weak topology.

**Proof.** Suppose that \(\mathcal{F}\) is equi-integrable, and let \(T = \phi^* \circ i : L^1(\mu) \to (L^\infty(\mu))^*\) be the isometric linear map in (1), for which

\[
T(f)(g) = \int_X fg d\mu, \quad f \in L^1(\mu), \quad g \in L^\infty(\mu).
\]

Then \(T(\mathcal{F})\) is a bounded subset of \((L^\infty(\mu))^*\), so is contained in some closed ball \(B\) in \((L^\infty(\mu))^*\). By the Banach-Alaoglu theorem, \(B\) is weak-* compact, and therefore the weak-* closure \(\mathcal{H}\) of \(T(\mathcal{F})\) is weak-* compact. Let \(F \in \mathcal{H}\).

There is a net $F_\alpha = T(f_\alpha)$ in $T(\mathcal{F})$, $\alpha \in I$, such that for each $g \in L^\infty(\mu)$, $F_\alpha(g) \to F(g)$, i.e.,

$$\int_X f_\alpha g d\mu \to F(g), \quad g \in L^\infty(\mu).$$  \hspace{1cm} (3)

Let $\epsilon > 0$. Because $\mathcal{F}$ is equi-integrable, there is some $\delta > 0$ such that when $A \in \Sigma$ and $\mu(A) \leq \delta$,

$$\sup_{\alpha \in I} \int_A |f_\alpha| d\mu \leq \epsilon,$$

which gives

$$|F(\chi_A)| = \lim_{\alpha} \left| \int_X f_\alpha \chi_A d\mu \right| = \lim_{\alpha} \left| \int_A f_\alpha d\mu \right| \leq \sup_{\alpha \in I} \int_A |f_\alpha| d\mu \leq \epsilon.$$

By Theorem 1, this tells us that there is some $f \in L^1(\mu)$ for which

$$F(g) = \int_X gf d\mu, \quad g \in L^\infty(\mu),$$

and hence $F = T(f)$. This shows that $\mathcal{H} \subset T(L^1(\mu))$, and

$$\int_X f_\alpha g d\mu \to \int_X fg d\mu, \quad g \in L^\infty(\mu)$$

tells us that $f_\alpha \to f$ in $\sigma(L^1(\mu),(L^1(\mu))^*)$, in other words $T^{-1}(F_\alpha)$ converges weakly to $T(F)$. Thus $T^{-1} : \mathcal{H} \to L^1(\mu)$ is continuous, where $\mathcal{H}$ has the subspace topology $\tau_{\mathcal{H}}$ inherited from $(L^\infty(\mu))^*$ with the weak-* topology and $L^1(\mu)$ has the weak topology. $(\mathcal{H},\tau_{\mathcal{H}})$ is a compact topological space, so $T^{-1}(\mathcal{H})$ is a weakly compact subset of $L^1(\mu)$. But $\mathcal{F} \subset T^{-1}(\mathcal{H})$, which establishes that $\mathcal{F}$ is a relatively weakly compact subset of $L^1(\mu)$.

Suppose that $\mathcal{F}$ is a relatively compact subset of $L^1(\mu)$ with the weak topology and suppose by contradiction that $\mathcal{F}$ is not equi-integrable. Then by (2), there is some $\eta > 0$ such that for all $C_0$ there is some $C \geq C_0$ such that

$$\sup_{f \in \mathcal{F}} \int_{\{|f| > C\}} |f| d\mu > \eta,$$

whence for each $n$ there is some $f_n \in \mathcal{F}$ with

$$\int_{\{|f_n| > n\}} |f_n| d\mu \geq \eta.$$  \hspace{1cm} (4)

On the other hand, because $\mathcal{F}$ is relatively weakly compact, the Eberlein-Smulian theorem tells us that $\mathcal{F}$ is relatively weakly sequentially compact, and so there is a subsequence $f_{a(n)}$ of $f_n$ and some $f \in L^1(\mu)$ such that $f_{a(n)}$ converges weakly to $f$. For $A \in \Sigma$, as $\chi_A \in L^\infty(\mu)$ we have

$$\lim_{n \to \infty} \int_A f_{a(n)} d\mu = \int_A f d\mu,$$

and thus Theorem 2 tells us that the collection $\{f_{a(n)}\}$ is equi-integrable, contradicting (4). Therefore, $\mathcal{F}$ is equi-integrable.
Corollary 4. Suppose that \((X, \Sigma, \mu)\) is a probability space. If \(\{f_n\} \subset L^1(\mu)\) is bounded and equi-integrable, then there is a subsequence \(f_{a(n)}\) of \(f_n\) and some \(f \in L^1(\mu)\) such that
\[
\int_X f_{a(n)} g d\mu \to \int_X f g d\mu, \quad g \in L^\infty(\mu).
\]

Proof. The Dunford-Pettis theorem tells us that \(\{f_n\}\) is relatively weakly compact, so by the Eberlein-Smulian theorem, \(\{f_n\}\) is relatively weakly sequentially compact, which yields the claim. \(\square\)

5 Separable topological spaces

It is a fact that if \(E\) is a separable topological vector space and \(K\) is a compact subset of \((E^*, \sigma(E^*, E))\), then \(K\) with the subspace topology inherited from \((E^*, \sigma(E^*, E))\) is metrizable. Using this and the Banach-Alaoglu theorem, if \(E\) is a separable normed space it follows that \(\{\lambda \in E^*: \|\lambda\|_{op} \leq 1\}\) with the subspace topology inherited from \((E, \sigma(E^*, E))\) is compact and metrizable, and hence is sequentially compact.\(^7\) In particular, when \(E\) is a separable normed space, a bounded sequence in \(E^*\) has a weak-* convergent subsequence.

If \(X\) is a separable metrizable space and \(\mu\) is a \(\sigma\)-finite Borel measure on \(X\), then the Banach space \(L^p(\mu)\) is separable for each \(1 \leq p < \infty\).\(^8\)

Theorem 5. Suppose that \(X\) is a separable metrizable space and \(\mu\) is a \(\sigma\)-finite Borel measure on \(X\). If \(\{g_n\}\) is a bounded subset of \(L^\infty(\mu)\), then there is a subsequence \(g_{a(n)}\) of \(g_n\) and some \(g \in L^\infty(\mu)\) such that
\[
\int_X f g_{a(n)} d\mu \to \int_X f g d\mu, \quad f \in L^1(\mu).
\]

\(^7\)A second-countable \(T_1\) space is compact if and only if it is sequentially compact: Stephen Willard, General Topology, p. 125, 17G.

\(^8\)René L. Schilling, Measures, Integrals and Martingales, p. 270, Corollary 23.20.