# Nonholomorphic Eisenstein series, the Kronecker limit formula, and the hyperbolic Laplacian

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## 1 Nonholomorphic Eisenstein series

Let $\mathbb{H} = \{x + iy \in \mathbb{C} : y > 0\}$ For $\tau = x + iy \in \mathbb{H}$ and $s = \sigma + it, \sigma > 1$, we define the **nonholomorphic Eisenstein series**

$$G(\tau, s) = \frac{1}{2} \sum_{(0,0) \neq (m,n) \in \mathbb{Z}^2} \frac{y^s}{|m\tau + n|^{2s}}.$$

The function $(\tau, a, b) \mapsto a\tau + b$ is continuous $\mathbb{H} \times S^1 \to \mathbb{C}$, and for all $\tau \in \mathbb{H}$ and $(a, b) \in S^1$ we have $a\tau + b \neq 0$. It follows that if $K$ is a compact subset of $\mathbb{H}$ then there is some $C_K > 0$ such that $|a\tau + b| \geq C_K$ for all $\tau \in K$, $(a, b) \in S^1$. Then, for all $\tau \in K$ and for all $(0, 0) \neq (m, n) \in \mathbb{Z}^2$,

$$|m\tau + n|^2 = \left| \frac{m}{\sqrt{m^2 + n^2}} \tau + \frac{n}{\sqrt{m^2 + n^2}} \right|^2 (m^2 + n^2) \geq C_K (m^2 + n^2),$$

and hence

$$\left| \frac{y^s}{|m\tau + n|^{2s}} \right| = \frac{y^\sigma}{|m\tau + n|^{2\sigma}} \leq \frac{y^\sigma}{(C_K (m^2 + n^2))^\sigma}.$$

Because $\sigma > 1$,

$$\sum_{(0,0) \neq (m,n) \in \mathbb{Z}^2} \frac{1}{(m^2 + n^2)^\sigma} < \infty.$$

It follows that for any $s = \sigma + it$ with $\sigma > 1$, the function $\tau \mapsto G(\tau, s)$ is continuous $\mathbb{H} \to \mathbb{C}$.

It is sometimes useful to write $G$ in another way. For $\tau = x + iy \in \mathbb{H}$ and $\operatorname{Re} s > 1$, define

$$E(\tau, s) = \frac{1}{2} \sum_{(c,d) \in \mathbb{Z}^2, \gcd(c,d) = 1} \frac{y^s}{|c\tau + d|^{2s}}.$$

**Theorem 1.** *For all $\tau \in \mathbb{H}$ and* $\operatorname{Re} s > 1$,

$$G(\tau, s) = \zeta(2s)E(\tau, s).$$

*Proof.* First we remark that for $0 \neq a \in \mathbb{Z}$, $\gcd(a, 0) = |a|$. For $(0, 0) \neq (m, n) \in \mathbb{Z}^2$, with $\nu = \gcd(m, n)$,

$$\gcd\left(\frac{m}{\nu}, \frac{n}{\nu}\right) = 1.$$

Then

$$\begin{aligned} G(\tau, s) &= \frac{1}{2}\sum_{\nu \geq 1} \sum_{(m,n)\in\mathbb{Z}^2, \gcd(m,n)=\nu} \frac{y^s}{|m\tau + n|^{2s}} \\ &= \frac{1}{2}\sum_{\nu \geq 1} \sum_{(c,d)\in\mathbb{Z}^2, \gcd(c,d)=1} \frac{y^s}{|\nu c\tau + \nu d|^{2s}} \\ &= \frac{1}{2} \sum_{(c,d)\in\mathbb{Z}^2, \gcd(c,d)=1} \frac{y^s}{|c\tau + d|^{2s}} \sum_{\nu \geq 0} \nu^{-2s} \\ &= \zeta(2s)E(\tau, s). \end{aligned}$$

□

# 2 Modular functions

**Theorem 2.** *For* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, $\tau \in \mathbb{H}$, *and* $\operatorname{Re} s > 1$,

$$G\left(\frac{a\tau + b}{c\tau + d}, s\right) = G(\tau, s).$$

*Proof.*

$$\frac{a\tau + b}{c\tau + d} = \frac{a\tau + b}{c\tau + d} \cdot \frac{c\overline{\tau} + d}{c\overline{\tau} + d} = \frac{ac|\tau|^2 + ad\tau + bc\overline{\tau} + bd}{|c\tau + d|^2},$$

so, for $\tau = x + iy$ and $\frac{a\tau+b}{c\tau+d} = u + iv$, using that $ad - bc = 1$,

$$u = \frac{ac|\tau|^2 + bd + x(ad + bc)}{|c\tau + d|^2}, \qquad v = \frac{y}{|c\tau + d|^2};$$

we shall only use the expression for $v$. Also, for $(m, n) \in \mathbb{Z}^2$,

$$m\left(\frac{a\tau + b}{c\tau + d}\right) + n = \frac{(ma + nc)\tau + mb + nd}{c\tau + d}.$$

Then,

$$\begin{aligned} G\left(\frac{a\tau + b}{c\tau + d}, s\right) &= \frac{1}{2} \sum_{(0,0)\neq(m,n)\in\mathbb{Z}^2} \left(\frac{y}{|c\tau + d|^2}\right)^s \left|\frac{(ma + nc)\tau + mb + nd}{c\tau + d}\right|^{-2s} \\ &= \frac{1}{2} \sum_{(0,0)\neq(m,n)\in\mathbb{Z}^2} \frac{y^s}{|(ma + nb)\tau + mb + nd|^{2s}}. \end{aligned}$$

But $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ implies that

$$(m,n) \mapsto (ma+nc, mb+nd)$$

is a bijection $\mathbb{Z}^2 \setminus \{(0,0)\} \to \mathbb{Z}^2 \setminus \{(0,0)\}$, so

$$\sum_{(0,0)\neq(m,n)\in\mathbb{Z}^2} \frac{1}{|(ma+nb)\tau+mb+nd|^{2s}} = \sum_{(0,0)\neq(\mu,\nu)\in\mathbb{Z}^2} \frac{1}{|\mu\tau+\nu|^{2s}},$$

and thus we get

$$G\left(\frac{a\tau+b}{c\tau+d}, s\right) = G(\tau, s),$$

completing the proof. □

# 3 Fourier expansion

We now derive the Fourier series of $G(\cdot, s)$.[1] $K_{s-\frac{1}{2}}$ denotes the Bessel function.

**Theorem 3.** *If $\tau \in \mathbb{H}$ and $\operatorname{Re} s > 1$, then*

$$G(\tau,s) = \zeta(2s)y^s + \pi^{\frac{1}{2}} \frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)} \zeta(2s-1) y^{-s+1} + 2\frac{\pi^s}{\Gamma(s)} \sum_{n=1}^{\infty} n^{s-1} \sigma_{-2s+1}(n) F_{s-\frac{1}{2}}(2\pi n y)\cos(2\pi n x),$$

*where*

$$F_{s-\frac{1}{2}}(w) = \left(\frac{2w}{\pi}\right)^{1/2} K_{s-\frac{1}{2}}(w).$$

*Proof.* Define

$$S(z,s) = \sum_{n\in\mathbb{Z}} \frac{|y|^s}{|z+n|^{2s}}, \qquad z = x+iy, y \neq 0, \quad \operatorname{Re} s > 1.$$

We can write $G(\tau, s)$ using this as

$$G(\tau,s) = \frac{1}{2}\sum_{n\neq 0} \frac{y^s}{|n|^{2s}} + \frac{1}{2}\sum_{m\neq 0}\sum_{n\in\mathbb{Z}} \frac{y^s}{|m\tau+n|^{2s}} = y^s\zeta(2s) + \frac{1}{2}\sum_{m\neq 0} \frac{S(m\tau,s)}{|m|^s}.$$

The Poisson summation formula[2] states that if $f : \mathbb{R} \to \mathbb{C}$ is continuous and of locally bounded variation, then for all $x \in \mathbb{R}$,

$$\sum_{n\in\mathbb{Z}} f(x+n) = \sum_{k\in\mathbb{Z}} \widehat{f}(k) e^{2\pi i k x},$$

[1] Henri Cohen, *Number Theory, vol. II: Analytic and Modern Tools*, p. 211, Theorem 10.4.3.

[2] Henri Cohen, *Number Theory, vol. I: Tools and Diophantine Equations*, p. 46, Corollary 2.2.17.

where

$$\widehat{f}(\xi) = \int_{\mathbb{R}} e^{-2\pi i \xi t} f(t) dt, \qquad \xi \in \mathbb{R}.$$

Let $z = x + iy, y \neq 0$, let $\operatorname{Re} s > 1$, and define $f_y : \mathbb{R} \to \mathbb{C}$ by

$$f_y(t) = |t + iy|^{-2s} = ((t+iy)(t-iy))^{-s} = (t^2 + y^2)^{-s}, \qquad t \in \mathbb{R}.$$

Applying the Poisson summation formula we get

$$\sum_{n \in \mathbb{Z}} |x + n + iy|^{-2s} = \sum_{k \in \mathbb{Z}} \widehat{f_y}(k) e^{2\pi i k x},$$

i.e.,

$$S(z, s) = |y|^s \sum_{k \in \mathbb{Z}} \widehat{f_y}(k) e^{2\pi i k x}, \tag{1}$$

with

$$\widehat{f_y}(k) = \int_{\mathbb{R}} e^{-2\pi i k t} (t^2 + y^2)^{-s} dt.$$

As $y \neq 0$, doing the change of variable $t = yu$ we get

$$\begin{aligned} \widehat{f_y}(k) &= \int_{\mathbb{R}} e^{-2\pi i k y u} (y^2 u^2 + y^2)^{-s} |y| du \\ &= |y|^{-2s+1} \int_{\mathbb{R}} e^{-2\pi i k y u} (u^2 + 1)^{-s} du \\ &= 2|y|^{-2s+1} \int_0^\infty \cos(2\pi k y u)(u^2+1)^{-s} du; \end{aligned}$$

the final equality is because the function $u \mapsto (u^2+1)^{-s}$ is even.

We use the following identity:[3] for $a > 0$ and $\operatorname{Re} s > \frac{1}{2}$,

$$\int_0^\infty \cos(au)(u^2+1)^{-s} du = \pi^{1/2} \cdot \left(\frac{a}{2}\right)^{s - \frac{1}{2}} \cdot \frac{1}{\Gamma(s)} \cdot K_{s-\frac{1}{2}}(a).$$

For $k \in \mathbb{Z} \setminus \{0\}$, using this with $a = 2\pi |ky| > 0$ gives

$$\begin{aligned} \widehat{f_y}(k) &= 2|y|^{-2s+1} \cdot \pi^{1/2} \cdot (\pi |ky|)^{s-\frac{1}{2}} \cdot \frac{1}{\Gamma(s)} \cdot K_{s-\frac{1}{2}}(2\pi |ky|) \\ &= 2|y|^{-s+\frac{1}{2}} \pi^s |k|^{s-\frac{1}{2}} \cdot \frac{1}{\Gamma(s)} \cdot K_{s-\frac{1}{2}}(2\pi|ky|). \end{aligned}$$

Therefore (1) becomes

$$\begin{aligned} S(z,s) &= |y|^s \widehat{f_y}(0) + |y|^s \sum_{k \neq 0} 2|y|^{-s+\frac{1}{2}} \pi^s |k|^{s-\frac{1}{2}} \cdot \frac{1}{\Gamma(s)} \cdot K_{s-\frac{1}{2}}(2\pi|ky|) \cdot e^{2\pi i k x} \\ &= |y|^s \widehat{f_y}(0) + 2|y|^{\frac{1}{2}} \pi^s \cdot \frac{1}{\Gamma(s)} \sum_{k \neq 0} |k|^{s-\frac{1}{2}} \cdot K_{s-\frac{1}{2}}(2\pi|ky|) \cdot e^{2\pi i k x} \\ &= |y|^s \widehat{f_y}(0) + 4|y|^{\frac{1}{2}} \pi^s \cdot \frac{1}{\Gamma(s)} \sum_{k=1}^{\infty} k^{s-\frac{1}{2}} \cdot K_{s-\frac{1}{2}}(2\pi k|y|) \cdot \cos(2\pi k x). \end{aligned}$$

[3] Henri Cohen, *Number Theory, vol. II: Analytic and Modern Tools*, p. 117, Theorem 9.8.9.

We use the following identity for the beta function:[4] For $\operatorname{Re} b > \frac{1}{2}\operatorname{Re} a > 0$,

$$\int_0^\infty u^{a-1}(u^2+1)^{-b}du = \frac{1}{2}B\left(\frac{a}{2}, b-\frac{a}{2}\right) = \frac{\Gamma\left(\frac{a}{2}\right)\Gamma\left(b-\frac{a}{2}\right)}{2\Gamma(b)}.$$

Using this with $a=1$ and $b=s$, and since $\Gamma\left(\frac{1}{2}\right) = \pi^{\frac{1}{2}}$,

$$\widehat{f_y}(0) = 2|y|^{-2s+1}\int_0^\infty (u^2+1)^{-s}du = 2|y|^{-2s+1}\frac{\pi^{\frac{1}{2}}\Gamma\left(s-\frac{1}{2}\right)}{2\Gamma(s)}.$$

Therefore

$$\begin{aligned} S(z,s) &= \pi^{\frac{1}{2}} \cdot |y|^{-s+1} \cdot \frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)} \\ &+ 4|y|^{\frac{1}{2}}\pi^s \cdot \frac{1}{\Gamma(s)}\sum_{k=1}^\infty k^{s-\frac{1}{2}} \cdot K_{s-\frac{1}{2}}(2\pi k|y|) \cdot \cos(2\pi kx) \\ &= \pi^{\frac{1}{2}} \cdot |y|^{-s+1} \cdot \frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)} \\ &+ 2\frac{\pi^s}{\Gamma(s)}\sum_{k=1}^\infty k^{s-1}F_{s-\frac{1}{2}}(2\pi k|y|) \cdot \cos(2\pi kx). \end{aligned}$$

We now express $G(\tau,s)$ using this formula for $S(z,s)$. For $\tau \in \mathbb{H}$ and $\operatorname{Re} s > 1$, since $S(z,s) = S(-z,s)$,

$$\begin{aligned} G(\tau,s) &= y^s\zeta(2s) + \frac{1}{2}\sum_{m\neq 0}\frac{S(m\tau,s)}{|m|^s} \\ &= y^s\zeta(2s) + \sum_{m=1}^\infty \frac{S(m\tau,s)}{m^s} \\ &= y^s\zeta(2s) + \pi^{\frac{1}{2}}\frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)}\sum_{m=1}^\infty \frac{(my)^{-s+1}}{m^s} \\ &+ 2\frac{\pi^s}{\Gamma(s)}\sum_{m=1}^\infty \frac{1}{m^s}\sum_{k=1}^\infty k^{s-1}F_{s-\frac{1}{2}}(2\pi kmy) \cdot \cos(2\pi kmx) \\ &= y^s\zeta(2s) + \pi^{\frac{1}{2}}\frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)}y^{-s+1}\zeta(2s-1) \\ &+ 2\frac{\pi^s}{\Gamma(s)}\sum_{k,m\geq 1}\frac{k^{s-1}}{m^s}F_{s-\frac{1}{2}}(2\pi kmy) \cdot \cos(2\pi kmx). \end{aligned}$$

As

$$\sum_{km=N}\frac{k^{s-1}}{m^s} = \sum_{km=N}\frac{(km)^{s-1}}{m^{2s-1}} = N^{s-1}\sum_{km=N}m^{-2s+1} = N^{s-1}\sigma_{-2s+1}(N),$$

[4] Henri Cohen, *Number Theory, vol. II: Analytic and Modern Tools*, p. 93, Corollary 9.6.40.

this can be written as

$$G(\tau, s) = y^s \zeta(2s) + \pi^{\frac{1}{2}} \frac{\Gamma\left(s - \frac{1}{2}\right)}{\Gamma(s)} y^{-s+1} \zeta(2s-1)$$
$$+ 2\frac{\pi^s}{\Gamma(s)} \sum_{N=1}^{\infty} N^{s-1} \sigma_{-2s+1}(N) F_{s-\frac{1}{2}}(2\pi N y) \cos(2\pi N x),$$

completing the proof. □

We use the above Fourier expansion to establish that for all $t \in \mathbb{H}$, $G(\tau, s)$ has a meromorphic continuation to $\mathbb{C}$ and satisfies a certain functional equation.[5] The meromorphic continuation and functional equation of $G(\tau, s)$ can also be obtained without using its Fourier expansion.[6]

**Theorem 4.** *For any $\tau \in \mathbb{H}$, $G(\tau, s)$ has a meromorphic continuation to $\mathbb{C}$ whose only pole is at $s = 1$, which is a simple pole with residue $\frac{\pi}{2}$. The function*

$$\mathcal{G}(\tau, s) = \pi^{-s} \Gamma(s) G(\tau, s)$$

*satisfies the functional equation*

$$\mathcal{G}(\tau, 1-s) = \mathcal{G}(\tau, s).$$

*Proof.* For $\nu \in \mathbb{C}$ and for $w > 0$ we have[7]

$$K_\nu(w) = \int_0^\infty e^{-w \cosh t} \cosh(\nu t) dt$$

and[8]

$$K_\nu(w) \sim \left(\frac{2w}{\pi}\right)^{-\frac{1}{2}} e^{-w}, \qquad w \to +\infty.$$

Using the above identity, one checks that for $w > 0$, the function $s \mapsto K_{s-\frac{1}{2}}(w)$ is entire, and that for any $s \in \mathbb{C}$, the function $w \mapsto K_{s-\frac{1}{2}}(w)$ belongs to $C^\infty(\mathbb{R}_{>0})$. We have a fortiori that for any $s \in \mathbb{C}$,

$$K_{s-\frac{1}{2}}(w) = O(e^{-w}), \qquad w \to +\infty.$$

[5]Henri Cohen, *Number Theory, vol. II: Analytic and Modern Tools*, p. 212, Corollary 10.4.4.

[6]Paul Garrett, *The simplest Eisenstein series*, http://www.math.umn.edu/~garrett/m/mfms/notes_c/simplest_eis.pdf

[7]Henri Cohen, *Number Theory, vol. II: Analytic and Modern Tools*, p. 113, Proposition 9.8.6.

[8]Henri Cohen, *Number Theory, vol. II: Analytic and Modern Tools*, p. 115, Proposition 9.8.7.

Let $\Lambda(s) = \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)$. The functional equation for the Riemann zeta function states that $\Lambda$ has a meromorphic continuation to $\mathbb{C}$ whose only poles are at $s = 0$ and $s = 1$, which are simple poles, and satisfies, for all $s \neq 0, 1$,

$$\Lambda(1-s) = \Lambda(s).$$

Using the Fourier series for $G(\cdot, s)$, we have that for $\tau \in \mathbb{H}$ and $\operatorname{Re} s > 1$,

$$\begin{aligned}\mathcal{G}(\tau, s) &= \pi^{-s}\Gamma(s)G(\tau, s)\\ &= \Lambda(2s)y^s + \Lambda(2s-1)y^{-s+1}\\ &+ 2\sum_{n=1}^{\infty} n^{s-1}\sigma_{-2s+1}(n)F_{s-\frac{1}{2}}(2\pi ny)\cos(2\pi nx).\end{aligned}$$

The residue of $\Lambda(2s)$ at $s = 0$ is $-\frac{1}{2}$; the residue of $\Lambda(2s)$ at $s = \frac{1}{2}$ is $\frac{1}{2}$; the residue of $\Lambda(2s-1)$ at $s = \frac{1}{2}$ is $-\frac{1}{2}$; and the residue of $\Lambda(2s-1)$ at $s = 1$ is $\frac{1}{2}$. It follows that the residue of $\mathcal{G}(\tau, s)$ at $s = 0$ is $-\frac{1}{2}$; the residue of $\mathcal{G}(\tau, s)$ at $s = \frac{1}{2}$ is $\frac{1}{2}y^{1/2} - \frac{1}{2}y^{1/2} = 0$; the residue of $\mathcal{G}(\tau, s)$ at $s = 1$ is $\frac{1}{2}$; and these are no other poles of $\mathcal{G}(\tau, s)$. Because $\Gamma(s)$ has a simple pole at $s = 0$, $G(\tau, s) = \frac{\pi^s}{\Gamma(s)}\mathcal{G}(\tau, s)$ does not have a pole at $s = 0$. The residue of $G(\tau, s)$ at $s = 1$ is $\frac{\pi}{\Gamma(1)} \cdot \frac{1}{2} = \frac{\pi}{2}$, and this is the only pole of $G(\tau, s)$.

For $s \in \mathbb{C}$,

$$\begin{aligned}n^{(1-s)-1}\sigma_{-2(1-s)+1}(n) &= n^{-s}\sigma_{2s-1}(n)\\ &= \sum_{ef=n}(ef)^{-s}e^{2s-1}\\ &= \sum_{ef=n}e^{s-1}f^{-s}\\ &= \sum_{ef=n}(ef)^{s-1}f^{-2s+1}\\ &= n^{s-1}\sigma_{-2s+1}(n).\end{aligned}$$

Generally, $K_\nu = K_{-\nu}$, so $F_{s-\frac{1}{2}} = F_{(1-s)-\frac{1}{2}}$. Thus each term in the series in the above formula for $\mathcal{G}(\tau, s)$ is unchanged if $s$ is replaced with $1 - s$, and together with $\Lambda(1-w) = \Lambda(w)$ this yields

$$\begin{aligned}\mathcal{G}(\tau, 1-s) &= \Lambda(2-2s)y^{1-s} + \Lambda(2-2s-1)y^{-(1-s)+1} \\ &+ 2\sum_{n=1}^{\infty} n^{s-1}\sigma_{-2s+1}(n)F_{s-\frac{1}{2}}(2\pi ny)\cos(2\pi nx) \\ &= \Lambda(1-(2s-1))y^{1-s} + \Lambda(1-2s)y^{s} \\ &+ 2\sum_{n=1}^{\infty} n^{s-1}\sigma_{-2s+1}(n)F_{s-\frac{1}{2}}(2\pi ny)\cos(2\pi nx) \\ &= \Lambda(2s-1)y^{1-s} + \Lambda(2s)y^{s} \\ &+ 2\sum_{n=1}^{\infty} n^{s-1}\sigma_{-2s+1}(n)F_{s-\frac{1}{2}}(2\pi ny)\cos(2\pi nx) \\ &= \mathcal{G}(\tau, s).\end{aligned}$$

□

# 4 Kronecker limit formula

For $\tau \in \mathbb{H}$, Theorem 4 shows that $G(\tau, s)$ is meromorphic and that its only pole is at $s = 1$, which is a simple pole with residue $\frac{\pi}{2}$. It follows that $G(\tau, s)$ has the Laurent expansion about $s = 1$,

$$G(\tau, s) = \frac{\pi}{2} \cdot \frac{1}{s-1} + a_0(\tau) + a_1(\tau) \cdot (s-1) + \cdots,$$

and so defining $\frac{\pi}{2}C(\tau) = a_0(\tau)$,

$$G(\tau, s) = \frac{\pi}{2}\left(\frac{1}{s-1} + C(\tau) + O(|s-1|)\right), \qquad s \to 1.$$

We define the **Dedekind eta function** $\eta : \mathbb{H} \to \mathbb{C}$ by

$$\eta(\tau) = e^{\frac{\pi i \tau}{12}} \prod_{n=1}^{\infty}(1-q^n), \qquad \tau \in \mathbb{H},$$

where $q = e^{2\pi i\tau} = e^{-2\pi y}e^{2\pi ix}$, for $\tau = x + iy$. We now prove the **Kronecker limit formula**,[9] which expresses $C(\tau)$ in terms of the Dedekind eta function.

**Theorem 5.** *For $\tau = x + iy \in \mathbb{H}$,*

$$G(\tau, s) = \frac{\pi}{2}\left(\frac{1}{s-1} + C(\tau) + O(|s-1|)\right), \qquad s \to 1,$$

*with*

$$C(\tau) = 2\gamma - 2\log 2 - \log y - 4\log|\eta(\tau)|.$$

[9] Henri Cohen, *Number Theory, vol. II: Analytic and Modern Tools*, p. 213, Theorem 10.4.6.

*Proof.* Define

$$G(s) = \pi^{\frac{1}{2}} \frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)} \zeta(2s-1) y^{-s+1}.$$

Then

$$\log G(s) = \frac{1}{2}\log\pi + \log\zeta(2s-1) + (-s+1)\log y + \log\left(\frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)}\right). \qquad (2)$$

We use the asymptotic formula

$$\zeta(s) = \frac{1}{s-1} + \gamma + O(|s-1|), \qquad s \to 1,$$

and with

$$\log(1+w) = w + O(|w|^2), \qquad w \to 1,$$

this gives, as $s \to 1$,

$$\begin{aligned}\log\zeta(s) &= \log\left(\frac{1}{s-1} + \gamma + O(|s-1|)\right)\\ &= -\log(s-1) + \log(1+\gamma(s-1) + O(|s-1|^2))\\ &= -\log(s-1) + \gamma(s-1) + O(|s-1|^2),\end{aligned}$$

and hence

$$\log\zeta(2s-1) = -\log(2s-2) + \gamma(2s-2) + O(|s-1|^2), \qquad s \to 1.$$

The Taylor series for $\log\Gamma(z)$ about $z = \frac{1}{2}$ is

$$\log\Gamma(z) = \frac{1}{2}\log\pi - (2\log 2 + \gamma)\left(z - \frac{1}{2}\right) + \sum_{k=2}^{\infty}(-1)^k(2^k-1)\frac{\zeta(k)}{k}\left(z-\frac{1}{2}\right)^k,$$

for $|z - \frac{1}{2}| < \frac{1}{2}$, and the Taylor series of $\log\Gamma(1+z)$ about $z = 0$ is

$$\log\Gamma(1+z) = -\gamma z + \sum_{k=2}^{\infty}(-1)^k\frac{\zeta(k)}{k}z^k,$$

for $|z| < 1$. Using these we have

$$\log\Gamma\left(s-\frac{1}{2}\right) = \frac{1}{2}\log\pi - (2\log 2+\gamma)(s-1) + O(|s-1|^2), \qquad s \to 1$$

and

$$\log\Gamma(s) = -\gamma(s-1) + O(|s-1|^2), \qquad s \to 1.$$

Applying these approximations with (2) we get, as $s \to 1$,

$$\begin{aligned}\log G(s) = \frac{1}{2}\log\pi - \log(2s-2) + \gamma(2s-2) + O(|s-1|^2) + (-s+1)\log y \\ + \frac{1}{2}\log\pi - (2\log 2 + \gamma)(s-1) + O(|s-1|^2) \\ + \gamma(s-1) + O(|s-1|^2) \\ = \log\pi - \log 2 - \log(s-1) + 2\gamma(s-1) + (-s+1)\log y \\ - (2\log 2 + \gamma)(s-1) + \gamma(s-1) + O(|s-1|^2) \\ = \log\frac{\pi}{2} - \log(s-1) + (2\gamma - 2\log 2 - \log y)(s-1) + O(|s-1|^2).\end{aligned}$$

Taking the exponential and using

$$e^w = 1 + w + O(|w|^2), \qquad w \to 0,$$

as $s \to 1$ we have

$$\begin{aligned}G(s) &= \frac{\pi}{2} \cdot \frac{1}{s-1} \cdot \big(1 + (2\gamma - 2\log 2 - \log y)(s-1) + O(|s-1|^2)\big) \\ &= \frac{\pi}{2} \cdot \frac{1}{s-1} + \frac{\pi}{2} \cdot (2\gamma - 2\log 2 - \log y) + O(|s-1|).\end{aligned}$$

Using this and the fact that

$$\zeta(2s)y^s = \frac{\pi^2}{6}y + O(|s-1|), \qquad s \to 1,$$

Theorem 3 thus yields that as $s \to 1$,

$$\begin{aligned}G(\tau, s) = \frac{\pi^2}{6}y + \frac{\pi}{2} \cdot \frac{1}{s-1} + \frac{\pi}{2} \cdot (2\gamma - 2\log 2 - \log y) + O(|s-1|) \\ + 2\frac{\pi^s}{\Gamma(s)} \sum_{n=1}^{\infty} n^{s-1}\sigma_{-2s+1}(n) F_{s-\frac{1}{2}}(2\pi n y)\cos(2\pi n x).\end{aligned}$$

We have

$$\frac{\pi^s}{\Gamma(s)} = \pi + O(|s-1|), \qquad s \to 1.$$

As well,

$$n^{s-1} = 1 + O(|s-1|), \qquad s \to 1,$$

and

$$\sigma_{-2s+1}(n) = \sum_{d|n} d^{-2s+1} = \sum_{d|n}(d^{-1} + O(|s-1|)) = \sigma_{-1}(n) + O(|s-1|), \qquad s \to 1.$$

Finally, we use the fact that that for all $t > 0$,[10]

$$K_{\frac{1}{2}}(t) = \sqrt{\frac{\pi}{2t}}e^{-t},$$

[10] Henri Cohen, *Number Theory, vol. II: Analytic and Modern Tools*, p. 112, Theorem 9.8.5.

giving

$$F_{\frac{1}{2}}(t) = \left(\frac{2t}{\pi}\right)^{\frac{1}{2}} K_{\frac{1}{2}}(t) = e^{-t},$$

and hence

$$F_{s-\frac{1}{2}}(2\pi ny) = e^{-2\pi ny} + O(|s-1|), \qquad s \to 1.$$

Therefore, as $s \to 1$,

$$\begin{aligned} G(\tau, s) = {} & \frac{\pi^2}{6} y + \frac{\pi}{2} \cdot \frac{1}{s-1} + \frac{\pi}{2} \cdot (2\gamma - 2\log 2 - \log y) \\ & + 2\pi \sum_{n=1}^{\infty} 1 \cdot \sigma_{-1}(n) e^{-2\pi ny} \cos(2\pi nx) + O(|s-1|). \end{aligned}$$

This implies that the constant term in the Laurent expansion of $G(\tau, s)$ about $s = 1$ is

$$a_0(\tau) = \frac{\pi^2}{6} y + \frac{\pi}{2} \cdot (2\gamma - 2\log 2 - \log y) + 2\pi \sum_{n=1}^{\infty} \sigma_{-1}(n) e^{-2\pi ny} \cos(2\pi nx).$$

But, with $q = e^{2\pi i \tau}$,

$$\begin{aligned} \mathrm{Re}\left(\sum_{n=1}^{\infty} \sigma_{-1}(n) q^n\right) &= \sum_{n=1}^{\infty} \sigma_{-1}(n) \mathrm{Re}\left(e^{2\pi i n\tau}\right) \\ &= \sum_{n=1}^{\infty} \sigma_{-1}(n) \mathrm{Re}\left(e^{-2\pi ny} e^{2\pi i nx}\right) \\ &= \sum_{n=1}^{\infty} \sigma_{-1}(n) e^{-2\pi ny} \cos(2\pi nx), \end{aligned}$$

so

$$a_0(\tau) = \frac{\pi^2}{6} y + \frac{\pi}{2} \cdot (2\gamma - 2\log 2 - \log y) + 2\pi S(\tau),$$

where

$$S(\tau) = \mathrm{Re}\left(\sum_{n=1}^{\infty} \sigma_{-1}(n) q^n\right).$$

Using the power series for $\log(1+z)$ about $z = 0$,

$$\begin{aligned} \log \prod_{n=1}^{\infty} (1 - q^n) &= \sum_{n=1}^{\infty} \log(1 - q^n) \\ &= -\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{q^{nm}}{m} \\ &= -\sum_{N=1}^{\infty} \sum_{d|N} \frac{q^N}{d} \\ &= -\sum_{N=1}^{\infty} \sigma_{-1}(N) q^N, \end{aligned}$$

so

$$S(\tau) = -\mathrm{Re}\left(\log \prod_{n=1}^{\infty}(1-q^n)\right).$$

Then, because

$$\mathrm{Re}\, \log z = \log |z|$$

and because

$$|\eta(\tau)| = \left| e^{\frac{\pi i \tau}{12}} \prod_{n=1}^{\infty}(1-q^n)\right| = e^{-\frac{\pi y}{12}} \prod_{n=1}^{\infty} |1-q^n|,$$

this becomes

$$S(\tau) = -\log \prod_{n=1}^{\infty} |1-q^n| = -\frac{\pi y}{12} - \log |\eta(\tau)|.$$

Thus

$$\begin{aligned} a_0(\tau) &= \frac{\pi^2}{6} y + \frac{\pi}{2} \cdot (2\gamma - 2\log 2 - \log y) - \frac{\pi^2}{6} y - 2\pi \log |\eta(\tau)| \\ &= \frac{\pi}{2} \cdot (2\gamma - 2\log 2 - \log y) - 2\pi \log |\eta(\tau)|, \end{aligned}$$

so

$$C(\tau) = 2\gamma - 2\log 2 - \log y - 4 \log |\eta(\tau)|,$$

completing the proof. □

# 5 Hyperbolic Laplacian

For $f \in C^2(\mathbb{H})$, we define $\Delta_{\mathbb{H}} f : \mathbb{H} \to \mathbb{C}$ by

$$(\Delta_{\mathbb{H}} f)(\tau) = -y^2(\partial_x^2 f + \partial_y^2 f)(\tau), \qquad \tau = x + iy \in \mathbb{H}.$$

For more on $\Delta_{\mathbb{H}}$ see the below references.[11]

Let $(0,0) \neq (m,n) \in \mathbb{Z}^2$ and $\mathrm{Re}\, s > 1$, and define $f : \mathbb{H} \to \mathbb{C}$ by

$$f(x,y) = y^s |mx + n + imy|^{-2s} = y^s (mx + n + imy)^{-s} (mx + n - imy)^{-s}.$$

Write

$$g(x,y) = (mx + n + imy)^{-s}(mx + n - imy)^{-s}.$$

[11] Daniel Bump, *Spectral Theory and the Trace Formula*, `http://sporadic.stanford.edu/bump/match/trace.pdf`; Fredrik Strömberg, *Spectral theory and Maass waveforms for modular groups– from a computational point of view*, `http://www.cams.aub.edu.lb/events/confs/modular2012/files/lecture_notes_spectral_theory.pdf`; cf. Anton Deitmar, *Automorphic Forms*, p. 54, Lemma 2.7.3.

We calculate

$$\begin{aligned}(\partial_x g)(x,y) &= -s(mx+n+imy)^{-s-1}m(mx+n-imy)^{-s} \\ &\quad - s(mx+n+imy)^{-s}(mx+n-imy)^{-s-1}m,\end{aligned}$$

and

$$\begin{aligned}(\partial_x^2 g)(x,y) &= s(s+1)(mx+n+imy)^{-s-2}m^2(mx+n-imy)^{-s} \\ &\quad + s^2(mx+n+imy)^{-s-1}(mx+n-imy)^{-s-1}m^2 \\ &\quad + s^2(mx+n+imy)^{-s-1}m^2(mx+n-imy)^{-s-1} \\ &\quad + s(s+1)(mx+n+imy)^{-s}(mx+n-imy)^{-s-2}m^2 \\ &= s(s+1)m^2(mx+n+imy)^{-2}g(x,y) \\ &\quad + 2s^2m^2(mx+n+imy)^{-1}(mx+n-imy)^{-1}g(x,y) \\ &\quad + s(s+1)m^2(mx+n-imy)^{-2}g(x,y),\end{aligned}$$

from which we have

$$\begin{aligned}(\partial_x^2 f)(x,y) &= s(s+1)m^2(mx+n+imy)^{-2}f(x,y) \\ &\quad + 2s^2m^2|m\tau+n|^{-2}f(x,y) \\ &\quad + s(s+1)m^2(mx+n-imy)^{-2}f(x,y).\end{aligned}$$

We also calculate

$$\begin{aligned}(\partial_y g)(x,y) &= -s(mx+n+imy)^{-s-1}im(mx+n-imy)^{-s} \\ &\quad - s(mx+n+imy)^{-s}(mx+n-imy)^{-s-1}(-im),\end{aligned}$$

$$\begin{aligned}(\partial_y^2 g)(x,y) &= s(s+1)(mx+n+imy)^{-s-2}(-m^2)(mx+n-imy)^{-s} \\ &\quad + s^2(mx+n+imy)^{-s-1}(mx+n-imy)^{-s-1}m^2 \\ &\quad + s^2(mx+n+imy)^{-s-1}(im)(mx+n-imy)^{-s-1}(-im) \\ &\quad + s(s+1)(mx+n+imy)^{-s}(mx+n-imy)^{-s-2}(-im)^2 \\ &= -s(s+1)m^2(mx+n+imy)^{-2}g(x,y) \\ &\quad + 2s^2m^2(mx+n+imy)^{-1}(mx+n-imy)^{-1}g(x,y) \\ &\quad - s(s+1)m^2(mx+n-imy)^{-2}g(x,y).\end{aligned}$$

Now,

$$(\partial_y f)(x,y) = sy^{s-1}g(x,y) + y^s(\partial_y g)(x,y)$$

and

$$\begin{aligned}(\partial_y^2 f)(x,y) &= s(s-1)y^{s-2}g(x,y) + 2sy^{s-1}(\partial_y g)(x,y) \\ &\quad + y^s(\partial_y^2 g)(x,y),\end{aligned}$$

from which we have

$$\begin{aligned}
(\partial_y^2 f)(x,y) &= s(s-1)y^{s-2}g(x,y) \\
&\quad - 2s^2 imy^{s-1}(mx+n+imy)^{-1}g(x,y) \\
&\quad + 2s^2 imy^{s-1}(mx+n-imy)^{-1}g(x,y) \\
&\quad - s(s+1)m^2y^s(mx+n+imy)^{-2}g(x,y) \\
&\quad + 2s^2m^2y^s(mx+n+imy)^{-1}(mx+n-imy)^{-1}g(x,y) \\
&\quad - s(s+1)m^2y^s(mx+n-imy)^{-2}g(x,y) \\
&= s(s-1)y^{-2}f(x,y) \\
&\quad - 2s^2 imy^{-1}(mx+n+imy)^{-1}f(x,y) \\
&\quad + 2s^2 imy^{-1}(mx+n-imy)^{-1}f(x,y) \\
&\quad - s(s+1)m^2(mx+n+imy)^{-2}f(x,y) \\
&\quad + 2s^2m^2|m\tau+n|^{-2}f(x,y) \\
&\quad - s(s+1)m^2(mx+n-imy)^{-2}f(x,y).
\end{aligned}$$

Combining the above expressions we get

$$\begin{aligned}
(\partial_x^2 f + \partial_y^2)(x,y) &= m^2 f(x,y)\cdot\left(2s^2|m\tau+n|^{-2} + 2s^2|m\tau+n|^{-2}\right) \\
&\quad + s(s-1)y^{-2}f(x,y) \\
&\quad - 2s^2 imy^{-1}(mx+n+imy)^{-1}f(x,y) \\
&\quad + 2s^2 imy^{-1}(mx+n-imy)^{-1}f(x,y) \\
&= m^2 f(x,y)\cdot\left(2s^2|m\tau+n|^{-2} + 2s^2|m\tau+n|^{-2}\right) \\
&\quad + s(s-1)y^{-2}f(x,y) - 4s^2m^2|m\tau+n|^{-2}f(x,y) \\
&= s(s-1)y^{-2}f(x,y).
\end{aligned}$$

Thus

$$(\Delta_{\mathbb{H}} f)(x,y) = s(s-1)f(x,y),$$

i.e.,

$$\Delta_{\mathbb{H}} f = s(s-1)f.$$

Thus we immediately get that for $\operatorname{Re} s > 1$,

$$\Delta_{\mathbb{H}} G(\cdot,s) = s(s-1)G(\cdot,s).$$

Because the coefficients of the differential operator $L = \Delta_{\mathbb{H}} - s(s-1)$ are real analytic, a function $f : \mathbb{H} \to \mathbb{C}$ satisfying $Lf = 0$ is real analytic.[12] Therefore, for $\operatorname{Re} s > 1$, $G(\cdot,s)$ is real analytic.

[12] Lipman Bers and Martin Schechter, *Elliptic Equations*, in Lipman Bers, Fritz John, and Martin Schechter, eds., *Partial Diferential Equations*, p. 207, Chapter 4, Appendix.