Self-adjoint linear operators on a finite dimensional complex vector space

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Let V be a finite dimensional vector space over $\mathbb{C}$ with an inner product $(\cdot,\cdot) : V \times V \to \mathbb{C}$. For $x \in V$, $|x|^2 = (x, x)$.

Let $A : V \times V \to \mathbb{C}$ be linear in its first argument and conjugate linear in its second argument. Then we check that for all $x, y \in V$,

$$\begin{aligned} A(x,y) &= \frac{1}{4}\Big(A(x+y,x+y) + iA(x+iy,x+iy) \\ &\quad -A(x-y,x-y) - iA(x-iy,x-iy)\Big). \end{aligned}$$

This is called the *polarization identity*, or the parallelogram law. A useful instance is for $A(x, y) = (x, y)$. Then,

$$(x,y) = \frac{1}{4}\Big(|x+y|^2 + i|x+iy|^2 - |x-y|^2 - i|x-iy|^2\Big).$$

This can be useful for proving a statement about an inner product that one has only verified for a norm.

If $T : V \to V$ is linear, one checks that there is a unique linear $T^* : V \to V$ such that if $x, y \in V$ then

$$(Tx, y) = (x, T^*y).$$

T^* is called the *adjoint* of T. If $T = T^*$ then we say that T is *self-adjoint*.

An operator being self-adjoint is similar to a complex number being real. Let $T : V \to V$ be linear and define $T_1 = \frac{T+T^*}{2}$ and $T_2 = \frac{T-T^*}{2i}$. Then T_1, T_2 are self-adjoint. This resembles writing a complex number as a sum of a real number and i times a real number.

We say that a self-adjoint operator $T : V \to V$ is *positive* if for all $x \in V$ we have $(Tx, x) \geq 0$. Like for complex numbers, for any linear $T : V \to V$, TT^* is positive (in particular it is self-adjoint).

We say that a linear $U : V \to V$ is *unitary* if $UU^* = U^*U = \mathrm{id}_V$. If $z \in \mathbb{C}$ and $z\overline{z} = 1$ then $|z| = 1$. An operator being unitary is similar to a complex number have absolute value 1, in other words a complex number being on the unit circle. For linear $T : V \to V$, the exponential $\exp(T) : V \to V$ is defined by

$$\exp(T)x = \sum_{k=0}^{\infty} \frac{T^k x}{k!}.$$

If T is self-adjoint then $\exp(iT)$ is unitary:

$$(\exp(iT))^* = \left(\sum_{k=0}^{\infty} \frac{(iT)^k}{k!} \right)^* = \sum_{k=0}^{\infty} \frac{(-i)^k T^k}{k!} = \sum_{k=0}^{\infty} \frac{(-iT)^k}{k!} = \exp(-iT),$$

hence

$$\exp(iT)^* = (\exp(iT))^{-1},$$

which shows that $\exp(iT)$ is unitary.

The eigenvalues of a self-adjoint operator are all real, and the eigenvalues of a unitary operator all have absolute value 1.

Fact: If H is positive then its eigenvalues are nonnegative, and if H is self-adjoint and has nonnegative eigenvalues then it is positive. Say V has dimension n. On the one hand, if H is positive, suppose that $Hv = \lambda v$. Then $(Hv, v) = (\lambda v, v) = \lambda(v, v)$. Since H is positive this is nonnegative, and (v, v) is nonnegative so it follows that λ is nonnegative too. On the other hand, let H be self-adjoint and let all the eigenvalues of H be nonnegative. Since H is self-adjoint it has an orthonormal eigenbasis with real eigenvalues (this is the spectral theorem), $He_j = \lambda_j e_j$ for $1 \leq j \leq n$. Let $x \in V$. We can write x as $x = a_1 e_1 + \cdots + a_n e_n$. We have, using that $e_1, \ldots, e_n$ is an orthonormal eigenbasis for H¡

$$\begin{aligned}
(Hx, x) &= (a_1 He_1 + \cdots + a_n He_n, a_1 e_1 + \cdots + a_n e_n) \\
&= (a_1 He_1, a_1 e_1) + \cdots + (a_n He_n, a_n e_n) \\
&= |a_1|^2 (He_1, e_1) + \cdots + |a_n|^2 (He_n, e_n) \\
&= \lambda_1 |a_1|^2 + \cdots + \lambda_n |a_n|^2 \\
&= \geq 0.
\end{aligned}$$

Thus H is positive.

Fact: If T is positive then there is a positive H such that $H^2 = T$. Let $Te_j = \lambda_j e_j$, $1 \leq j \leq n$. T has an eigenbasis with real eigenvalues because T is self-adjoint, and by the above fact the eigenvalues are nonnegative since T is positive. Define H by $He_j = \sqrt{\lambda_j} e_j$. Then H is positive, and $H^2 = T$. Thus, if an operator is positive then it has a positive square root.

Now, let $T : V \to V$ be linear and invertible. TT^* is positive so it has a positive square root H. Since T is invertible, so is T^* and thus so is TT^* and so is H. Define $U = H^{-1}T$.

$$UU^* = H^{-1}T(H^{-1}T)^* = H^{-1}TT^*(H^{-1})^* = H^{-1}TT^*(H^*)^{-1} = H^{-1}H^2H^{-1} = \mathrm{id}_V.$$

Hence U is unitary. Thus we can write any invertible linear $T : V \to V$ as a HU where H is positive and U is unitary, like how we can write a nonzero complex number as the product of a positive real number and a complex number that has absolute value 1.