Self-adjoint linear operators on a finite dimensional complex vector space

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April 3, 2014

Let $V$ be a finite dimensional vector space over $\mathbb{C}$ with an inner product $(\cdot, \cdot) : V \times V \to \mathbb{C}$. For $x \in V$, $|x|^2 = (x, x)$.

Let $A : V \times V \to \mathbb{C}$ be linear in its first argument and conjugate linear in its second argument. Then we check that for all $x, y \in V$,

$$A(x, y) = \frac{1}{4} \left( A(x + y, x + y) + iA(x + iy, x + iy) \
- A(x - y, x - y) - iA(x - iy, x - iy) \right).$$

This is called the polarization identity, or the parallelogram law. A useful instance is for $A(x, y) = (x, y)$. Then,

$$(x, y) = \frac{1}{4} \left( |x + y|^2 + i|x + iy|^2 - |x - y|^2 - i|x - iy|^2 \right).$$

This can be useful for proving a statement about an inner product that one has only verified for a norm.

If $T : V \to V$ is linear, one checks that there is a unique linear $T^* : V \to V$ such that if $x, y \in V$ then

$$(Tx, y) = (x, T^*y).$$

$T^*$ is called the adjoint of $T$. If $T = T^*$ then we say that $T$ is self-adjoint.

An operator being self-adjoint is similar to a complex number being real. Let $T : V \to V$ be linear and define $T_1 = \frac{T + T^*}{2}$ and $T_2 = \frac{T - T^*}{2i}$. Then $T_1, T_2$ are self-adjoint. This resembles writing a complex number as a sum of a real number and $i$ times a real number.

We say that a self-adjoint operator $T : V \to V$ is positive if for all $x \in V$ we have $(Tx, x) \geq 0$. Like for complex numbers, for any linear $T : V \to V$, $TT^*$ is positive (in particular it is self-adjoint).

We say that a linear $U : V \to V$ is unitary if $UU^* = U^*U = \text{id}_V$. If $z \in \mathbb{C}$ and $\bar{z}z = 1$ then $|z| = 1$. An operator being unitary is similar to a complex
number have absolute value 1, in other words a complex number being on the
unit circle. For linear $T : V \to V$, the exponential $\exp(T) : V \to V$ is defined by

$$\exp(T)x = \sum_{k=0}^{\infty} \frac{T^k x}{k!}.$$ 

If $T$ is self-adjoint then $\exp(iT)$ is unitary:

$$(\exp(iT))^* = \left( \sum_{k=0}^{\infty} \frac{(iT)^k}{k!} \right)^* = \sum_{k=0}^{\infty} \frac{(-i)^k T^k}{k!} = \sum_{k=0}^{\infty} \frac{(-iT)^k}{k!} = \exp(-iT),$$

hence

$$\exp(iT)^* = (\exp(iT))^{-1},$$

which shows that $\exp(iT)$ is unitary.

The eigenvalues of a self-adjoint operator are all real, and the eigenvalues of
a unitary operator all have absolute value 1.

Fact: If $H$ is positive then its eigenvalues are nonnegative, and if $H$ is
self-adjoint and has nonnegative eigenvalues then it is positive. Say $V$ has
dimension $n$. On the one hand, if $H$ is positive, suppose that $Hv = \lambda v$. Then
$(Hv,v) = (\lambda v,v) = \lambda (v,v)$. Since $H$ is positive this is nonnegative, and $(v,v)$
is nonnegative so it follows that $\lambda$ is nonnegative too. On the other hand, let $H$
be self-adjoint and let all the eigenvalues of $H$ be nonnegative. Since $H$ is
self-adjoint it has an orthonormal eigenbasis with real eigenvalues (this is the
spectral theorem), $He_j = \lambda_j e_j$ for $1 \leq j \leq n$. Let $x \in V$. We can write $x$
as $x = a_1 e_1 + \cdots + a_n e_n$. We have, using that $e_1, \ldots, e_n$ is an orthonormal
eigenbasis for $H_1$

$$(Hx,x) = (a_1 H e_1 + \cdots + a_n H e_n, a_1 e_1 + \cdots + a_n e_n)$$
$$= (a_1 H e_1, a_1 e_1) + \cdots + (a_n H e_n, a_n e_n)$$
$$= |a_1|^2 (H e_1, e_1) + \cdots + |a_n|^2 (H e_n, e_n)$$
$$= \lambda_1 |a_1|^2 + \cdots + \lambda_n |a_n|^2$$
$$\geq 0.$$

Thus $H$ is positive.

Fact: If $T$ is positive then there is a positive $H$ such that $H^2 = T$. Let $T e_j = \lambda_j e_j$, $1 \leq j \leq n$. $T$ has an eigenbasis with real eigenvalues because $T$
is self-adjoint, and by the above fact the eigenvalues are nonnegative since $T$ is
positive. Define $H$ by $H e_j = \sqrt{\lambda_j} e_j$. Then $H$ is positive, and $H^2 = T$. Thus,
if an operator is positive then it has a positive square root.

Now, let $T : V \to V$ be linear and invertible. $TT^*$ is positive so it has
a positive square root $H$. Since $T$ is invertible, so is $T^*$ and thus so is $TT^*$ and
so is $H$. Define $U = H^{-1} T$.

$$UU^* = H^{-1} T (H^{-1} T)^* = H^{-1} T T^* (H^{-1})^* = H^{-1} T T^* (H^*)^{-1} = H^{-1} H^2 H^{-1} = \text{id}_V.$$
Hence $U$ is unitary. Thus we can write any invertible linear $T : V \to V$ as a $HU$ where $H$ is positive and $U$ is unitary, like how we can write a nonzero complex number as the product of a positive real number and a complex number that has absolute value 1.