Finite-dimensional distributions of stochastic processes

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1 Products of probability spaces

Let I be a nonempty set, and for each $i \in I$, let $(E_i, \mathscr{E}_i, \mu_i)$ be a probability space. Let

$$E = \prod_{i \in I} E_i,$$

the collection of functions $e : I \to \bigcup_{i \in I} E_i$ such that for each i, $e(i) \in E_i$. Let

$$\pi_i : E \to E_i$$

be the **projection map**: $\pi_i(e) = e(i)$. Let

$$\mathscr{E} = \bigotimes_{i \in I} \mathscr{E}_i,$$

the **product σ-algebra**, which is the coarsest σ-algebra on E such that each π_i is measurable $\mathscr{E} \to \mathscr{E}_i$.[1]

Let $\mu = \prod_{i \in I} \mu_i$, the unique probability measure[2] on $\mathscr{E}$ such that when J is a finite nonempty subset of I and $A_i \in \mathscr{E}_i$ for each $i \in J$,

$$\mu\left(\prod_{i \in J} A_i \times \prod_{i \in I \setminus J} E_i\right) = \prod_{i \in J} \mu_i(A_i).$$

2 Joint distributions

Let $(\Omega, \mathscr{F}, P)$ be a probability space, and for each $i \in I$ let $X_i : (\Omega, \mathscr{F}) \to (E_i, \mathscr{E}_i)$ be measurable. Let $P_{X_i} = X_{i*}P$, the **distribution** of X_i, which is a probability measure on $\mathscr{E}_i$: for $A \in \mathscr{E}_i$,

$$P_{X_i}(A) = (X_{i*}P)(A) = P(X_i^{-1}(A)).$$

[1] See http://individual.utoronto.ca/jordanbell/notes/kolmogorov.pdf
[2] See http://individual.utoronto.ca/jordanbell/notes/productmeasure.pdf

In other words, P_{X_i} is the pushforward of the probability measure P by the random variable X_i.

A family of σ-algebras $(\mathscr{F}_i)_{i\in I}$, each contained in $\mathscr{F}$, is said to be **independent** when for any finite nonempty subset J of I and for $A_i \in \mathscr{F}_i$ for $i \in J$,

$$P\left(\bigcap_{i\in J} A_i\right) = \prod_{i\in J} P(A_i).$$

The family of random variables $(X_i)_{i\in I}$ is called **independent** when the family of σ-algebras $(\sigma(X_i))_{i\in I}$ is independent.

Define

$$X = \bigotimes_{i\in I} X_i : \Omega \to E$$

by $X(\omega)(i) = X_i(\omega)$, i.e. $\pi_i \circ X = X_i$. X is measurable $\mathscr{F} \to \mathscr{E}$ because for each i, $\pi_i \circ X$ is measurable $\mathscr{F} \to \mathscr{E}_i$. The **joint distribution of the family** $(X_i)_{i\in I}$ is

$$P_X = X_* P,$$

the pushforward of the measure P by the random variable X. The joint distribution P_X is a probability measure on the product σ-algebra $\mathscr{E}$.

If J is a finite nonempty subset of I, $A_i \in \mathscr{E}_i$ for $i \in J$, and $A = \prod_{i\in J} A_i \times \prod_{i\in I\setminus J} E_i$, on the one hand

$$X^{-1}(A) = \bigcap_{i\in I} X_i^{-1}(\pi_i(A)) = \bigcap_{i\in J} X_i^{-1}(A_i)$$

and thus

$$P_X(A) = (X_*P)(A) = P(X^{-1}(A)) = P\left(\bigcap_{i\in J} X_i^{-1}(A_i)\right),$$

and on the other hand

$$\left(\prod_{i\in I} P_{X_i}\right)(A) = \prod_{i\in J} P_{X_i}(A_i).$$

The following theorem states that the joint distribution P_X is equal to the product of the distributions P_{X_i} if and only if the family $(X_i)_{i\in I}$ is independent.[3]

Theorem 1. *The family of random variables $(X_i)_{i\in I}$ is independent if and only if*

$$P_X = \prod_{i\in I} P_{X_i}.$$

[3] Heinz Bauer, *Probability Theory*, p. 62, Theorem 9.4.

3 Stochastic processes and projective families

Let $(E, \mathscr{E})$ be a measurable space let I be a nonempty set. For a nonempty subset J of I, let

$$E^J = \prod_{t \in J} E$$

and let

$$\mathscr{E}^J = \bigotimes_{t \in J} \mathscr{E}.$$

For nonempty subsets J and K of I with $J \subset K$, let

$$\pi_{K,J} : E^K \to E^J$$

be the projection map: $\pi_{K,J}(f)(t) = f(t)$ for $t \in J$. $\pi_{K,J}$ is measurable $\mathscr{E}^K \to \mathscr{E}^J$. We write

$$\pi_J = \pi_{I,J}.$$

A **stochastic process** is a family $(X_t)_{t \in I}$ of random variables $(\Omega, \mathscr{F}, P) \to (E, \mathscr{E})$. For a finite nonempty subset J of I, define

$$X_J = \bigotimes_{t \in J} X_t : \Omega \to E^J$$

by $X_J(\omega)(i) = X_i(\omega)$ for $i \in J$, i.e. $\pi_i \circ X_J = X_i$ for $i \in J$. X_J is measurable $\mathscr{F} \to \mathscr{E}^J$. Let P_J be the joint distribution of the family of random variables $(X_t)_{t \in J}$, the probability measure on $\mathscr{E}^J$ defined by, for $A \in \mathscr{E}^J$,

$$P_J(A) = (X_{J*} P)(A) = P(X_J^{-1}(A)).$$

For finite nonempty subsets J and K of I with $J \subset K$,

$$X_J = \pi_{K,J} \circ X_K.$$

Then for $A \in \mathscr{E}^J$,

$$((\pi_{K,J})_* P_K)(A) = P_K(\pi_{K,J}^{-1}(A)) = P(X_K^{-1}(\pi_{K,J}^{-1}(A))) = P(X_J^{-1}(A)) = P_J(A),$$

thus

$$(\pi_{K,J})_* P_K = P_J. \tag{1}$$

Let

$$\mathscr{K}(I)$$

be the collection of all finite nonempty subsets of I. We call $(E^J, \mathscr{E}^J, P_J)_{J \in \mathscr{K}(I)}$ the **family of finite-dimensional distributions** of the stochastic process $(X_t)_{t \in I}$.

On the other hand, if I is a nonempty set, $(E, \mathscr{E})$ is a measurable space, and for each $J \in \mathscr{K}(I)$, P_J is a probability measure on $\mathscr{E}^J$ such that (1) is true for all $J, K \in \mathscr{K}(I)$ with $J \subset K$, we say that $(E^J, \mathscr{E}^J, P_J)_{J \in \mathscr{K}(I)}$ is a **projective family**. The following gives a weaker condition that in fact is sufficient for $(P_J)_{J \in \mathscr{K}(I)}$ to be a projective family.[4]

[4] Heinz Bauer, *Probability Theory*, p. 300, Remark 1.

Lemma 2. *If* (1) *is true for all* $J, K \in \mathscr{K}(I)$ *with* $J \subset K$ *and* $K \setminus J$ *a singleton, then* $(P_J)_{J \in \mathscr{K}(I)}$ *is a projective family.*

Proof. If $J, K \in \mathscr{K}(I)$, $J \subset K$, and $K \setminus J$ consists of n distinct elements, write

$$K_0 = J \subset K_1 \subset \cdots \subset K_n = K,$$

where $K_j \setminus K_{j-1}$ is a singleton for each j. By hypothesis, $(\pi_{K_j,K_{j-1}})_* P_{K_j} = P_{K_{j-1}}$ for each j. Because

$$\pi_{K_n,K_0} = \pi_{K_n,K_{n-1}} \circ \cdots \circ \pi_{K_1,K_0},$$

it follows that

$$(\pi_{K_n,K_0})_*(K_n) = K_0.$$

□

The following is the **Kolmogorov extension theorem**.[5]

Theorem 3 (Kolmogorov extension theorem)**.** *If* E *is a Polish space,* $\mathscr{E}$ *is the Borel* σ*-algebra of* E*,* I *is a nonempty set, and* $(E^J, \mathscr{E}^J, P_J)_{J \in \mathscr{K}(I)}$ *is a projective family, then there is a unique probability measure* P_I *on* $\mathscr{E}^I$ *such that for any* $J \in \mathscr{K}(I)$,

$$\pi_{J*} P_I = P_J.$$

P_I is called the **projective limit** of the projective family.

Suppose that E is a Polish space with Borel σ-algebra $\mathscr{E}$, that I is a nonempty set, and that $(E^J, \mathscr{E}^J, P_J)_{J \in \mathscr{K}(I)}$ is a projective family, and let

$$\Omega = E^I, \qquad \mathscr{F} = \mathscr{E}^I, \qquad P = P_I.$$

$(\Omega, \mathscr{F}, P)$ is a probability space. For $t \in I$, define $X_t : \Omega \to E$ by

$$X_t(\omega) = \pi_t(\omega) = \omega(t),$$

which is measurable $\mathscr{F} \to \mathscr{E}$. The family $(X_t)_{t \in I}$ is a stochastic process. For $J \in \mathscr{K}(I)$,

$$X_J = \pi_J,$$

and thus for $A \in \mathscr{E}^J$, using that $P = P_I$ is the projective limit of P_J,

$$(X_{J*} P)(A) = P_I(\pi_J^{-1}(A)) = (\pi_{J*} P_I)(A) = P_J(A),$$

which shows that P_J is the joint distribution of the family $(X_t)_{t \in J}$. Thus, $(X_t)_{t \in I}$ is a stochastic process whose family of finite-dimensional distributions is $(P_J)_{J \in \mathscr{K}(I)}$.

[5] `http://individual.utoronto.ca/jordanbell/notes/kolmogorov.pdf`; Heinz Bauer, *Probability Theory*, p. 301, Theorem 35.3.