The Fredholm determinant

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1 Introduction

By $\mathbb{N}$ we mean the set of positive integers. In this note we write inner products as conjugate linear in the first variable, following the notation of Reed and Simon. The purpose of this note is to make sense of $\det(I + A)$ for bounded trace class operators on a Hilbert space. This definition is consistent with the definition of the determinant for a finite dimensional Hilbert space.

2 Singular value decomposition

Let H be a Hilbert space with an inner product that is conjugate linear in the first variable. We do not presume unless we say so that H is separable.[1]

We denote by $\mathscr{B}(H)$ the set of bounded linear operators $H \to H$. For any $A \in \mathscr{B}(H)$, A^*A is positive and one proves that it has a unique positive square root $|A| \in \mathscr{B}(H)$. We call $|A|$ the *absolute value of* A.

We say that $U \in \mathscr{B}(H)$ is a *partial isometry* if there is a closed subspace X of H such that the restriction of U to X is an isometry $X \to U(X)$ and $\ker U = X^\perp$. One proves that for any $A \in \mathscr{B}(H)$, there is a unique partial isometry U satisfying both $\ker U = \ker A$ and $A = U|A|$, and $A = U|A|$ is called the *polar decomposition of* A. Some useful identities that the polar decomposition satisfies are

$$U^*U|A| = |A|, \quad U^*A = |A|, \quad UU^*A = A.$$

If $A \in \mathscr{B}(H)$ is compact and self-adjoint, the spectral theorem tells us that there is an orthonormal set $\{e_n : n \in \mathbb{N}\}$ in H and $\lambda_n \in \mathbb{R}$, $|\lambda_1| \geq |\lambda_2| \geq \cdots$, such that

$$Ax = \sum_{n\in\mathbb{N}} \lambda_n \langle e_n, x\rangle e_n, \qquad x \in H.$$

[1] Often one assumes separability not because statements are false for nonseparable Hilbert spaces but because it is notationally easier to talk about separable Hilbert spaces, and this indolence hides where separability matters. Moreover, whether or not H is separable, it does not take long to prove that the image of a compact linear operator is separable.

If $A \in \mathscr{B}(H)$ is compact, then $|A|$ is compact (the compact operators are an ideal and $|A| = U^*A$), so the spectral theorem tells us that there is an orthonormal set $\{e_n : n \in \mathbb{N}\}$ in H and $\lambda_n \in \mathbb{R}$, $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$, such that

$$|A|x = \sum_{n\in\mathbb{N}} \lambda_n \langle e_n, x\rangle e_n, \qquad x \in H.$$

Write $\sigma(A) = \lambda_n(|A|)$. $\sigma_n(A)$ are called the *singular values of* A.

Using the spectral theorem and the polar decomposition, one proves that for any compact $A \in \mathscr{B}(H)$, there are orthonormal sets $\{e_n : n \in \mathbb{N}\}$ and $\{f_n : n \in \mathbb{N}\}$ in H such that

$$Ax = \sum_{n\in\mathbb{N}} \sigma_n(A) \langle f_n, x\rangle e_n, \qquad x \in H.$$

This is called the *singular value decomposition of* A.

3 Trace class operators

We denote by $\mathscr{B}_1(H)$ the set of those $A \in \mathscr{B}(H)$ that are compact and such that

$$\|A\|_1 = \sum_{n\in\mathbb{N}} \sigma_n(A) < \infty.$$

Elements of $\mathscr{B}_1(H)$ are called *trace class operators*. It can be proved that $\mathscr{B}_1(H)$ with the norm $\|\cdot\|_1$ is a Banach space.

Let $\mathscr{E}$ be an orthonormal basis for H. We define $\mathrm{tr} : \mathscr{B}_1(H) \to \mathbb{C}$ by

$$\mathrm{tr}\, A = \sum_{e\in\mathscr{E}} \langle e, Ae\rangle, \qquad A \in \mathscr{B}_1(H).$$

One proves that the value of this sum is the same for any orthonormal basis of H, and that tr is a bounded linear operator.

It is a fact that if $A \in \mathscr{B}_1(H)$ then $A^* \in \mathscr{B}_1(H)$, $\mathrm{tr}\, A^* = \overline{\mathrm{tr}\, A}$, and $\|A^*\|_1 = \|A\|_1$. Another fact is that if $A \in \mathscr{B}_1(H)$ and $B \in \mathscr{B}(H)$, then $AB, BA \in \mathscr{B}_1(H)$, $\mathrm{tr}\,(AB) = \mathrm{tr}\,(BA)$, $|\mathrm{tr}\,(BA)| \leq \|B\| \|A\|_1$, and $\|AB\|_1 \leq \|A\|_1 \|B\|$, $\|BA\|_1 \leq \|B\| \|A\|_1$. Finally, if $A \in \mathscr{B}_1(H)$ then $\|A\| \leq \|A\|_1$.

4 Logarithms

We say that $A \in \mathscr{B}(H)$ is *invertible* if $A^{-1} \in \mathscr{B}(H)$. If $A \in \mathscr{B}(H)$ and $\|A\| < 1$, one checks that $I - A$ is invertible: $(I-A)^{-1} = \sum_{n=0}^{\infty} A^n$. Furthermore, suppose that $A \in \mathscr{B}_1(H)$ and $\|A\| < 1$. Then,

$$\log(I - A) = \sum_{n=1}^{\infty} -\frac{A^n}{n},$$

hence

$$\operatorname{tr}\,\log(I-A)=\sum_{n=1}^{\infty}-\frac{\operatorname{tr}(A^n)}{n},$$

and

$$\exp\operatorname{tr}\,\log(I-A)=\prod_{n=1}^{\infty}\exp\left(-\frac{\operatorname{tr}(A^n)}{n}\right). \tag{1}$$

Because $\det\exp B=\exp\operatorname{tr}B$ for any $B\in\mathscr{B}(\mathbb{R}^N)$, if $A\in\mathscr{B}(\mathbb{R}^N)$ satisfies $\|A\|<1$ then $\det(I-A)=\exp\operatorname{tr}\,\log(I-A)$. The above expression makes sense for any $A\in\mathscr{B}_1(H)$ with $\|A\|<1$, so it makes sense to define

$$\det(I-A)=\exp\operatorname{tr}\,\log(I-A)$$

in this case, for which we have the explicit formula (1). We have not shown that det has the properties we might expect it to have, but at least in the case $H=\mathbb{R}^N$ it is equal to the ordinary determinant, which is the least we could demand of a function which we denote by det.

5 Tensor products

Suppose that H is a Hilbert space over K with inner product $\langle\cdot,\cdot\rangle$ that is conjugate linear in the first variable. For each $n\in\mathbb{N}$ and $x_1,\ldots,x_n\in H$, let $x_1\otimes\cdots\otimes x_n$ be the multilinear function on H^n defined by

$$(x_1\otimes\cdots\otimes x_n)(y_1,\ldots,y_n)=\prod_{k=1}^{n}\langle x_k,y_k\rangle, \qquad (y_1,\ldots,y_n)\in H^n.$$

The set of all finite linear combinations of such multilinear functions is a vector space for which there is a unique inner product that satisfies

$$\langle x_1\otimes\cdots\otimes x_n, y_1\otimes\cdots\otimes y_n\rangle=\prod_{k=1}^{n}\langle x_k,y_k\rangle, \qquad x_1,\ldots,x_n,y_1,\ldots,y_n\in H.$$

We denote by $\bigotimes^n H$ the completion of this vector space using this inner product.[2] Thus, $\bigotimes^n H$ is a Hilbert space. We call this a *tensor product*, but it does not have the universal property of tensor products, so is not a categorical tensor product.[3]

If $A\in\mathscr{B}(H)$, one proves that there is a unique $T\in\mathscr{B}(\bigotimes^n H)$ that satisfies

$$T(x_1\otimes\cdots\otimes x_n)=Ax_1\otimes\cdots\otimes Ax_n, \qquad x_1\otimes\cdots\otimes x_n\in\bigotimes^n H,$$

and we write $T=\bigotimes^n A$. For $A,B\in\mathscr{B}(H)$ and $x_1\otimes\cdots\otimes x_n\in\bigotimes^n H$, it is straightforward to check that

$$(\bigotimes^n(AB))(x_1\otimes\cdots\otimes x_n)=(\bigotimes^n A)(\bigotimes^n B)(x_1\otimes\cdots\otimes x_n),$$

from which it follows that $\bigotimes^n(AB)=\bigotimes^n A\bigotimes^n B$.

[2]See Michael Reed and Barry Simon, *Methods of Modern Mathematical Physics, volume I: Functional Analysis*, revised and enlarged ed., p. 50.

[3]Paul Garrett, `http://www.math.umn.edu/~garrett/m/v/nonexistence_tensors.pdf`

6 Exterior powers

Let S_n denote the group of permutations on n symbols and let $\operatorname{sgn}(\pi)$ denote the sign of the permutation π. For $x_1, \ldots, x_n \in H$, we define $x_1 \wedge \cdots \wedge x_n \in \bigotimes^n H$ by

$$x_1 \wedge \cdots \wedge x_n = (n!)^{-1/2} \sum_{\pi \in S_n} \operatorname{sgn}(\pi) x_{\pi(1)} \otimes \cdots \otimes x_{\pi(n)}.$$

We define $\bigwedge^n H$ to be the closure in $\bigotimes^n H$ of the set of all finite linear combinations of elements of $\bigotimes^n H$ of the form $x_1 \wedge \cdots \wedge x_n$.[4] Thus, $\bigwedge^n H$ is a Hilbert space.

For $x_1 \wedge \cdots \wedge x_n, y_1 \wedge \cdots \wedge y_n \in \bigwedge^n H$, one proves that[5]

$$\langle x_1 \wedge \cdots \wedge x_n, y_1 \wedge \cdots \wedge y_n \rangle = \det(\langle x_i, y_j \rangle),$$

where

$$\det(a_{i,j}) = \sum_{\pi \in S_n} \operatorname{sgn}(\pi) a_{1,\pi(1)} \cdots a_{n,\pi(n)}.$$

In particular, if $\{x_1, \ldots, x_n\}$ is an orthonormal set in H, then $\|x_1 \wedge \cdots \wedge x_n\| = 1$.

For $A \in \mathscr{B}(H)$ and $x_1 \wedge \cdots \wedge x_n \in \bigwedge^n H$, it is apparent that

$$(\bigotimes^n A) x_1 \wedge \cdots \wedge x_n = Ax_1 \wedge \cdots \wedge Ax_n.$$

It follows that $\bigotimes^n A$ sends an element of $\bigwedge^n H$ to an element $\bigwedge^n H$, and hence that the restriction of $\bigotimes^n A$ to $\bigwedge^n H$ belongs to $\mathscr{B}(\bigwedge^n H)$. We denote this restriction by $\bigwedge^n A$. Because $\bigotimes^n (AB) = \bigotimes^n A \bigotimes^n B$, we also have $\bigwedge^n (AB) = \bigwedge^n A \bigwedge^n B$.

7 Finite dimensional Hilbert spaces

Suppose that H is an n-dimensional Hilbert space and that $A \in \mathscr{B}(H)$. If H has dimension n and $\{e_1, \ldots, e_n\}$ is an orthonormal basis for H, one proves that

$$\{e_{i_1} \wedge \cdots \wedge e_{i_k} : 1 \leq i_1 < \cdots < i_k \leq n\}$$

is an orthonormal basis for $\bigwedge^k H$, and hence that $\bigwedge^k H$ has dimension $\binom{n}{k}$. So $\bigwedge^n H$ has dimension 1, and as $\bigwedge^n A \in \mathscr{B}(\bigwedge^n H)$, there is some scalar α such that $(\bigwedge^n A)v = \alpha v$ for all $v \in \bigwedge^n H$. (A linear map from a one dimensional vector space to itself is multiplication by a scalar.) On the one hand,

$$\begin{aligned}
\langle e_1 \wedge \cdots \wedge e_n, (\textstyle\bigwedge^n A)(e_1 \wedge \cdots \wedge e_n) \rangle &= \langle e_1 \wedge \cdots \wedge e_n, \alpha e_1 \wedge \cdots \wedge e_n \rangle \\
&= \alpha \langle e_1 \wedge \cdots \wedge e_n, e_1 \wedge \cdots \wedge e_n \rangle \\
&= \alpha.
\end{aligned}$$

[4]cf. Paul Garrett, `http://www.math.umn.edu/~garrett/m/algebra/notes/28.pdf`

[5]Michael Reed and Barry Simon, *Methods of Modern Mathematical Physics, volume IV: Analysis of Operators*, p. 321.

On the other hand,

$$\begin{aligned}\langle e_1 \wedge \cdots \wedge e_n, (\textstyle\bigwedge^n A)(e_1 \wedge \cdots \wedge e_n)\rangle &= \langle e_1 \wedge \cdots \wedge e_n, Ae_1 \wedge \cdots \wedge Ae_n\rangle \\ &= \det A.\end{aligned}$$

Thus, $\bigwedge^n A$ is the map $v \mapsto \det(A)v$.[6]

Still taking H to be n-dimensional, take $A \in \mathscr{B}(H)$, and let $e_1, \ldots, e_n$ be an orthonormal basis for H such that $V_k = \operatorname{span}\{e_1, \ldots, e_k\}$ and such that $A(V_k) = V_k$ for $k = 1, \ldots, n$; such a basis is obtained using the *Schur decomposition* of A.

The equality

$$\begin{aligned}\operatorname{tr}(\textstyle\bigwedge^k A) &= \displaystyle\sum_{1 \le i_1 < \cdots < i_k \le n} \left\langle e_{i_1} \wedge \cdots \wedge e_{i_k}, (\textstyle\bigwedge^k A)(e_{i_1} \wedge \cdots \wedge e_{i_k})\right\rangle \\ &= \sum_{1 \le i_1 < \cdots i_k \le n} \langle e_{i_1} \wedge \cdots \wedge e_{i_k}, Ae_{i_1} \wedge \cdots \wedge Ae_{i_k}\rangle \\ &= \sum_{1 \le i_1 < \cdots < i_k \le n} \det(\langle e_{i_i}, Ae_{i_j}\rangle) \\ &= \sum_{1 \le i_1 < \cdots < i_k \le n} \lambda_{i_1} \cdots \lambda_{i_k}\end{aligned}$$

and the equality

$$\begin{aligned}\det(I + A) &= \langle e_1 \wedge \cdots \wedge e_n, (I + A)e_1 \wedge \cdots \wedge (I + A)e_n\rangle \\ &= \det(\langle e_i, (I + A)e_j\rangle) \\ &= \prod_{j=1}^{n} (1 + \lambda_j)\end{aligned}$$

together give

$$\det(I + A) = \sum_{j=0}^{n} \operatorname{tr}(\textstyle\bigwedge^j A).$$

If H is infinite dimensional we define $\det(I+A)$ for $A \in \mathscr{B}_1(H)$ following the above formula. For this definition to make sense we use the following lemma.[7] It is stated in Reed and Simon for separable Hilbert spaces. In reading the proof I don't see how separability is essential to proving the result, but without carefully working out the proof and refreshing myself about the singular value decomposition, it would be dishonest to assert that the result is true for nonseparable Hilbert spaces.

[6] Michael Reed and Barry Simon, *Methods of Modern Mathematical Physics, volume IV: Analysis of Operators*, p. 321, Lemma 2.

[7] Michael Reed and Barry Simon, *Methods of Modern Mathematical Physics, volume IV: Analysis of Operators*, p. 323, Lemma 3.

Lemma 1. *Let H be a separable Hilbert space and let $A \in \mathscr{B}_1(H)$. For any $k \in \mathbb{N}$ we have $\bigwedge^k A \in \mathscr{B}_1(\bigwedge^k H)$, and*

$$\left\| \bigwedge^k A \right\|_1 = \sum_{i_1 < \cdots < i_k} \mu_{i_1} \cdots \mu_{i_k},$$

and

$$\left\| \bigwedge^k A \right\|_1 \leq \frac{\|A\|_1^k}{k!}.$$

Definition 2. If H is a separable Hilbert space and $A \in \mathscr{B}_1(H)$, we define

$$\det(I + A) = \sum_{k=0}^{\infty} \operatorname{tr} (\bigwedge^k A).$$

We call det the *Fredholm determinant on H*.

For a separable Hilbert space, Reed and Simon also prove that[8]

$$|\det(I + A)| \leq \exp(\|A\|_1), \qquad A \in \mathscr{B}_1(H),$$

that

$$\det(I + A) \det(I + B) = \det(I + A + B + AB), \qquad A, B \in \mathscr{B}_1(H),$$

and that for $A \in \mathscr{B}_1(H)$, $I + A$ is invertible if and only if $\det(I + A) \neq 0$.[9]

[8] Michael Reed and Barry Simon, *Methods of Modern Mathematical Physics, volume IV: Analysis of Operators*, p. 323, Lemma 4.

[9] Michael Reed and Barry Simon, *Methods of Modern Mathematical Physics, volume IV: Analysis of Operators*, p. 325, Theorem XIII.105.