Unbounded operators and the Friedrichs extension

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1 Introduction

In this note, by $A \subset B$, I mean that A is contained in B, and it may be that $A = B$; usually I write this by $A \subseteq B$, but $A \subset B$ fits with the usual notation for saying that an operator is an extension of another.

In this note, unless we say otherwise H denotes a Hilbert space over $\mathbb{C}$, and we do not presume H to be separable. We shall write the inner product $\langle \cdot, \cdot \rangle$ on H as conjugate linear in the second argument.

We say that T *is an operator in* H if there is a linear subspace $\mathscr{D}(T)$ of H such that $T : \mathscr{D}(T) \to H$ is a linear map. We call $\mathscr{D}(T)$ the *domain of* T and $\mathscr{R}(T) = T(\mathscr{D}(T))$ the *range of* T. We do not presume unless we say so that $\mathscr{D}(T)$ is dense in H, and we say that T is *densely defined* when this is so.

Define

$$\mathscr{G}(T) = \{(x, Tx) : x \in \mathscr{D}(T)\},$$

called the *graph of* T. We say that an operator S is an *extension* of an operator T if $\mathscr{G}(T) \subset \mathscr{G}(S)$, and we write $T \subset S$. The set of all extensions of an operator is a partially ordered set, so it makes sense to talk about a maximal extension. In particular, if $\mathscr{D}(T) = H$ then T is maximal.

$H \times H$ is a Hilbert space with the inner product

$$\langle (x, y), (v, w) \rangle = \langle x, v \rangle + \langle y, w \rangle.$$

We say that an operator T is *closed* if $\mathscr{G}(T)$ is a closed subset of $H \times H$. Thus, to say that T is a closed operator means that if x_n is a sequence in $\mathscr{D}(T)$ and $(x_n, Tx_n) \to (x, y) \in H \times H$, then $(x, y) \in \mathscr{G}(T)$, i.e. $x \in \mathscr{D}(T)$ and $y = Tx$. It is apparent that if $T \in \mathscr{B}(H)$ then T is closed. On the other hand, if T is closed and $\mathscr{D}(T) = H$, then the closed graph theorem tells us that $T \in \mathscr{B}(H)$.

2 Adjoints

Following Garrett,[1] we say that an operator T' is a *sub-adjoint* of an operator T if

$$\langle Tv, w\rangle = \langle v, T'w\rangle, \qquad v \in \mathscr{D}(T), w \in \mathscr{D}(T').$$

Obviously, $T' = 0$ with $\mathscr{D}(T') = \{0\}$ is a sub-adjoint of any operator. As well, if T is densely defined, then for any linear subspace V of H there is at most one sub-adjoint of T with domain V.

We define $J : H \times H \to H \times H$ by $J(v, w) = (-w, v)$. J is unitary. We follow Garrett's proof of the following theorem.[2]

Theorem 1. *If T is densely defined, then it has a a unique maximal sub-adjoint, denoted T^* and called the* adjoint *of T. T^* is closed, with*

$$\mathscr{G}(T^*) = J(\mathscr{G}(T))^{\perp}.$$

Proof. Write $X = J(\mathscr{G}(T))^{\perp}$. Suppose that T' is a sub-adjoint of T. For $w \in \mathscr{D}(T')$ and $v \in \mathscr{D}(T)$,

$$\langle J(v, Tv), (w, T'w)\rangle = \langle(-Tv, v), (w, T'w)\rangle = \langle -Tv, w\rangle + \langle v, T'w\rangle = 0,$$

showing that $\mathscr{G}(T') \subset X$.

For any $w \in H$, suppose that $(w, w_1), (w, w_2) \in X$. This means that for all $v \in \mathscr{D}(T)$, $\langle(w, w_1), (-Tv, v)\rangle = 0$ and $\langle(w, w_2), (-Tv, v)\rangle = 0$, i.e. $\langle w, -Tv\rangle + \langle w_1, v\rangle = 0$ and $\langle w, -Tv\rangle + \langle w_2, v\rangle = 0$, so $\langle w_1 - w_2, v\rangle = 0$. Because $\mathscr{D}(T)$ is dense in H, this implies that $w_1 - w_2 = 0$ (lest a sequence of things that are each 0 converge to something that is not 0). Therefore for any $w \in H$ there is at most one $w' \in H$ such that $(w, w') \in X$, and we define

$$W = \{w \in H : \text{there is some } w' \in H \text{ such that } (w, w') \in X\}.$$

We define $T^* : W \to H$ by $Tw = w'$, so T^* is an operator with $\mathscr{D}(T^*) = W$. It is apparent that $\mathscr{G}(T^*) = X$.

For $v \in \mathscr{D}(T)$ and $w \in \mathscr{D}(T^*)$, using $(w, T^*w) \in X$ and $(-Tv, v) \in J(\mathscr{G}(T))$ we get

$$0 = \langle(-Tv, v), (w, T^*w)\rangle = \langle -Tv, w\rangle + \langle v, T^*w\rangle,$$

showing that T^* is a sub-adjoint of T.

If T' is a sub-adjoint of T, we have shown that $\mathscr{G}(T') \subset X$ and that $\mathscr{G}(T^*) = X$, giving $T \subset T^*$. Hence T^* is the unique maximal sub-adjoint of T.

$\mathscr{G}(T^*) = X$ is an orthogonal complement hence closed in $H \times H$, meaning that T^* is a closed operator, completing the proof. □

[1] Paul Garrett, *Unbounded operators, Friedrichs extension theorem*, `http://www.math.umn.edu/~garrett/m/v/friedrichs.pdf`

[2] Paul Garrett, *Unbounded operators, Friedrichs extension theorem*, `http://www.math.umn.edu/~garrett/m/v/friedrichs.pdf`

Using the expression in the above theorem for the graph of the adjoint as an orthogonal complement, if T_1, T_2 are densely defined and $T_1 \subset T_2$, then $T_2^* \subset T_1^*$.

Definition 2. *Suppose that T is an operator in H. We say that T is* self-adjoint *if T is densely defined and $T = T^*$, i.e. $\mathscr{G}(T) = \mathscr{G}(T^*)$.*

Theorem 3. *If T is a densely defined closed operator in H, then*

$$H \times H = J(\mathscr{G}(T)) \oplus \mathscr{G}(T^*) = \mathscr{G}(T) \oplus J(\mathscr{G}(T^*)).$$

Proof. Because T is densely defined we have $\mathscr{G}(T^*) = J(\mathscr{G}(T))^\perp$. Then taking orthogonal complements, $\overline{J(\mathscr{G}(T))} = \mathscr{G}(T^*)^\perp$. But T is closed and J is unitary, so $\overline{J(\mathscr{G}(T))} = J(\mathscr{G}(T))$, giving $J(\mathscr{G}(T)) = \mathscr{G}(T^*)^\perp$. Moreover, $\mathscr{G}(T^*)$ is a closed linear subspace of $H \times H$, so

$$H \times H = \mathscr{G}(T^*) \oplus \mathscr{G}(T^*)^\perp = \mathscr{G}(T^*) \oplus J(\mathscr{G}(T)).$$

Because J is unitary and $J^2 = I$,

$$H \times H = J(H \times H) = J(\mathscr{G}(T^*)) \oplus J^2(\mathscr{G}(T)) = J(\mathscr{G}(T^*)) \oplus \mathscr{G}(T).$$

□

We now use the above orthogonal direct sum to show that the adjoint of a densely defined closed operator is itself densely defined; then since T^* is densely defined it makes sense to talk about T^{**}, and this is equal to T. The proof follows Rudin.[3]

Theorem 4. *If T is a densely defined closed operator in H, then $\mathscr{D}(T^*)$ is dense in H and $T^{**} = T$.*

Proof. Suppose that $z \in \mathscr{D}(T^*)^\perp$. For all $y \in \mathscr{D}(T^*)$, $\langle z, y\rangle = 0$, which can be written as $\langle (0, z), (-T^*y, y)\rangle = 0$, which means that $(0, z) \in (J(\mathscr{G}(T^*))^\perp$. But by Theorem 3, $(J(\mathscr{G}(T^*))^\perp = \mathscr{G}(T)$, so $(0, z) \in \mathscr{G}(T)$. That is, $T(0) = z$, hence $z = 0$. Therefore $\mathscr{D}(T^*)^\perp = \{0\}$, which implies that $\mathscr{D}(T^*)$ is dense in H.

Because T^* is densely defined, we can apply Theorem 3 to get

$$H \times H = J(\mathscr{G}(T^*)) \oplus \mathscr{G}(T^{**}).$$

But we also have

$$H \times H = \mathscr{G}(T) \oplus J(\mathscr{G}(T^*)).$$

Therefore $\mathscr{G}(T^{**}) = \mathscr{G}(T)$. □

[3] Walter Rudin, *Functional Analysis*, second ed., p. 354, Theorem 13.12.

3 Symmetric operators

We say that an operator T in H is *symmetric* if

$$\langle Tx, y\rangle = \langle x, Ty\rangle, \qquad x, y \in \mathscr{D}(T).$$

Theorem 5. *Suppose that T is a densely defined operator in H. Then T is symmetric if and only if $T \subset T^*$.*

Proof. Suppose that T is symmetric. For $v, w \in \mathscr{D}(T)$,

$$\langle(-Tv, v), (w, Tw)\rangle = \langle -Tv, w\rangle + \langle v, Tw\rangle = 0,$$

showing by Theorem 1 that $(w, Tw) \in \mathscr{G}(T^*)$. Therefore $\mathscr{G}(T) \subset \mathscr{G}(T^*)$, i.e. $T \subset T^*$.

Suppose that $T \subset T^*$. For $x, y \in \mathscr{D}(T)$, the fact that $T \subset T^*$ gives $Tx = T^*x$, so

$$\langle x, Ty\rangle = \langle x, T^*y\rangle = \langle Tx, y\rangle,$$

showing that T is symmetric. □

Lemma 6. *If T is a symmetric operator in H and $\lambda \in \mathbb{C}$ is an eigenvalue of T, then $\lambda \in \mathbb{R}$.*

Proof. Let $v \in \mathscr{D}(T)$, $v \neq 0$ and $Tv = \lambda v$. T being symmetric gives $\langle Tv, v\rangle = \langle v, Tv\rangle$, hence $\langle \lambda v, v\rangle = \langle v, \lambda v\rangle$ and thus $\lambda\|v\| = \overline{\lambda}\|v\|$, and $\|v\| \neq 0$ so $\lambda = \overline{\lambda}$, meaning $\lambda \in \mathbb{R}$. □

Definition 7. *An operator T in H is called* positive *if it is symmetric and if*

$$\langle Tv, v\rangle \geq 0, \qquad v \in \mathscr{D}(T);$$

we stipulate that T is symmetric so that the left-hand side of the above inequality is real.

4 The Hellinger-Toeplitz theorem

The Hellinger-Toeplitz theorem is the statement that if an operator in a Hilbert space is defined everywhere and is symmetric, then it is in fact bounded. Our proofs follows Rudin.[4]

Theorem 8 (Hellinger-Toeplitz theorem)**.** *If T is a symmetric operator in H with $\mathscr{D}(T) = H$, then $T \in \mathscr{B}(H)$.*

Proof. Because $\mathscr{D}(T) = H$, of course T is densely defined, so because T is symmetric, by Theorem 5 we have $T \subset T^*$; it makes sense to talk about T^* because T is densely defined. $T \subset T^*$ and $\mathscr{D}(T) = H$ together imply $T = T^*$. But from Theorem 1, $\mathscr{G}(T^*)$ is closed, and hence $\mathscr{G}(T)$ is closed too. Then, because $\mathscr{D}(T) = H$ and $\mathscr{G}(T)$ is closed, the closed graph theorem tells us that T is continuous. □

[4] Walter Rudin, *Functional Analysis*, second ed., p. 353, Theorem 13.11.

5 Friedrichs extension

The proof of the following theorem expands on Garrett.[5]

Theorem 9 (Friedrichs extension)**.** *If T is densely defined and positive, then there is an operator in H that is self-adjoint and positive and whose restriction to $\mathscr{D}(T)$ is equal to T.*

Proof. Define

$$\langle v, w\rangle_1 = \langle v, w\rangle + \langle Tv, w\rangle, \qquad v, w \in \mathscr{D}(T).$$

It is apparent that $\langle\cdot,\cdot\rangle_1$ is a Hermitian form on the vector space $\mathscr{D}(T)$, conjugate linear in the second argument. Moreover, for $v \in \mathscr{D}(T)$,

$$\langle v, v\rangle_1 = \langle v, v\rangle + \langle Tv, v\rangle \geq 0$$

because T is positive. Therefore $\langle\cdot,\cdot\rangle_1$ is an inner product on $\mathscr{D}(T)$.

Let K be the completion of $\mathscr{D}(T)$ with respect to the inner product $\langle\cdot,\cdot\rangle_1$. That is, K is a Hilbert space, and there is a one-to-one linear map $k : \mathscr{D}(T) \to K$ such that $\langle kv, kw\rangle_K = \langle v, w\rangle_1$ for all $v, w \in \mathscr{D}(T)$, and $k(\mathscr{D}(T))$ is dense in K. k is an isometry, so it makes sense to define $j : k(\mathscr{D}(T)) \to H$ by $j(k(x)) = x$, and j is itself an isometry. Because $k(\mathscr{D}(T))$ is dense in K and j is a bounded linear map, there is a unique bounded linear map $\hat{j} : K \to H$ whose restriction to $k(\mathscr{D}(T))$ is equal to j, and $\|\hat{j}\| = \|j\| \leq 1$. Suppose that $\hat{j}(\phi) = 0$ for some $\phi \in K$. As $k(\mathscr{D}(T))$ is dense in K, there is a sequence $v_n \in \mathscr{D}(T)$ such that $\|kv_n - \phi\|_K \to 0$, and as

$$\|v_n\| \leq \|v_n\|_1 = \|j(kv_n)\|_1 = \|\hat{j}(kv_n)\|_1 = \|\hat{j}(kv_n - \phi)\|_1 \leq \|kv_n - \phi\|_K,$$

this means that $\|v_n\| \to 0$. Then,

$$\begin{aligned}
\|\phi\|_K^2 &= \langle \phi, \phi\rangle_K \\
&= \lim_{n\to\infty} \langle \phi, kv_n\rangle_K \\
&= \lim_{m\to\infty}\lim_{n\to\infty} \langle kv_m, kv_n\rangle_K \\
&= \lim_{m\to\infty}\lim_{n\to\infty} \langle v_m, v_n\rangle_1 \\
&= \lim_{m\to\infty}\lim_{n\to\infty} (\langle v_m, v_n\rangle + \langle Tv_m, v_n\rangle) \\
&\leq \limsup_{m\to\infty}\limsup_{n\to\infty} (\|v_m\|\|v_n\| + \|Tv_m\|\|v_n\|) \\
&= \limsup_{m\to\infty} 0 \\
&= 0;
\end{aligned}$$

this uses $\|v_n\| \to 0$, and does not presume that T is bounded. Hence $\|\phi\|_K = 0$, so $\phi = 0$. This shows

$$\hat{j} : K \to H \text{ is one-to-one.}$$

[5] Paul Garrett, *Unbounded operators, Friedrichs extension theorem*, `http://www.math.umn.edu/~garrett/m/v/friedrichs.pdf`

For $h \in H$, define $\lambda_h : K \to \mathbb{C}$ by

$$\lambda_h \phi = \langle \hat{j}\phi, h \rangle, \qquad \phi \in K,$$

which satisfies

$$|\lambda_h \phi| \le \|\hat{j}\phi\| \|h\| \le \|\phi\|_K \|h\|, \qquad \phi \in K,$$

and hence $\|\lambda_h\| \le \|h\|$, so by the Riesz representation theorem there is a unique $Ch \in K$ satisfying $\|Ch\|_1 = \|\lambda_h\| \le \|h\|$ and

$$\lambda_h \phi = \langle \phi, Ch \rangle_K, \qquad \phi \in K.$$

For $h_1, h_2 \in H$, $\alpha \in \mathbb{C}$, and $\phi \in K$,

$$\begin{aligned}
\langle \phi, C(\alpha h_1 + h_2) \rangle_K &= \lambda_{\alpha h_1 + h_2} \phi \\
&= \langle \hat{j}\phi, \alpha h_1 + h_2 \rangle \\
&= \overline{\alpha} \langle \hat{j}\phi, h_1 \rangle + \langle \hat{j}\phi, h_2 \rangle \\
&= \overline{\alpha} \lambda_{h_1} \phi + \lambda_{h_2} \phi \\
&= \overline{\alpha} \langle \phi, Ch_1 \rangle_K + \langle \phi, Ch_2 \rangle_K \\
&= \langle \phi, \alpha Ch_1 + Ch_2 \rangle_K.
\end{aligned}$$

This being true for all $\phi \in K$ implies that $C(\alpha h_1 + h_2) = \alpha Ch_1 + Ch_2$. Thus, $C : H \to K$ is linear, and $\|C\| \le 1$. Define $B : H \to H$ by

$$B = \hat{j} \circ C,$$

which satisfies $\|B\| \le \|\hat{j}\| \|C\| \le 1$, so $B \in \mathscr{B}(H)$.

For $v, w \in H$,

$$\begin{aligned}
\langle Bv, w \rangle &= \langle \hat{j}(Cv), w \rangle \\
&= \lambda_w(Cv) \\
&= \langle Cv, Cw \rangle_K \\
&= \overline{\langle Cw, Cv \rangle_K} \\
&= \overline{\lambda_v(Cw)} \\
&= \overline{\langle \hat{j}(Cw), v \rangle} \\
&= \overline{\langle Bw, v \rangle} \\
&= \langle v, Bw \rangle,
\end{aligned}$$

so B is self-adjoint. Moreover, for $v \in H$,

$$\langle Bv, v \rangle = \langle \hat{j}(Cv), v \rangle = \lambda_v(Cv) = \langle Cv, Cv \rangle_K \ge 0.$$

Therefore, B is a positive operator. As well, if $Bv = 0$ then $(\hat{j} \circ C)(v) = 0$, and as $\hat{j}$ is one-to-one this means that $Cv = 0$, and hence for all $\phi \in K$,

$$0 = \langle \phi, Cv \rangle_K = \lambda_v \phi = \langle \hat{j}\phi, v \rangle.$$

But $\mathscr{D}(T) \subset \hat{j}(K)$, so the above holding for all $\phi \in K$ means in particular that $\langle w, v \rangle = 0$ for all $w \in \mathscr{D}(T)$. As $\mathscr{D}(T)$ is dense in H, this implies that $v = 0$. This shows that B is one-to-one. Finally, suppose that $\phi \in C(H)^\perp$, so for all $h \in H$,

$$0 = \langle \phi, Ch \rangle_K = \lambda_h \phi = \langle \hat{j}\phi, h \rangle.$$

Because this holds for all $h \in H$, we have $\hat{j}\phi = 0$, and $\hat{j}$ is one-to-one so $\phi = 0$. This shows that $C(H)$ is dense in K. Then, as $\hat{j} : K \to \hat{j}(K)$ is a bijection, we have $B(H)$ is dense in $\hat{j}(K)$. But $\mathscr{D}(T) \subset \hat{j}(K)$ and $\mathscr{D}(T)$ is dense in H, so $B(H)$ is dense in H.

We define $\mathscr{D}(A) = B(H)$, and define $A : \mathscr{D}(A) \to H$ by $Av = B^{-1}v$, which makes sense because B is one-to-one. We have just shown that $B(H)$ is dense in H, so A is a densely defined operator in H. As well, $A : \mathscr{D}(A) \to H$ is onto, because $B \in \mathscr{B}(H)$. A is symmetric: for $v, w \in \mathscr{D}(A) = B(H)$, there are $x, y \in H$ with $Bx = v$ and $By = w$, and using the fact that B is self-adjoint,

$$\langle Av, w \rangle = \langle A(Bx), By \rangle = \langle x, By \rangle = \langle Bx, y \rangle = \langle v, Aw \rangle.$$

Moreover, A is positive: for $v \in \mathscr{D}(A)$ there is some $x \in H$ with $Bx = v$, and the fact that B is positive gives

$$\langle Av, v \rangle = \langle A(Bx), Bx \rangle = \langle x, Bx \rangle \geq 0.$$

In this paragraph we show that A is self-adjoint; because A is densely defined it indeed has an adjoint A^*. Define $U : H \times H \to H \times H$ by $U(v, w) = (w, v)$, which is unitary. It is apparent that

$$\mathscr{G}(A) = U(\mathscr{G}(B)).$$

For any linear subspace X of $H \times H$,

$$(UX)^\perp = U(X^\perp).$$

Then using $J \circ U = -U \circ J$ and Theorem 1 we obtain (since $-X = X$ when X is a vector space)

$$\begin{aligned}
\mathscr{G}(A^*) &= J(\mathscr{G}(A))^\perp \\
&= J(U(\mathscr{G}(B)))^\perp \\
&= U(J(\mathscr{G}(B)))^\perp \\
&= U(J(\mathscr{G}(B))^\perp) \\
&= U(\mathscr{G}(B^*)) \\
&= U(\mathscr{G}(B)) \\
&= \mathscr{G}(A).
\end{aligned}$$

Thus A is self-adjoint.

Define $S : \mathscr{D}(T) \to H$ by $S = \mathrm{id}_H + T$. For $v, w \in \mathscr{D}(T)$,

$$\langle v, Sw \rangle = \langle \hat{j}(kv), Sw \rangle = \lambda_{Sw}(kv) = \langle kv, C(Sw) \rangle_K =$$

and also

$$\langle v, Sw\rangle = \langle v, w\rangle + \langle v, Tw\rangle = \langle v, w\rangle + \langle Tv, w\rangle = \langle v, w\rangle_1 = \langle kv, kw\rangle_K.$$

Because $k(\mathscr{D}(T))$ is dense in K, it follows that, for all $w \in \mathscr{D}(T)$, $C(Sw) = kw$, or $\hat{j}(C(Sw)) = \hat{j}(kw)$, i.e., $B(Sw) = w$. This shows that $w \in B(H) = \mathscr{D}(T)$, so

$$\mathscr{D}(T) \subset \mathscr{D}(A),$$

and we can apply A to $B(Sw) = w$ and get $Sw = Aw$. Therefore,

$$S \subset A.$$

Define $\mathscr{D}(F) = \mathscr{D}(A)$ and $F = A - \mathrm{id}_H$. We verify that F is self-adjoint:

$$=$$

For $v \in \mathscr{D}(A) = B(H)$, there is some $x \in H$ with $Bx = v$, and

$$\langle v, Av\rangle = \langle Bx, B^{-1}Bx\rangle = \langle Bx, x\rangle = \langle \hat{j}(Cx), x\rangle = \lambda_x(Cx) = \langle Cx, Cx\rangle_K,$$

but $\|v\| = \|Bx\| = \|\hat{j}(Cx)\| \leq \|Cx\|_K$. This shows that

$$\langle v, Av - v\rangle \geq 0, \qquad v \in \mathscr{D}(A),$$

or

$$\langle v, Fv\rangle \geq 0.$$

Therefore, F is positive. For $v \in \mathscr{D}(T)$, which is contained in $\mathscr{D}(F)$, using the fact that $S \subset A$,

$$Fv = Av - v = Sv - v = v + Tv - v = Tv.$$

Therefore,

$$T \subset F.$$

We have established that T is self-adjoint and positive, and thus F is the operator we wish to obtain. $\square$