Gaussian Hilbert spaces

Jordan Bell
jordan.bell@gmail.com
Department of Mathematics, University of Toronto

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1 Gaussian measures

Let γ be a Borel probability measure on $\mathbb{R}$. For $a \in \mathbb{R}$, if $\gamma = \delta_a$ then we call γ a **Gaussian measure with mean** a **and variance** 0. If $\sigma > 0$ and γ has density

$$p(t, a, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(t-a)^2}{2\sigma^2}\right), \qquad t \in \mathbb{R},$$

with respect to Lebesgue measure λ_1 on $\mathbb{R}$, then we call γ a **Gaussian measure with mean** a **and variance** σ^2.

A Borel probability measure γ on $\mathbb{R}^n$ is called **Gaussian** if for each $\xi \in (\mathbb{R}^n)^*$, the pushforward measure $\xi_*\gamma$ is a Gaussian measure on $\mathbb{R}$. The characteristic function of a Borel probability measure μ on $\mathbb{R}^n$ is

$$\widetilde{\mu}(y) = \int_{\mathbb{R}^n} e^{i\langle y,x\rangle} d\mu(x), \qquad y \in \mathbb{R}^n.$$

We call a linear operator $C \in \mathscr{L}(\mathbb{R}^n)$ **positive** when $\langle Cx, x\rangle \geq 0$ for all $x \in \mathbb{R}^n$.[1] It can be proved[2] that a Borel probability measure γ on $\mathbb{R}^n$ is Gaussian if and only if there is some $a \in \mathbb{R}^n$ and some positive self-adjoint $C \in \mathscr{L}(\mathbb{R}^n)$ such that

$$\widetilde{\gamma}(y) = \exp\left(i\langle y, a\rangle - \frac{1}{2}\langle Cy, y\rangle\right), \qquad y \in \mathbb{R}^n.$$

We say that γ has **mean** a and **covariance operator** C. If C is invertible (which is equivalent to $\langle Cx, x\rangle > 0$ for all nonzero $x \in \mathbb{R}^n$), then the density of γ with respect to Lebesgue measure λ_n on $\mathbb{R}^n$ is

$$x \mapsto \frac{1}{\sqrt{(2\pi)^n \det C}} \exp\left(-\frac{1}{2}\langle C^{-1}(x-a), x-a\rangle\right), \qquad \mathbb{R}^n \to \mathbb{R}.$$

[1] We remark that $\mathbb{R}^n$ is a real Hilbert space, and differently than a complex Hilbert space it need not be true that a positive linear operator is self-adjoint.

[2] http://individual.utoronto.ca/jordanbell/notes/gaussian.pdf, Theorem 5.

The **standard Gaussian measure on** $\mathbb{R}^n$, denoted γ_n, is the Gaussian measure on $\mathbb{R}^n$ with mean 0 and covariance operator I:

$$d\gamma_n(x) = (2\pi)^{-n/2} \exp\left(-\frac{1}{2}\langle x, x\rangle\right) d\lambda_n(x),$$

where λ_n is Lebesgue measure on $\mathbb{R}^n$. Throughout the remainder of this note, when we speak of Gaussian measures we assume unless we say otherwise that they have mean 0.

2 $\mathbb{R}^I$

Let I be the positive integers. For nonempty subsets J and K of I with $J \subset K$, let $\pi_{K,J} : \mathbb{R}^K \to \mathbb{R}^J$ be the projection map, and for $i \in I$ let $\pi_i = \pi_{I,\{i\}}$. For a topological space X, let $\mathscr{B}_X$ be the Borel σ-algebra of X.

If $B_i \in \mathscr{B}_{\mathbb{R}}$ for each $i \in I$ and $\{i \in I : B_i \neq \mathbb{R}\}$ is finite, we call

$$\prod_{i \in I} B_i \subset \mathbb{R}^I$$

a **cylinder set**. The σ-algebra generated by the collection of all cylinder sets is called the **product σ-algebra**, and is denoted by $\mathscr{B}_{\mathbb{R}}^I$. The product measure $\gamma_I = \prod_{i \in I} \gamma_1$ is the unique probability measure[3] on the product σ-algebra $\mathscr{B}_{\mathbb{R}}^I$ such that for each cylinder set $\prod_{i \in I} B_i$,

$$\gamma_I\left(\prod_{i \in I} B_i\right) = \prod_{i \in I} \gamma_1(B_i).$$

Because I is countable and $\mathbb{R}$ is a second-countable topological space, the Borel σ-algebra of $\mathbb{R}^I$, with the product topology, is equal to the product σ-algebra on $\mathbb{R}^I$:[4]

$$\mathscr{B}_{\mathbb{R}^I} = \mathscr{B}_{\mathbb{R}}^I.$$

Thus γ_I is a Borel probability measure on $\mathbb{R}^I$.

On the one hand, $\mathbb{R}^I$ is a real vector space. On the other hand, with the product topology it is a topological space. It can be proved that with the separating family of seminorms $\{|\pi_i| : i \in I\}$ it is a Fréchet space,[5] whose topology (the product topology) is induced by the complete translation invariant metric

$$d(x, y) = \sum_{i \in I} 2^{-i} \frac{|x_i - y_i|}{1 + |x_i - y_i|}, \qquad x, y \in \mathbb{R}^I.$$

We remark that because $\mathbb{R}^I$ is a countable product of second-countable topological spaces, it is itself a second-countable topological space, and so is separable.

[3] `http://individual.utoronto.ca/jordanbell/notes/productmeasure.pdf`

[4] `http://individual.utoronto.ca/jordanbell/notes/kolmogorov.pdf`, Theorem 7.

[5] Charalambos D. Aliprantis and Kim C. Border, *Infinite Dimensional Analysis: A Hitchhiker's Guide*, third ed., p. 207, Example 5.78.

The **dual space of** $\mathbb{R}^I$, denoted $(\mathbb{R}^I)^*$, is the collection of continuous linear maps $\mathbb{R}^I \to \mathbb{R}$. It turns out that the dual space $(\mathbb{R}^I)^*$ of $\mathbb{R}^I$ is equal to the collection of those $x \in \mathbb{R}^I$ such that $\{i \in I : \pi_i(x) \neq 0\}$ is finite,[6] with the dual pair

$$\langle x, y \rangle = \sum_{i \in I} x_i y_i, \qquad x \in \mathbb{R}^I, \quad y \in (\mathbb{R}^I)^*.$$

3 Gaussian Hilbert spaces

Because $\mathbb{R}^I$ is a second-countable topological space, its Borel σ-algebra $\mathscr{B}_{\mathbb{R}^I}$ is countably generated, and because γ_I is a probability measure on $\mathscr{B}_{\mathbb{R}^I}$ it is a fortiori σ-finite, so the real Hilbert space $L^2(\gamma_I)$ is separable.[7]

Let $\mathscr{H}$ be a real separable Hilbert space, with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, let e_i, $i \in I$, be an orthonormal basis for $\mathscr{H}$, and let V be the linear span of this basis. In particular, V is a dense linear subspace of $\mathscr{H}$. For $v \in V$, define

$$\phi(v) = \sum_{i \in I} \langle v, e_i \rangle \pi_i.$$

Using

$$\int_{\mathbb{R}} x_i d\gamma_1(x_i) = 0, \qquad \int_{\mathbb{R}} x_i^2 d\gamma_1(x_i) = 1,$$

we calculate[8]

$$\begin{aligned}
\|\phi(v)\|_{L^2(\gamma_I)}^2 &= \int_{\mathbb{R}^I} |\phi(v)(x)|^2 d\gamma_I(x) \\
&= \sum_{i \in I} \int_{\mathbb{R}^I} \langle v, e_i \rangle^2 \pi_i(x)^2 d\gamma_I(x) \\
&+ \sum_{i \in I} \int_{\mathbb{R}^I} \langle v, e_i \rangle \langle v, e_j \rangle \pi_i(x)\pi_j(x) d\gamma_I(x) \\
&= \sum_{i \in I} \langle v, e_i \rangle^2 \\
&= \|v\|^2,
\end{aligned}$$

showing that $\phi : V \to L^2(\gamma_I)$ is a linear isometry. Because (i) V is a dense linear subspace of $\mathscr{H}$, (ii) the operator $\phi : V \to L^2(\gamma_I)$ is bounded, and (iii) $L^2(\gamma_I)$ is a Hilbert space, there is a unique bounded linear operator $\Phi : \mathscr{H} \to L^2(\gamma_I)$ whose restriction to V is equal to ϕ.[9] For $v \in \mathscr{H}$ there is a sequence v_n in V that tends to v, and because Φ is continuous and $\|\cdot\| : \mathscr{H} \to \mathbb{R}$ is continuous,

$$\|\Phi(v)\| = \lim_{n \to \infty} \|\Phi(v_n)\| = \lim_{n \to \infty} \|\phi(v_n)\| = \lim_{n \to \infty} \|v_n\| = \|v\|,$$

[6] Charalambos D. Aliprantis and Kim C. Border, *Infinite Dimensional Analysis: A Hitchhiker's Guide*, third ed., p. 528, Theorem 16.3.

[7] Donald L. Cohn, *Measure Theory*, second ed., p. 102, Proposition 3.4.5.

[8] `http://individual.utoronto.ca/jordanbell/notes/parseval.pdf`

[9] Walter Rudin, *Functional Analysis*, second ed., p. 39, Exercise 19.

showing that Φ is a linear isometry.

Define $F : \mathscr{H} \to \mathbb{C}$ by

$$F(v) = \int_{\mathbb{R}^I} \exp(i\Phi(v)(x))d\gamma_I(x).$$

Because $|e^{is} - e^{it}| \leq |s-t|$, and using the Cauchy-Schwarz inequality, for $v, w \in \mathscr{H}$,

$$\begin{aligned}
|F(v) - F(w)| &\leq \int_{\mathbb{R}^I} |\exp(i\Phi(v)(x)) - \exp(i\Phi(w)(x))|\, d\gamma_I(x) \\
&\leq \int_{\mathbb{R}^I} |\Phi(v)(x) - \Phi(w)(x)| d\gamma_I(x) \\
&= \int_{\mathbb{R}^I} |\Phi(v-w)(x)| d\gamma_I(x) \\
&\leq \|\Phi(v-w)\|_{L^2(\gamma_I)} \\
&= \|v - w\|,
\end{aligned}$$

which shows in particular that F is continuous. For $v \in V$, let $I_v = \{i \in I : \langle v, e_i \rangle\} \neq 0$, which is finite. We calculate using Fubini's theorem

$$\begin{aligned}
F(v) &= \int_{\mathbb{R}^I} \exp\left(i \sum_{i \in I} \langle v, e_i \rangle \pi_i(x) \right) d\gamma_I(x) \\
&= \int_{\mathbb{R}^I} \prod_{i \in I} \exp(i \langle v, e_i \rangle \pi_i(x)) d\gamma_I(x) \\
&= \prod_{i \in I_v} \int_{\mathbb{R}} \exp(i \langle v, e_i \rangle t) d\gamma_1(t) \\
&= \prod_{i \in I_v} \widetilde{\gamma}_1(\langle v, e_i \rangle) \\
&= \prod_{i \in I_v} \exp\left(-\frac{1}{2} | \langle v, e_i \rangle |^2 \right) \\
&= \exp\left(-\frac{1}{2} \|v\|^2 \right).
\end{aligned}$$

For $v \in \mathscr{H}$, there is a sequence $v_n \in V$ that tends to v, and because F and $w \mapsto \exp\left(-\frac{1}{2}\|w\|^2\right)$ are continuous,

$$F(v) = \lim_{n \to \infty} F(v_n) = \lim_{n \to \infty} \exp\left(-\frac{1}{2} \|v_n\|^2 \right) = \exp\left(-\frac{1}{2} \|v\|^2 \right).$$

That is, for all $v \in \mathscr{H}$,

$$\int_{\mathbb{R}^I} \exp(i\Phi(v)(x)) d\gamma_I(x) = \exp\left(-\frac{1}{2} \|v\|^2 \right).$$

For distinct $v_1, \ldots, v_n \in \mathscr{H}$, write $X = \Phi(v_1) \otimes \cdots \otimes \Phi(v_n)$, which is measurable $\mathbb{R}^I \to \mathbb{R}^n$, and let μ be the pushforward measure of γ_I by X, namely the joint distribution of the random variables $\Phi(v_1), \ldots, \Phi(v_n)$. For $y \in \mathbb{R}^n$ we calculate using the change of variables theorem[10] and Fubini's theorem

$$\begin{aligned}
\widetilde{\mu}(y) &= \int_{\mathbb{R}^n} e^{i\langle y,u\rangle} d\mu(u) \\
&= \int_{\mathbb{R}^I} e^{i\langle y,X(x)\rangle} d\gamma_I(x) \\
&= \int_{\mathbb{R}^I} e^{i(y_1\Phi(v_1)(x)+\cdots+y_n\Phi(v_n)(x))} d\gamma_I(x) \\
&= \int_{\mathbb{R}^I} e^{i\Phi(y_1v_1+\cdots+y_nv_n)(x)} d\gamma_I(x) \\
&= \exp\left(-\frac{1}{2}\|y_1v_1+\cdots+y_nv_n\|^2\right) \\
&= \exp\left(-\frac{1}{2}\sum_{i,j}\langle v_i, v_j\rangle y_iy_j\right),
\end{aligned}$$

which shows that μ is a Gaussian measure on $\mathbb{R}^n$ with covariance matrix $C_{i,j} = \langle v_i, v_j\rangle$.[11] Thus $\{\Phi(v)\}_{v\in\mathscr{H}}$ is a stochastic process with sample space $(\mathbb{R}^I, \mathscr{B}_{\mathbb{R}^I}, \gamma_I)$, index set $\mathscr{H}$, and state space $\mathbb{R}$, which we call the **Gaussian process with covariance** $\langle\cdot,\cdot\rangle$.

Let T be a separable metric space and suppose that $c : T \times T \to \mathbb{R}$ is continuous and that for any $t_1, \ldots, t_n \in T$, $\{c(t_i, t_j)\}_{1\leq i,j\leq n}$ is a symmetric positive semidefinite matrix. For each $t \in T$ let δ_t be a formal symbol, and let V be the linear span of $\{\delta_t : t \in T\}$. For $v, w \in V$, there are distinct $t_1, \ldots, t_n \in T$ and real numbers $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n$ such that $v = \sum_{i=1}^n \alpha_i\delta_{t_i}$ and $w = \sum_{i=1}^n \beta_i\delta_{t_i}$, and we define

$$[v, w] = \sum_{1\leq i,j\leq n} \alpha_i\beta_j c(t_i, t_j).$$

For $a \in \mathbb{R}$,

$$[av, w] = \sum_{1\leq i,j\leq n} (a\alpha_i)\beta_j c(t_i, t_j) = a\sum_{1\leq i,j\leq n} \alpha_i\beta_j c(t_i, t_j) = a[v, w].$$

For $u, v, w \in V$ there are distinct $t_1, \ldots, t_n \in T$ and real numbers

$$\alpha_1, \ldots, \alpha_n, \qquad \beta_1, \ldots, \beta_n, \qquad \gamma_1, \ldots, \gamma_n$$

such that $v = \sum_{i=1}^n \alpha_i\delta_{t_i}$, $w = \sum_{i=1}^n \beta_i\delta_{t_i}$, $u = \sum_{i=1}^n \gamma_i\delta_{t_i}$, and

$$[u + v, w] = \sum_{1\leq i,j\leq n} (\alpha_i + \gamma_i)\beta_j c(t_i, t_j) = [u, w] + [v, w].$$

[10] Charalambos D. Aliprantis and Kim C. Border, *Infinite Dimensional Analysis: A Hitchhiker's Guide*, third ed., p. 484, Theorem 13.46.

[11] See Barry Simon, *Functional Integration and Quantum Physics*, p. 16, Theorem 2.3A.

Because $\{c(t_i,t_j)\}_{1\le i,j\le n}$ is symmetric, $[v,w]=[w,v]$. Finally, because the matrix $\{c(t_i,t_j)\}_{1\le i,j\le n}$ is positive semidefinite,

$$[v,v]=\sum_{1\le i,j\le n}\alpha_i\alpha_j c(t_i,t_j)\ge 0.$$

This establishes that $[\cdot,\cdot]$ is a positive semidefinite inner product on V. Then $|v|=\sqrt{[v,v]}$,

$$|v|^2=\sum_{1\le i,j\le n}\alpha_i\alpha_j c(t_i,t_j),$$

is a seminorm on V, and $N=\{v\in V:|v|=0\}$ is a closed linear subspace of V. Let

$$V/N=\{v+N:v\in V\},$$

For $v,w\in V$ and $r,s\in N$, because $|r|=0$ and $|s|=0$, by the Cauchy-Schwarz inequality (which is indeed true for a positive semidefinite inner product)[12] we have $|[v,s]|^2\le|v||s|=0$ and $|[r,w]|^2\le|r||w|=0$, hence

$$[v+r,w+s]=[v,w]+[v,s]+[r,w]+[r,s]=[v,w].$$

Therefore it makes sense to define

$$\langle v+N,w+N\rangle=[v,w].$$

If $\langle v+N,w+N\rangle=0$ then $[v,v]=0$, i.e. $|v|=0$ and so $v\in N$, i.e. $v+N=0\in V/N$, and therefore $\langle\cdot,\cdot\rangle$ is an inner product on V/N. Then there is a Hilbert space $(\mathscr{H},\langle\cdot,\cdot\rangle)$ and a linear isometry $i:V/N\to\mathscr{H}$ whose image is a dense subset of $\mathscr{H}$, called a **completion** of V/H, and this completion is unique up to a unique isomorphism of Hilbert spaces.[13]

T is separable so it has a countable dense subset S. For $t\in T$, either $t\in S$ or $t\notin S$. If $t\notin S$, there is a sequence of distinct s_k in S that tends to t, and because c is continuous,

$$|\delta_{s_k}-\delta_t|^2=c(s_k,s_k)-c(s_k,t)-c(t,s_k)+c(t,t)\to c(t,t)-c(t,t)-c(t,t)+c(t,t),$$

so $\delta_{s_k}\to\delta_t$ in V, which shows that $\delta:T\to V$ is continuous. Let W be the $\mathbb{Q}$-linear span of $\{\delta_s:s\in S\}$. W is countable, and it follows from the above that W is dense in V. Define $\pi:V\to V/N$ by $\pi(v)=v+N$, which is an onto continuous linear map. Then the image of W under $i\circ\pi:V\to\mathscr{H}$ is dense in $\mathscr{H}$, thus $(\mathscr{H},\langle\cdot,\cdot\rangle)$ is a separable Hilbert space.

Now let $\{\Phi(v)\}_{v\in\mathscr{H}}$ be the **Gaussian process with covariance** $\langle\cdot,\cdot\rangle$, and define $q:T\to L^2(\gamma_I)$ by

$$q=\Phi\circ i\circ\pi\circ\delta.$$

[12] A. Ya. Helemskii, *Lectures and Exercises on Functional Analysis*, p. 68, Theorem 1.
[13] A. Ya. Helemskii, *Lectures and Exercises on Functional Analysis*, p. 172, Proposition 3.

For distinct $t_1, \ldots, t_n \in T$, the vectors $v_j = (i \circ \pi \circ \delta)(t_j) \in \mathscr{H}$, $1 \le j \le n$, are distinct. Then $(\Phi(v_1) \otimes \cdots \otimes \Phi(v_n))_* \gamma_I$ is the Gaussian measure on $\mathbb{R}^n$ with covariance matrix

$$C_{i,j} = \langle v_i, v_j \rangle = \langle (i \circ \pi \circ \delta)(t_i), (i \circ \pi \circ \delta)(t_j) \rangle = [\delta_{t_i}, \delta_{t_j}] = c(t_i, t_j).$$

That is, the joint distribution of the random variables $q_{t_1}, \ldots, q_{t_n}$ is the Gaussian measure on $\mathbb{R}^n$ with covariance matrix $C_{i,j} = c(t_i, t_j)$.