

The Gauss map

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1 Koopman operator

For a function $T : X \rightarrow X$ and for a function $f : X \rightarrow \mathbb{C}$, define

$$C_T f = f \circ T.$$ 

We call $C_T$ the Koopman operator of $T$. For $x \in X$ and $j \geq 0$, $(C^j_T f)(x) = (f \circ T^j)(x)$.

Let $\mathcal{A}$ be a $\sigma$-algebra on a set $X$ and let $\mu$ be a probability measure on $\mathcal{A}$. For a measurable function $T : X \rightarrow X$, let $T^* \mu$ be the pushforward of $\mu$ by $T$:

$$(T^* \mu)(E) = \mu(T^{-1}(E)).$$

2 Transfer operator

Let $(X, \mathcal{A}, \mu)$ be a probability space and let $T : X \rightarrow X$ be measurable. Denote by $T^* \mu$ the pushforward of $\mu$ by $T$. We call $T$ nonsingular if $T^* \mu$ be absolutely continuous with respect to $\mu$. For $f \in L^1(\mu)$ let $\mu_f$ be the measure on $\mathcal{A}$ whose Radon-Nikodym derivative with respect to $\mu$ is $f$: $d\mu_f = f d\mu$. The transfer operator of $T$ is $L_T : L^1(\mu) \rightarrow L^1(\mu)$ defined by $L_T f = \frac{d(T^* \mu_f)}{d\mu}$. Thus for $g \in L^\infty(\mu)$,

$$\int_X g \cdot L_T f d\mu = \int_X gd(T^* \mu_f)$$

$$= \int_X g \circ T d\mu_f$$

$$= \int_X (g \circ T) \cdot f d\mu$$

$$= \int_X f \cdot C_T g d\mu.$$ 

We remark that we merely suppose $T$ be nonsingular, not that $T$ be measure preserving.
3 Gauss map

Let $\mathcal{B}$ be the Borel $\sigma$-algebra of the compact metric space $[0, 1]$ and let $\mu$ be Lebesgue measure on $\mathcal{B}$. For $x \in \mathbb{R}$ let $\lfloor x \rfloor$ be the greatest integer $\leq x$ and let $\{x\} = x - \lfloor x \rfloor$, for which $\{x\} \in [0, 1)$. Define $T: [0, 1] \to [0, 1]$ by

$$T(x) = \begin{cases} 0 & x = 0, \\ \{1/x\} & x \neq 0, \end{cases}$$

called the Gauss map. Let

$$I_k = \left( \frac{1}{k+1}, \frac{1}{k} \right), \quad k \geq 1.$$  

For $k \geq 1$, if $x \in I_k$ then $\lfloor 1/x \rfloor = k$ so $\{1/x\} = \frac{1}{x} - k$. Thus

$$T(x) = \sum_{k=1}^{\infty} 1_{I_k}(x)(x^{-1} - k).$$

For $F = \{0\} \cup \{k^{-1} : k \geq 1\}$, $U = [0, 1] \setminus F = \bigcup_{k \geq 1} I_k$, and for $x \in U$,

$$T'(x) = -\sum_{k=1}^{\infty} 1_{I_k}(x)x^{-2}.$$  

It is apparent that $T \in C^\infty(U)$. Now, for $x \in I_k$, $k^2 < |T'(x)| < (k+1)^2$. Define $\phi_k : (0, 1) \to I_k$ by

$$\phi_k(x) = \frac{1}{x + k},$$

which is a diffeomorphism. For $x \in (0, 1)$ and $k \geq 1$,

$$(T \circ \phi_k)(x) = \frac{1}{\phi_k(x)} - k = x + k - k = x.$$  

For $f : [0, 1] \to \mathbb{C}$ and $\mu_f(A) = \int_A f \, d\mu$, and for $A$ an open subset of $[0, 1]$,

$$\mu_f(T^{-1}A) = \int_{T^{-1}(A)} f \, d\mu = \sum_{k=1}^{\infty} \int_{\phi_k(A)} f \, d\mu.$$  

But for $k \geq 1$, using the change of variables formula and $\phi'_k(x) = -\frac{1}{(x+k)^2}$,

$$\int_{\phi_k(A)} f \, d\mu = \int_A (f \circ \phi_k)(x) \cdot |\phi'_k(x)| \, dx = \int_A f \left( \frac{1}{x+k} \right) \frac{1}{(x+k)^2} \, dx;$$

2
we will impose some conditions on $f$ after we play around with things. Then

$$\mu_f(T^{-1} A) = \sum_{k=1}^{\infty} \int_A f\left(\frac{1}{x+k}\right) \frac{1}{(x+k)^2} dx = \int_A \sum_{k=1}^{\infty} f\left(\frac{1}{x+k}\right) \frac{1}{(x+k)^2} dx.$$ 

Define $\mathcal{G} f : [0, 1] \to \mathbb{C}$ by

$$(\mathcal{G} f)(x) = \sum_{k=1}^{\infty} f\left(\frac{1}{x+k}\right) \frac{1}{(x+k)^2}.$$ 

We call $\mathcal{G}$ the **Gauss-Kuzmin-Wirsing operator**. If we want $T$ to preserve the measure $\mu_f$ then it must be the case that $\mathcal{G} f = f$ almost everywhere. In fact, for $f(x) = c(1+x)^{-1}$, $c > 0$, 

$$\begin{align*}
(\mathcal{G} f)(x) &= c \sum_{k=1}^{\infty} \frac{1}{1 + \frac{1}{x+k}} \frac{1}{(x+k)^2} \\
&= c \sum_{k=1}^{\infty} \frac{1}{(x+k+1)(x+k)} \\
&= c \sum_{k=1}^{\infty} \left(\frac{1}{x+k} - \frac{1}{x+k+1}\right) \\
&= c \frac{1}{x+1} \\
&= f(x).
\end{align*}$$

Now, $\mu_f$ being a probability measure is equivalent with $\mu_f([0, 1]) = 1$, i.e.

$$1 = \mu_f([0, 1]) = c \int_0^1 \frac{1}{x+1} dx = c \log 2.$$ 

Thus take $c = \frac{1}{\log 2}$, $f(x) = \frac{1}{(1+x)\log 2}$. Then $d\nu(x) = \frac{1}{(1+x)\log 2} d\mu(x)$ is a probability measure for which the Gauss map $T$ is measure preserving. We call $\nu$ the **Gauss measure**.

### 4 Dynamical zeta function

Let $M$ be a set, let $f : M \to M$ be a function, and

$$\text{Fix } f = \{ x \in M : fx = x \}.$$ 

Let $M_d(\mathbb{C})$ be the set of $n \times n$ matrices over $\mathbb{C}$. For $m \geq 1$, if Fix $f^m$ is finite let

$$a_m = \sum_{x \in \text{Fix } f^m} \text{tr} \prod_{k=0}^{m-1} \phi(f^k x).$$
If each \( \text{Fix} f^m \) is finite, then define
\[
\zeta(f, \phi, z) = \sum_{m=1}^{\infty} \frac{a_m}{m} z^m,
\]
where the series converges.

## 5 Continued fractions

For irrational \( x \in [0, 1] \) let \( a_n(x) \) be the \( n \)th partial quotient of its continued fraction. It satisfies
\[
a_n(x) = \left\lfloor \frac{1}{T_n-1}x \right\rfloor, \quad n \geq 1.
\]

For positive integers \( a_1, \ldots, a_m \), let \( x = [a_1, \ldots, a_m] \) satisfy \( a_1(x) = a_1, \ldots, a_m(x) = a_m \) and \( a_{j+m}(x) = a_j(x) \). Namely, \( [a_1, \ldots, a_m] \) is a purely periodic continued fraction. For \( m \geq 1 \),
\[
\text{Fix}(T^m) = \{0\} \cup \{[a_1, \ldots, a_m] : a_1, \ldots, a_m \geq 1\}.
\]

We remark that a nonzero element of \( \text{Fix}(T^m) \) is a quadratic irrational.

For \( m \geq 1 \) and for positive integers \( a_1, \ldots, a_m \) define
\[
w[a_1, \ldots, a_m] = \prod_{k=1}^{m} ([a_k, \ldots, a_m, a_1, \ldots, a_{k-1}])^{-2}.
\]

Define
\[
Z_m(s) = \sum_{(a_1, \ldots, a_m) \in \mathbb{Z}_{\geq 1}^m} (w[a_1, \ldots, a_m])^{-s}.
\]

When it converges, define
\[
\zeta_{CF}(z, s) = \exp \left( \sum_{m=1}^{\infty} \frac{z^m}{m} Z_m(s) \right).
\]

Let \( \Delta = \{ z \in \mathbb{C} : |z-1| < \frac{3}{2} \} \). Let \( X \) be the collection of continuous functions \( \phi : \Delta \to \mathbb{C} \) whose restriction to \( \Delta \) is holomorphic.