The Gauss map

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1 Koopman operator

For a function $T : X \to X$ and for a function $f : X \to \mathbb{C}$, define

$$C_T f = f \circ T.$$

We call C_T the **Koopman operator of** T. For $x \in X$ and $j \geq 0$, $(C_T^j f)(x) = (f \circ T^j)(x)$.

Let $\mathscr{A}$ be a σ-algebra on a set X and let μ be a probability measure on $\mathscr{A}$. For a measurable function $T : X \to X$, let $T_* \mu$ be the pushforward of μ by T:

$$(T_* \mu)(E) = \mu(T^{-1}(E)).$$

2 Transfer operator

Let $(X, \mathscr{A}, \mu)$ be a probability space and let $T : X \to X$ be measurable. Denote by $T_* \mu$ the pushforward of μ by T. We call T **nonsingular** if $T_* \mu$ be absolutely continuous with respect to μ. For $f \in L^1(\mu)$ let μ_f be the measure on $\mathscr{A}$ whose Radon-Nikodym derivative with respect to μ is f: $d\mu_f = f d\mu$. The **transfer operator of** T is $L_T : L^1(\mu) \to L^1(\mu)$ defined by $L_T f = \frac{d(T_* \mu_f)}{d\mu}$. Thus for $g \in L^\infty(\mu)$,

$$\begin{aligned}
\int_X g \cdot L_T f d\mu &= \int_X g d(T_* \mu_f) \\
&= \int_X g \circ T d\mu_f \\
&= \int_X (g \circ T) \cdot f d\mu \\
&= \int_X f \cdot C_T g d\mu.
\end{aligned}$$

We remark that we merely suppose T be nonsingular, not that T be measure preserving.

3 Gauss map

Let $\mathscr{B}$ be the Borel σ-algebra of the compact metric space $[0,1]$ and let μ be Lebesgue measure on $\mathscr{B}$. For $x \in \mathbb{R}$ let $[x]$ be the greatest integer $\leq x$ and let $\{x\} = x - [x]$, for which $\{x\} \in [0,1)$. Define $T : [0,1] \to [0,1]$ by

$$T(x) = \begin{cases} 0 & x = 0, \\ \{1/x\} & x \neq 0, \end{cases}$$

called the **Gauss map**. Let

$$I_k = \left(\frac{1}{k+1}, \frac{1}{k} \right), \qquad k \geq 1.$$

For $k \geq 1$, if $x \in I_k$ then $[1/x] = k$ so $\{1/x\} = \frac{1}{x} - k$. Thus

$$T(x) = \sum_{k=1}^{\infty} 1_{I_k}(x)(x^{-1} - k).$$

For

$$F = \{0\} \cup \{k^{-1} : k \geq 1\}, \qquad U = [0,1] \setminus F = \bigcup_{k \geq 1} I_k,$$

and for $x \in U$,

$$T'(x) = -\sum_{k=1}^{\infty} 1_{I_k}(x) x^{-2}.$$

It is apparent that $T \in C^\infty(U)$. Now, for $x \in I_k$, $k^2 < |T'(x)| < (k+1)^2$. Define $\phi_k : (0,1) \to I_k$ by

$$\phi_k(x) = \frac{1}{x+k},$$

which is a diffeomorphism. For $x \in (0,1)$ and $k \geq 1$,

$$(T \circ \phi_k)(x) = \frac{1}{\phi_k(x)} - k = x + k - k = x.$$

For $f : [0,1] \to \mathbb{C}$ and $\mu_f(A) = \int_A f d\mu$, and for A an open subset of $[0,1]$,

$$\mu_f(T^{-1}A) = \int_{T^{-1}(A)} f d\mu = \sum_{k=1}^{\infty} \int_{\phi_k(A)} f d\mu.$$

But for $k \geq 1$, using the change of variables formula and $\phi_k'(x) = -\frac{1}{(x+k)^2}$,

$$\int_{\phi_k(A)} f d\mu = \int_A (f \circ \phi_k)(x) \cdot |\phi_k'(x)| dx = \int_A f\left(\frac{1}{x+k} \right) \frac{1}{(x+k)^2} dx;$$

we will impose some conditions on f after we play around with things. Then

$$\mu_f(T^{-1}A) = \sum_{k=1}^{\infty} \int_A f\left(\frac{1}{x+k}\right) \frac{1}{(x+k)^2} dx = \int_A \sum_{k=1}^{\infty} f\left(\frac{1}{x+k}\right) \frac{1}{(x+k)^2} dx.$$

Define $\mathscr{G} f : [0,1] \to \mathbb{C}$ by

$$(\mathscr{G} f)(x) = \sum_{k=1}^{\infty} f\left(\frac{1}{x+k}\right) \frac{1}{(x+k)^2}.$$

We call $\mathscr{G}$ the **Gauss-Kuzmin-Wirsing operator**. If we want T to preserve the measure μ_f then it must be the case that $\mathscr{G} f = f$ almost everywhere. In fact, for $f(x) = c(1+x)^{-1}$, $c > 0$,

$$\begin{aligned}
(\mathscr{G} f)(x) &= c \sum_{k=1}^{\infty} \frac{1}{1 + \frac{1}{x+k}} \frac{1}{(x+k)^2} \\
&= c \sum_{k=1}^{\infty} \frac{1}{(x+k+1)(x+k)} \\
&= c \sum_{k=1}^{\infty} \left(\frac{1}{x+k} - \frac{1}{x+k+1} \right) \\
&= c \frac{1}{x+1} \\
&= f(x).
\end{aligned}$$

Now, μ_f being a probability measure is equivalent with $\mu_f([0,1]) = 1$, i.e.

$$1 = \mu_f([0,1]) = c \int_0^1 \frac{1}{x+1} dx = c \log 2.$$

Thus take $c = \frac{1}{\log 2}$, $f(x) = \frac{1}{(1+x)\log 2}$. Then $d\nu(x) = \frac{1}{(1+x)\log 2} d\mu(x)$ is a probability measure for which the Gauss map T is measure preserving. We call ν the **Gauss measure**.

4 Dynamical zeta function

Let M be a set, let $f : M \to M$ be a function, and

$$\mathrm{Fix}\, f = \{x \in M : fx = x\}.$$

Let $M_d(\mathbb{C})$ be the set of $n \times n$ matrices over $\mathbb{C}$. For $m \geq 1$, if $\mathrm{Fix}\, f^m$ is finite let

$$a_m = \sum_{x \in \mathrm{Fix}\, f^m} \mathrm{tr} \prod_{k=0}^{m-1} \phi(f^k x).$$

If each $\mathrm{Fix}\, f^m$ is finite, then define

$$\zeta(f, \phi, z) = \sum_{m=1}^{\infty} \frac{a_m}{m} z^m,$$

where the series converges.

5 Continued fractions

For irrational $x \in [0, 1]$ let $a_n(x)$ be the nth partial quotient of its continued fraction. It satisfies

$$a_n(x) = \left[\frac{1}{T^{n-1}x} \right], \qquad n \geq 1.$$

For positive integers $a_1, \ldots, a_m$, let $x = [a_1, \ldots, a_m]$ satisfy $a_1(x) = a_1, \ldots, a_m(x) = a_m$ and $a_{j+m}(x) = a_j(x)$. Namely, $[a_1, \ldots, a_m]$ is a **purely periodic continued fraction**. For $m \geq 1$,

$$\mathrm{Fix}\,(T^m) = \{0\} \cup \{[a_1, \ldots, a_m] : a_1, \ldots, a_m \geq 1\}.$$

We remark that a nonzero element of $\mathrm{Fix}\,(T^m)$ is a quadratic irrational.

For $m \geq 1$ and for positive integers $a_1, \ldots, a_m$ define

$$w[a_1, \ldots, a_m] = \prod_{k=1}^{m} ([a_k, \ldots, a_m, a_1, \ldots, a_{k-1}])^{-2}.$$

Define

$$Z_m(s) = \sum_{(a_1, \ldots, a_m) \in \mathbb{Z}_{\geq 1}^m} (w[a_1, \ldots, a_m])^{-s}.$$

When it converges, define

$$\zeta_{CF}(z, s) = \exp\left(\sum_{m=1}^{\infty} \frac{z^m}{m} Z_m(s) \right).$$

Let $\Delta = \{z \in \mathbb{C} : |z - 1| < \frac{3}{2}\}$. Let X be the collection of continuous functions $\phi : \overline{\Delta} \to \mathbb{C}$ whose restriction to Δ is holomorphic.