# Hausdorff measure

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## 1 Outer measures and metric outer measures

Suppose that $X$ is a set. A function $\nu : \mathscr{P}(X) \to [0,\infty]$ is said to be an **outer measure** if (i) $\nu(\emptyset) = 0$, (ii) $\nu(A) \leq \nu(B)$ when $A \subset B$, and, (iii) for any countable collection $\{A_j\} \subset \mathscr{P}(X)$,

$$\nu\left(\bigcup_{j=1}^{\infty} A_j\right) \leq \sum_{j=1}^{\infty} \nu(A_j).$$

We say that a subset $A$ of $X$ is **$\nu$-measurable** if

$$\nu(E) = \nu(E \cap A) + \nu(E \cap A^c), \qquad E \in \mathscr{P}(X). \tag{1}$$

Here, instead of taking a $\sigma$-algebra as given and then defining a measure on this $\sigma$-algebra (namely, on the measurable sets), we take an outer measure as given and then define measurable sets using this outer measure. **Carathéodory's theorem**[1] states that the collection $\mathscr{M}$ of $\nu$-measurable sets is a $\sigma$-algebra and that the restriction of $\nu$ to $\mathscr{M}$ is a complete measure.

Suppose that $(X,\rho)$ is a metric space. An outer measure $\nu$ on $X$ is said to be a **metric outer measure** if

$$\rho(A,B) = \inf\{\rho(a,b) : a \in A, b \in B\} > 0$$

implies that

$$\nu(A \cup B) = \nu(A) + \nu(B).$$

We prove that the Borel sets are $\nu$-measurable.[2] That is, we prove that the Borel $\sigma$-algebra is contained in the $\sigma$-algebra of $\nu$-measurable sets.

**Theorem 1.** *If $\nu$ is a metric outer measure on a metric space $(X,\rho)$, then every Borel set is $\nu$-measurable.*

[1] Gerald B. Folland, *Real Analysis*, second ed., p. 29, Theorem 1.11.
[2] Gerald B. Folland, *Real Analysis*, second ed., p. 349, Proposition 11.16.

*Proof.* Because $\nu$ is an outer measure, by Carathéodory's theorem the collection $\mathscr{M}$ of $\nu$-measurable sets is a $\sigma$-algebra, and hence to prove that $\mathscr{M}$ contains the Borel $\sigma$-algebra it suffices to prove that $\mathscr{M}$ contains all the closed sets. Let $F$ be a closed set in $X$, and let $E$ be a subset of $X$. Because $\nu$ is an outer measure,

$$\nu(E) = \nu((E \cap F) \cup (E \cap F^c)) \leq \nu(E \cap F) + \nu(E \cap F^c).$$

In the case $\nu(E) = \infty$, certainly $\nu(E) \geq \nu(E \cap F) + \nu(E \cap F^c)$. In the case $\nu(E) < \infty$, for each $n$ let

$$E_n = \{x \in E \setminus F : \rho(x, F) \geq n^{-1}\},$$

which satisfies $\rho(E_n, F) \geq n^{-1}$. Because $\rho(E_n, E \cap F) \geq \rho(E_n, F) \geq n^{-1}$, the fact that $\nu$ is a metric outer measure tells us that

$$\nu((E \cap F) \cup E_n) = \nu(E \cap F) + \nu(E_n). \tag{2}$$

Because $F$ is closed, for any $x \in E \setminus F$ we have $\rho(x, F) > 0$, and hence

$$E \setminus F = \bigcup_{n=1}^{\infty} E_n. \tag{3}$$

Therefore

$$E = (E \cap F) \cup (E \cap F^c) = (E \cap F) \cup \bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} ((E \cap F) \cup E_n),$$

hence for each $n$, using this and (2) we have

$$\nu(E) \geq \nu((E \cap F) \cup E_n) = \nu(E \cap F) + \nu(E_n).$$

To prove that $\nu(E) \geq \nu(E \cap F) + \nu(E \cap F^c)$, it now suffices to prove that

$$\lim_{n \to \infty} \nu(E_n) = \nu(E \cap F^c).$$

Let $D_n = E_{n+1} \setminus E_n$. For $x \in D_{n+1}$ and $y \in X$ satisfying $\rho(x, y) < ((n+1)n)^{-1}$, we have

$$\rho(y, F) \leq \rho(x, y) + \rho(x, F) < \frac{1}{n(n+1)} + \frac{1}{n+1} = \frac{1}{n},$$

which implies that $y \notin E_n$. Thus,

$$\rho(D_{n+1}, E_n) \geq \frac{1}{n(n+1)}. \tag{4}$$

For any $n$, using (4) and the fact that $\nu$ is a metric outer measure,

$$\begin{aligned}
\nu(E_{2n+1}) &= \nu(D_{2n} \cup E_{2n}) \\
&\geq \nu(D_{2n} \cup E_{2n-1}) \\
&= \nu(D_{2n}) + \nu(E_{2n-1}) \\
&\geq \cdots \\
&= \nu(D_{2n}) + \nu(D_{2n-2}) + \cdots + \nu(D_2) + \nu(E_1) \\
&\geq \sum_{j=1}^{n} \nu(D_{2j}),
\end{aligned}$$

and

$$\begin{aligned}
\nu(E_{2n}) &= \nu(D_{2n-1} \cup E_{2n-1}) \\
&\geq \nu(D_{2n-1} \cup E_{2n-2}) \\
&= \nu(D_{2n-1}) + \nu(E_{2n-2}) \\
&\geq \cdots \\
&= \nu(D_{2n-1}) + \nu(D_{2n-3}) + \cdots + \nu(D_3) + \nu(D_1) + \nu(E_0) \\
&= \sum_{j=1}^{n} \nu(D_{2j-1}).
\end{aligned}$$

But $E_n \subset E$ so $\nu(E_n) \leq \nu(E)$, and hence each of the series $\sum_{j=1}^{\infty} \nu(D_{2j})$ and $\sum_{j=1}^{\infty} \nu(D_{2j-1})$ converges to a value $\leq \nu(E)$. Thus the series $\sum_{j=1}^{\infty} \nu(D_j)$ converges to a value $\leq 2\nu(E)$. But for any $n$,

$$\nu(E \setminus F) = \nu\left(E_n \cup \bigcup_{j=n}^{\infty} D_j\right) \leq \nu(E_n) + \sum_{j=n}^{\infty} \nu(D_j).$$

Because the series $\sum_{j=1}^{\infty} \nu(D_j)$ converges, the sum on the right-hand side of the above tends to 0 as $n \to \infty$, so

$$\nu(E \setminus F) \leq \liminf_{n\to\infty} \nu(E_n) \leq \limsup_{n\to\infty} \nu(E_n) \leq \nu(E \setminus F);$$

the last inequality is due to (3), which tells us $\nu(E_n) \leq \nu(E \setminus F)$. Therefore,

$$\lim_{n\to\infty} \nu(E_n) = \nu(E \setminus F) = \nu(E \cap F^c),$$

which completes the proof. □

We shall use the following.[3]

[3]Gerald B. Folland, *Real Analysis*, second ed., p. 29, Proposition 1.10.

**Lemma 2.** *Let $(X, \rho)$ be a metric space. Suppose that $\mathscr{E} \subset \mathscr{P}(X)$ satisfies $\emptyset, X \in \mathscr{E}$ and that $d : \mathscr{E} \to [0, \infty]$ satisfies $d(\emptyset) = 0$. Then the function $\nu : \mathscr{P}(X) \to [0, \infty]$ defined by*

$$\nu(A) = \inf \left\{ \sum_{j=1}^{\infty} d(E_j) : E_j \in \mathscr{E} \text{ and } A \subset \bigcup_{j=1}^{\infty} E_j \right\}, \qquad A \in \mathscr{P}(X)$$

*is an outer measure.*

We remark that if there is no covering of a set $A$ by countably many elements of $\mathscr{E}$ then $\nu(A)$ is an infinimum of an empty set and is thus equal to $\infty$.

## 2 Hausdorff measure

Suppose that $(X, \rho)$ is a metric space and let $p \geq 0$, $\delta > 0$. Let $\mathscr{E}$ be the collection of those subsets of $X$ with diameter $\leq \delta$ together with the set $X$, and define $d(A) = (\operatorname{diam} A)^p$. By Lemma 2, the function $H_{p,\delta} : \mathscr{P}(X) \to [0, \infty]$ defined by

$$H_{p,\delta}(A) = \inf \left\{ \sum_{j=1}^{\infty} d(E_j) : E_j \in \mathscr{E} \text{ and } A \subset \bigcup_{j=1}^{\infty} E_j \right\}, \qquad A \in \mathscr{P}(X)$$

is an outer measure. If $\delta_1 \leq \delta_2$ then $H_{p,\delta_1}(A) \geq H_{p,\delta_2}(A)$, from which it follows that for each $A \in \mathscr{P}(X)$, as $\delta$ tends to 0, $H_{p,\delta}(A)$ tends to some element of $[0, \infty]$. We define $H_p = \lim_{\delta \to 0} H_{p,\delta}$ and show that this is a metric outer measure.[4]

**Theorem 3.** *Suppose that $(X, \rho)$ is a metric space and let $p \geq 0$. Then $H_p : \mathscr{P}(X) \to [0, \infty]$ defined by*

$$H_p(A) = \lim_{\delta \to 0} H_{p,\delta}(A), \qquad A \in \mathscr{P}(X).$$

*is a metric outer measure.*

*Proof.* First we establish that $H_p$ is an outer measure. It is apparent that $H_p(\emptyset) = 0$. If $A \subset B$, then, using that $H_{p,\delta}$ is a metric outer measure,

$$H_p(A) = \lim_{\delta \to 0} H_{p,\delta}(A) \leq \lim_{\delta \to 0} H_{p,\delta}(B) = H_p(B).$$

[4]Gerald B. Folland, *Real Analysis*, second ed., p. 350, Proposition 11.17.

If $\{A_j\} \subset \mathscr{P}(X)$ is countable then, using that $H_{p,\delta}$ is a metric outer measure,

$$\begin{aligned} H_p\left(\bigcup_{j=1}^{\infty} A_j\right) &= \lim_{\delta\to 0} H_{p,\delta}\left(\bigcup_{j=1}^{\infty} A_j\right) \\ &\leq \lim_{\delta\to 0} \sum_{j=1}^{\infty} H_{p,\delta}(A_j) \\ &= \sum_{j=1}^{\infty} \lim_{\delta\to 0} H_{p,\delta}(A_j) \\ &= \sum_{j=1}^{\infty} H_p(A_j). \end{aligned}$$

Hence $H_p$ is an outer measure.

To obtain that $H_p$ is a metric outer measure, we must show that if $\rho(A,B) > 0$ then $H_p(A \cup B) \geq H_p(A) + H_p(B)$. Let $0 < \delta < \rho(A,B)$ and let $\mathscr{E}$ be the collection of those subsets of $X$ with diameter $\leq \delta$ together with the set $X$. If there is no covering of $A \cup B$ by countably many elements of $\mathscr{E}$, then $H_p(A \cup B) \geq H_{p,\delta}(A \cup B) = \infty$. Otherwise, let $\{E_j\} \subset \mathscr{E}$ be a covering of $A \cup B$. For each $j$, because $\operatorname{diam} E_j \leq \delta < \rho(A,B)$, it follows that $E_j$ does not intersect both $A$ and $B$. Write

$$\mathscr{E} = \{E_{a_j}\} \cup \{E_{b_j}\},$$

where $E_{a_j} \cap B = \emptyset$ and $E_{b_j} \cap A = \emptyset$. Then $A \subset \bigcup E_{a_j}$ and $B \subset \bigcup E_{b_j}$, so

$$\sum_{j=1}^{\infty} (\operatorname{diam} E_j)^p = \sum_{j=1}^{\infty} (\operatorname{diam} E_{a_j})^p + \sum_{j=1}^{\infty} (\operatorname{diam} E_{j_b})^p \geq H_{p,\delta}(A) + H_{p,\delta}(B).$$

This is true for any covering of $A \cup B$ by countably many element of $\mathscr{E}$, so

$$H_{p,\delta}(A \cup B) \geq H_{p,\delta}(A) + H_{p,\delta}(B).$$

The above inequality is true for any $0 < \delta < \rho(A,B)$, and taking $\delta \to 0$ yields

$$H_p(A \cup B) \geq H_p(A) + H_p(B),$$

completing the proof. □

We call the metric outer measure $H_p : \mathscr{P}(X) \to [0,\infty]$ in the above theorem the **$p$-dimensional Hausdorff outer measure**. From Theorem 1 it follows that the restriction of $H_p$ to the Borel $\sigma$-algebra $\mathscr{B}_X$ of a metric space is a meausure. We call this restriction the **$p$-dimensional Hausdorff measure**, and denote it also by $H_p$.

It is straightforward to verify that if $T : X \to X$ is an isometric isomorphism then $H_p \circ T = H_p$. In particular, for $X = \mathbb{R}^n$, $H_p$ is invariant under translations.

We will use the following inequality when talking about Hausdorff measure on $\mathbb{R}^n$.[5]

**Lemma 4.** *Let $Y$ be a set and $(X,\rho)$ be a metric space. If $f,g : Y \to X$ satisfy*

$$\rho(f(y),f(z)) \leq C\rho(g(y),g(z)), \qquad y,z \in Y,$$

*then for any $A \in \mathscr{P}(Y)$,*

$$H_p(f(A)) \leq C^p H_p(g(A)).$$

*Proof.* Take $\delta > 0$ and $\epsilon > 0$. There are countably many sets $E_j$ that cover $g(A)$ each with diameter $\leq C^{-1}\delta$ and such that

$$\sum (\operatorname{diam} E_j)^p \leq H_p(g(A)) + \epsilon.$$

Let $a \in A$. There is some $j$ with $g(a) \in E_j$, so $a \in g^{-1}(E_j)$ and then $f(a) \in f(g^{-1}(E_j))$. Therefore the sets $f(g^{-1}(E_j))$ cover $f(A)$. For $u,v \in f(g^{-1}(E_j))$, there are $y,z \in g^{-1}(E_j)$ with $u = f(y), v = f(z)$. Because $g(y), g(z) \in E_j$,

$$\rho(u,v) = \rho(f(y),f(z)) \leq C\rho(g(y),g(z)) \leq C \operatorname{diam} E_j,$$

hence

$$\operatorname{diam} f(g^{-1}(E_j)) \leq C \operatorname{diam} E_j.$$

Since the sets $f(g^{-1}(E_j))$ cover $f(A)$ and each has diameter $\leq C \operatorname{diam} E_j \leq \delta$,

$$H_{p,\delta}(f(A)) \leq \sum (\operatorname{diam} f(g^{-1}(E_j)))^p \leq \sum C^p (\operatorname{diam} E_j)^p \leq C^p (H_p(g(A)) + \epsilon).$$

This is true for all $\delta > 0$, so taking $\delta \to 0$,

$$H_p(f(A)) \leq C^p (H_p(g(A)) + \epsilon).$$

This is true for all $\epsilon > 0$, so taking $\epsilon \to 0$,

$$H_p(f(A)) \leq C^p H_p(g(A)).$$

□

# 3 Hausdorff dimension

**Theorem 5.** *If $H_p(A) < \infty$ then $H_q(A) = 0$ for all $q > p$.*

[5] Gerald B. Folland, *Real Analysis*, second ed., p. 350, Proposition 11.18.

*Proof.* Let $\delta > 0$. Then $H_{p,\delta}(A) \leq H_p(A) < \infty$ Let $\{E_j\}$ be countably many sets each with diameter $\leq \delta$ such that $A \subset \bigcup E_j$ and

$$\sum (\operatorname{diam} E_j)^p \leq H_{p,\delta}(A) + 1 \leq H_p(A) + 1.$$

This gives us

$$\begin{aligned} H_{q.\delta}(A) \leq \sum (\operatorname{diam} E_j)^q &= \sum (\operatorname{diam} E_j)^{q-p} (\operatorname{diam} E_j)^p \\ &\leq \delta^{q-p} \sum (\operatorname{diam} E_j)^p \\ &\leq \delta^{q-p} (H_p(A) + 1). \end{aligned}$$

This is true for any $\delta > 0$ and $q - p > 0$, so taking $\delta \to 0$ we obtain $H_q(A) = 0$. □

For $A \in \mathscr{P}(X)$, we define the **Hausdorff dimension of** $A$ to be

$$\inf\{q \geq 0 : H_q(A) = 0\}.$$

If the set whose infimum we are taking is empty, then the Hausdorff dimension of $A$ is $\infty$.

# 4 Radon measures and Haar measures

Before speaking about Hausdorff measure on $\mathbb{R}^n$, we remind ourselves of some material about Radon measures and Haar measures. Let $X$ be a locally compact Hausdorff space. A Borel measure $\mu$ on $X$ is said to be a **Radon measure** if (i) it is finite on each compact set, (ii) for any Borel set $E$,

$$\mu(E) = \inf\{\mu(U) : U \text{ open and } E \subset U\},$$

and (iii) for any open set $E$,

$$\mu(E) = \sup\{\mu(K) : K \text{ compact and } K \subset E\}.$$

It is a fact that if $X$ is a locally compact Hausdorff space in which every open set is $\sigma$-compact, then every Borel measure on $X$ that is finite on compact sets is a Radon measure.[6]

Suppose that $G$ is a locally compact group. A Borel measure $\mu$ on $G$ is said to be **left-invariant** if for all $x \in G$ and $E \in \mathscr{B}_G$,

$$\mu(xE) = \mu(E).$$

A **left Haar measure on** $G$ is a nonzero left-invariant Radon measure on $G$. It is a fact that if $\mu$ and $\nu$ are left Haar measures on $G$ then there is some $c > 0$ such that $\mu = c\nu$.[7]

[6]Gerald B. Folland, *Real Analysis*, second ed., p. 217, Theorem 7.8.
[7]Gerald B. Folland, *Real Analysis*, second ed., p. 344, Theorem 11.9.

## 5 Hausdorff measure in $\mathbb{R}^n$

Let $m_n$ denote Lebesgue measure on $\mathbb{R}^n$.

**Lemma 6.** *If $E$ is a Borel set in $\mathbb{R}^n$, then*

$$H_n(E) \geq 2^n m_n(E).$$

*Proof.* Let $\epsilon > 0$ and let $\{E_j\}$ be countably many closed sets that cover $E$ and such that

$$\sum (\operatorname{diam} E_j)^n \leq H_n(E) + \epsilon.$$

The **isodiametric inequality** (which one proves using the Brunn-Minkowski inequality) states that if $A$ is a Borel set in $\mathbb{R}^n$, then

$$m_n(A) \leq \left( \frac{\operatorname{diam} A}{2} \right)^n.$$

Using this gives

$$\sum 2^n m_n(E_j) \leq H_n(E) + \epsilon.$$

But because the sets $E_j$ cover $E$ we have $m_n(E) \leq m_n(\bigcup E_j) \leq \sum m_n(E_j)$, so we get

$$m_n(E) \leq \frac{H_n(E) + \epsilon}{2^n}.$$

This expression does not involve the sets $E_j$ (which depend on $\epsilon$), and since this expression is true for any $\epsilon > 0$, taking $\epsilon \to 0$ yields

$$m_n(E) \leq \frac{H_n(E)}{2^n}.$$

□

Let

$$Q = \left\{ x \in \mathbb{R}^n : |x_1| \leq \frac{1}{2}, \ldots, |x_n| \leq \frac{1}{2} \right\}.$$

**Lemma 7.** *$0 < H_n(Q) < \infty$.*

*Proof.* For any $m \geq 1$, the cube $Q$ can be covered by $m^n$ cubes $q_1, \ldots, q_{m^n}$ of side length $\frac{1}{m}$. Let $0 < \delta < 1$ and let $m > \frac{1}{\delta}$. The distance from the center of $q_j$ to one of the vertices of $q_j$ is

$$r = \sqrt{\left(\frac{1}{2m}\right)^2 + \cdots + \left(\frac{1}{2m}\right)^2} = \frac{\sqrt{n}}{2m}.$$

Inscribe $q_j$ in a closed ball $b_j$ with the same center as $q_j$ and radius $r$. These balls cover $Q$. Hence

$$H_{p,\delta}(Q) \leq \sum_{j=1}^{m^n} (\operatorname{diam} b_j)^n = \sum_{j=1}^{m^n} (2r)^n = (2r)^n \cdot m^n = n^{n/2}.$$

Taking $\delta \to 0$ gives $H_p(Q) \leq n^{n/2} < \infty$.

On the other hand, by Lemma 6,

$$H_n(Q) \geq 2^n m_n(Q) = 2^n > 0.$$

□

**Theorem 8.** *There is some constant $c_n > 0$ such that*

$$H_n = c_n m_n.$$

*Proof.* $\mathbb{R}^n$ is a locally compact Hausdorff space in which every open set in $\mathbb{R}^n$ is $\sigma$-compact. Therefore, to show that $H_n$ is a Radon measure it suffices to show that $H_n$ is finite on every compact set. If $K$ is a compact subset of $\mathbb{R}^n$, there is some $r > 0$ such that $K \subset rQ$. By Lemma 4 and Lemma 7 we get $H_n(rQ) < \infty$, so $H_n(K) < \infty$. Therefore $H_n$ is a Radon measure.

Because $H_n(Q) > 0$, $H_n$ is not the zero measure. Any translation is an isometric isomorphism $\mathbb{R}^n \to \mathbb{R}^n$, so $H_n$ is invariant under translations. Thus $H_n$ is a left Haar measure on $\mathbb{R}^n$. But Lebesgue measure $m_n$ is also a left Haar measure on $\mathbb{R}^n$, so there is some $c_n > 0$ such that

$$H_n = c_n m_n,$$

proving the claim. □