The heat kernel on the torus

Jordan Bell
jordan.bell@gmail.com
Department of Mathematics, University of Toronto
October 7, 2014

1 Heat kernel on \( T \)

For \( t > 0 \), define \( k_t : \mathbb{R} \to (0, \infty) \) by

\[
k_t(x) = (4\pi t)^{-1/2} \exp \left( -\frac{x^2}{4t} \right), \quad x \in \mathbb{R}.
\]

For \( t > 0 \), define \( g_t : \mathbb{R} \to (0, \infty) \) by

\[
g_t(x) = 2\pi \sum_{k \in \mathbb{Z}} k_t(x + 2\pi k), \quad x \in \mathbb{R},
\]

which one checks indeed converges for all \( x \in \mathbb{R} \). Of course, \( g_t(x + 2\pi k) = g_t(x) \)
for any \( k \in \mathbb{Z} \), so we can interpret \( g_t \) as a function on \( T \), where \( T = \mathbb{R}/2\pi \mathbb{Z} \).

Let \( m \) be Haar measure on \( T \): \( dm(x) = (2\pi)^{-1} dx \), and so \( m(T) = 1 \). With
\n\[
\|f\|_1 = \int_T |f| dm \quad \text{for} \quad f : T \to \mathbb{C},
\]
we have, because \( g_t > 0 \),

\[
\|g_t\|_1 = \sum_{k \in \mathbb{Z}} \int_T k_t(x + 2\pi k) dx = \int_\mathbb{R} k_t(x) dx = 1.
\]

\footnote{Most of this note is my working through of notes by Patrick Maheux. \texttt{http://www.univ-orleans.fr/mapmo/membres/maheux/InfiniteTorusV2.pdf}}
Hence $g_t \in L^1(T)$. For $\xi \in \mathbb{Z}$, we compute
\[
\hat{g}_t(\xi) = \int_T g_t(x) e^{-i\xi x} \, dm(x) \\
= \sum_{k \in \mathbb{Z}} \int_T k_t(x + 2\pi k) e^{-i\xi x} \, dx \\
= \sum_{k \in \mathbb{Z}} \int_T k_t(x + 2\pi k) e^{-i\xi(x + 2\pi k)} \, dx \\
= \int_{\mathbb{R}} k_t(x) e^{-i\xi x} \, dx
\]
\[
= \hat{k}_t \left( \frac{\xi}{2\pi} \right) \\
= e^{-\xi^2 t}.
\]

**Lemma 1.** For $t > 0$ and $x \in \mathbb{R}$,
\[
g_t(x) = \sqrt{\frac{\pi}{t}} \exp \left( -\frac{x^2}{4t} \right) \left( 1 + 2 \sum_{k \geq 1} \exp \left( -\frac{\pi^2 k^2}{t} \right) \cosh \left( \frac{\pi k x}{t} \right) \right).
\]

**Proof.** Using the definition of $g_t$,
\[
g_t(x) = 2\pi \sum_{k \in \mathbb{Z}} k_t(x + 2\pi k) \\
= 2\pi \sum_{k \in \mathbb{Z}} (4\pi t)^{-1/2} \exp \left( -\frac{(x + 2\pi k)^2}{4t} \right) \\
= \sqrt{\frac{\pi}{t}} \exp \left( -\frac{x^2}{4t} \right) \sum_{k \in \mathbb{Z}} \exp \left( -\frac{\pi k x}{t} \right) \exp \left( -\frac{\pi^2 k^2}{t} \right) \\
= \sqrt{\frac{\pi}{t}} \exp \left( -\frac{x^2}{4t} \right) \left( 1 + \sum_{k \geq 1} \left( \exp \left( \frac{\pi k x}{t} \right) + \exp \left( -\frac{\pi k x}{t} \right) \right) \exp \left( -\frac{\pi^2 k^2}{t} \right) \right),
\]
which gives the claim, using $\cosh y = \frac{e^y + e^{-y}}{2}$. \hfill \Box

**Definition 2.** For $x \in \mathbb{R}$, let $\|x\| = \inf \{|x - 2\pi k| : k \in \mathbb{Z}\}$.

For $k \in \mathbb{Z}$, $\|x + 2\pi k\| = \|x\|$, so it makes sense to talk about $\|x\|$ for $x \in T$.

**Theorem 3.** For $t > 0$ and $x \in \mathbb{R}$,
\[
\exp \left( -\frac{\|x\|^2}{4t} \right) g_t(0) \leq g_t(x) \leq \exp \left( -\frac{\|x\|^2}{4t} \right) \left( \sqrt{\frac{\pi}{t}} + g_t(0) \right).
\]
Proof. Let \( x = 2\pi m + \theta \) with \( |\theta| \leq \pi \), so that \( \|x\| = \|\theta\| = |\theta| \), and \( g_t(x) = g_t(\theta) \). Using Lemma 1 and the fact that \( \cosh y \geq 1 \), we get

\[
g_t(\theta) \geq \exp \left( -\frac{\theta^2}{4t} \right) \sqrt{\frac{\pi}{t}} \left( 1 + 2 \sum_{k \geq 1} \exp \left( -\frac{\pi^2 k^2}{t} \right) \right) = \exp \left( -\frac{\theta^2}{4t} \right) g_t(0),
\]

hence

\[
g_t(x) \geq \exp \left( -\frac{\|x\|^2}{4t} \right) g_t(0),
\]

the lower bound we wanted to prove.

Write

\[
S = 1 + 2 \sum_{k \geq 1} \exp \left( -\frac{\pi^2 k^2}{t} \right) \cosh \left( \frac{\pi k \theta}{t} \right).
\]

For any \( k \geq 1 \), using \( |\theta| \leq \pi \),

\[
2 \cosh \left( \frac{\pi k \theta}{t} \right) \leq 2 \cosh \left( \frac{\pi^2 k}{t} \right) = \exp \left( \frac{\pi^2 k}{t} \right) + \exp \left( -\frac{\pi^2 k}{t} \right) \leq 1 + \exp \left( \frac{\pi^2 k}{t} \right).
\]

Hence

\[
S \leq 1 + \sum_{k \geq 1} \exp \left( -\frac{\pi^2 k^2}{t} \right) \left( 1 + \exp \left( \frac{\pi^2 k}{t} \right) \right)
= 1 + \sum_{k \geq 1} \exp \left( -\frac{\pi^2 k^2}{t} \right) + \exp \left( -\frac{\pi^2 k(k-1)}{t} \right)
\leq 1 + \sum_{k \geq 1} \exp \left( -\frac{\pi^2 k^2}{t} \right) + \exp \left( -\frac{\pi^2 (k-1)^2}{t} \right)
= 2 + 2 \sum_{k \geq 1} \exp \left( -\frac{\pi^2 k^2}{t} \right)
= 1 + \sqrt{\frac{t}{\pi}} g_t(0).
\]

But \( g_t(\theta) = \sqrt{\frac{\pi}{t}} \exp \left( -\frac{\theta^2}{4t} \right) S \), so

\[
g_t(\theta) \leq \exp \left( -\frac{\theta^2}{4t} \right) \left( \sqrt{\frac{\pi}{t}} + g_t(0) \right) = \exp \left( -\frac{\|x\|^2}{4t} \right) \left( \sqrt{\frac{\pi}{t}} + g_t(0) \right),
\]

the upper bound we wanted to prove. \( \square \)

Applying Lemma 1 with \( x = 0 \) gives \( g_t(0) \geq \sqrt{\frac{\pi}{t}} \), and using this with the above theorem we obtain

\[
g_t(x) \leq 2 \exp \left( -\frac{\|x\|^2}{4t} \right) g_t(0).
\]

(1)
Theorem 4. For \( t > 0 \),

\[
\sqrt{\frac{\pi}{t}} \leq g_t(0) \leq 1 + \sqrt{\frac{\pi}{t}}
\]

and

\[
2e^{-t} \leq g_t(0) - 1 \leq \frac{2e^{-t}}{1 - e^{-t}}.
\]

Proof. Using Lemma 1 we have

\[
g_t(0) \geq \sqrt{\frac{\pi}{t}}.
\]

For each \( x \in \mathbb{R} \) we have

\[
g_t(x) = \sum_{k \in \mathbb{Z}} \hat{g}_t(k)e^{ikx} = \sum_{k \in \mathbb{Z}} e^{-k^2t}e^{ikx} = 1 + 2\sum_{k \geq 1} e^{-k^2t} \cos(kx).
\]

Writing \( \phi(t) = \sum_{k \geq 1} e^{-k^2t} \), we then have

\[
g_t(0) = 1 + 2\phi(t).
\]

But as \( e^{-x^2t} \) is positive and decreasing, bounding a sum by an integral we get

\[
\phi(t) \leq \int_0^{\infty} e^{-x^2t}dx = \frac{1}{\sqrt{t}} \int_0^{\infty} e^{-x^2}dx = \frac{1}{2} \sqrt{\frac{\pi}{t}},
\]

hence

\[
g_t(0) = 1 + 2\phi(t) \leq 1 + \sqrt{\frac{\pi}{t}}.
\]

Moreover, because \( \phi(t) \geq e^{-t} \) (lower bounding the sum by the first term), we have

\[
g_t(0) = 1 + 2\phi(t) \geq 1 + 2e^{-t}.
\]

Finally, because \( e^{-tk^2} \leq e^{-tk} \) for \( k \geq 1 \),

\[
\phi(t) \leq \sum_{k \geq 1} e^{-tk} = e^{-t} \frac{1}{1 - e^{-t}},
\]

thus

\[
g_t(0) \leq 1 + \frac{2e^{-t}}{1 - e^{-t}}.
\]

Taking \( t \to 0 \) and \( t \to \infty \) in the above theorem gives the following asymptotics.

Corollary 5.

\[
g_t(0) \sim \sqrt{\frac{\pi}{t}}, \quad t \to 0
\]

and

\[
g_t(0) - 1 \sim 2e^{-t}, \quad t \to \infty.
\]
2 Heat kernel on $\mathbb{T}^n$

Fix $n \geq 1$, and let $\mathcal{A} = (a_1, \ldots, a_n)$, $a_i$ positive real numbers. We define $g^{\mathcal{A}}_t : \mathbb{R}^n \rightarrow (0, \infty)$ by

$$g^{\mathcal{A}}_t(x) = \prod_{k=1}^n g_{a_k t}(x_k), \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^n.$$  

For $x \in \mathbb{R}^n$ and $\xi \in \mathbb{Z}^n$ we have

$$g^{\mathcal{A}}_t(x + 2\pi \xi) = \prod_{k=1}^n g_{a_k t}(x_k + 2\pi \xi_k) = \prod_{k=1}^n g_{a_k t}(x_k) = g^{\mathcal{A}}_t(x),$$

so $g^{\mathcal{A}}_t$ can be interpreted as a function on $\mathbb{T}^n$.

Let $m_n$ be Haar measure on $\mathbb{T}^n$:

$$dm_n(x) = \prod_{k=1}^n dm(x_k) = \prod_{k=1}^n (2\pi)^{-1} dx_k = (2\pi)^{-n} dx,$$

which satisfies $m_n(\mathbb{T}^n) = 1$. Define $\mu^{\mathcal{A}}_t$ to be the measure on $\mathbb{T}^n$ whose density with respect to $m_n$ is $g^{\mathcal{A}}_t$:

$$d\mu^{\mathcal{A}}_t = g^{\mathcal{A}}_t dm_n.$$  

We now calculate the Fourier coefficients of $g^{\mathcal{A}}_t$. For $\xi \in \mathbb{Z}^n$,

$$\mathcal{F}(g^{\mathcal{A}}_t)(\xi) = \int_{\mathbb{T}^n} g^{\mathcal{A}}_t(x) e^{-i\xi \cdot x} dm_n(x)$$

$$= \int_{\mathbb{T}^n} \prod_{k=1}^n g_{a_k t}(x_k) e^{-i\xi_1 x_1 - \cdots - i\xi_n x_n} dm_n(x)$$

$$= \prod_{k=1}^n \int_{\mathbb{T}} g_{a_k t}(x_k) e^{-i\xi_k x_k} dm(x_k)$$

$$= \prod_{k=1}^n \hat{g}_{a_k t}(\xi_k)$$

$$= \prod_{k=1}^n e^{-\xi_k^2 a_k t}$$

$$= e^{-tq(\xi)},$$

where

$$q(\xi) = \sum_{k=1}^n a_k \xi_k^2, \quad \xi \in \mathbb{Z}^n.$$  

**Definition 6.** For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ we define

$$\|x\|_{\mathcal{A}}^2 = \frac{1}{a_1} \|x_1\|^2 + \cdots + \frac{1}{a_n} \|x_n\|^2,$$

with $\mathcal{A} = (a_1, \ldots, a_n)$.
For \( \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{Z}^n \), because \( \|x_k + 2\pi\xi_k\| = \|x_k\| \), we have \( \|x + 2\pi\xi\|_{A^r} = \|x\|_{A^r} \), so it makes sense to talk about \( \|\cdot\|_{A^r} \) on \( \mathbb{T}^n \).

Using Theorem 3 and (1) we get the following.

**Theorem 7.** For \( t > 0 \) and \( x \in \mathbb{R}^n \),

\[
\exp \left( -\frac{\|x\|^2}{4t} \right) g_t^{A^r}(0) \leq g_t^{A^r}(x) \leq \exp \left( -\frac{\|x\|^2}{4t} \right) g_t^{A^r}(0).
\]

Combining this with Theorem 4 we obtain the following. The first inequality is appropriate for \( t \to 0^+ \) and the second inequality for \( t \to \infty \).

**Theorem 8.** For \( t > 0 \) and \( x \in \mathbb{R}^n \),

\[
\exp \left( -\frac{\|x\|^2}{4t} \right) \prod_{k=1}^n \sqrt{\frac{\pi}{a_k t}} \leq g_t^{A^r}(x) \leq \exp \left( -\frac{\|x\|^2}{4t} \right) \prod_{k=1}^n \left( 1 + \sqrt{\frac{\pi}{a_k t}} \right)
\]

and

\[
\exp \left( -\frac{\|x\|^2}{4t} \right) \prod_{k=1}^n \left( 1 + 2e^{-a_k t} \right) \leq g_t^{A^r}(x) \leq \exp \left( -\frac{\|x\|^2}{4t} \right) \prod_{k=1}^n \left( 1 + \frac{2e^{-a_k t}}{1 - e^{-a_k t}} \right).
\]

### 3 The infinite-dimensional torus

\( \mathbb{T}^\infty \) with the product topology is a compact abelian group. Let \( m_\infty \) be Haar measure on \( \mathbb{T}^\infty \):

\[
dm_\infty(x) = \prod_{k=1}^\infty dm(x_k), \quad x = (x_1, x_2, \ldots) \in \mathbb{T}^\infty,
\]

where \( m \) is Haar measure on \( \mathbb{T} \).

For \( t > 0 \), let \( \mu_t \) be the measure on \( \mathbb{T} \) whose density with respect to Haar measure \( m \) is \( g_t \):

\[
d\mu_t = g_t dm.
\]

This is a probability measure on \( \mathbb{T} \).

Let \( \mathcal{A} = (a_1, a_2, \ldots) \in \mathbb{N}^\infty \). For \( t > 0 \) we define

\[
\mu_t^\mathcal{A} = \prod_{k=1}^\infty \mu_{a_k t}^r.
\]

This is a probability measure on \( \mathbb{T}^{\infty} \).

\(^2\)Christian Berg determines conditions on \( \mathcal{A} \) and \( t \) so that \( \mu_t^\mathcal{A} \) is absolutely continuous with respect to Haar measure \( m_\infty \) on \( \mathbb{T}^\infty \): *Potential theory on the infinite dimensional torus*, Invent. Math. 32 (1976), no. 1, 49–100.
The Pontryagin dual of $\mathbb{T}^\infty$ is the direct sum $\bigoplus_{k=1}^\infty \mathbb{Z}$, which we denote by $\mathbb{Z}^{(\infty)}$, which is a discrete abelian group. For $\xi \in \mathbb{Z}^{(\infty)}$ and $x \in \mathbb{T}^\infty$, we write
\[ e_\xi(x) = \exp \left( i \sum_{k=1}^\infty \xi_k x_k \right). \]

The Fourier transform of $\mu_t^\infty$ is $\mathcal{F}(\mu_t^\infty) : \mathbb{Z}^{(\infty)} \to \mathbb{C}$ defined by
\[ \mathcal{F}(\mu_t^\infty)(\xi) = \int_{\mathbb{T}^\infty} e^{-\xi(x)} dm_\infty(x), \quad \xi \in \mathbb{Z}^{(\infty)}, \]
which is
\[ \int_{\mathbb{T}^\infty} e^{-\xi(x)} dm_\infty(x) = \int_{\mathbb{T}^\infty} \exp \left( -i \sum_{k=1}^\infty \xi_k x_k \right) d\mu_t^\infty(x) \]
\[ = \int_{\mathbb{T}^\infty} \prod_{k=1}^\infty \exp(-i\xi_k x_k) d\mu_t^\infty(x) \]
\[ = \prod_{k=1}^\infty \int_{\mathbb{T}} \exp(-i\xi_k x_k) g_{a_k t}(x_k) dm(x_k) \]
\[ = \prod_{k=1}^\infty g_{a_k t}(\xi_k) \]
\[ = \prod_{k=1}^\infty \exp(-\xi_k^2 a_k t) \]
\[ = \exp \left( -t \sum_{k=1}^\infty a_k \xi_k^2 \right). \]

4 Convergence of infinite products

If $c_k \geq 0$, then for any $n$,
\[ 1 + \frac{1}{n} \sum_{k=1}^n c_k \leq \prod_{k=1}^n (1 + c_k) \leq \exp \left( \sum_{k=1}^n c_k \right). \]

Thus, the limit of $\prod_{k=1}^n (1 + c_k)$ as $n \to \infty$ exists if and only if
\[ \sum_{k=1}^\infty c_k < \infty. \]

For the second inequality in Theorem 8, the limit of $\prod_{k=1}^n (1 + 2e^{-a_k t})$ as $n \to \infty$ exists if and only if
\[ \sum_{k=1}^\infty 2e^{-a_k t} < \infty. \]