# The Banach algebra of functions of bounded variation and the pointwise Helly selection theorem

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## 1 $BV[a,b]$

Let $a < b$. For $f : [a,b] \to \mathbb{R}$, we define[1]

$$\|f\|_\infty = \sup_{t \in [a,b]} |f(t)|,$$

and if $\|f\|_\infty < \infty$ we say that $f$ is **bounded**. We define $B[a,b]$ to be the set of bounded functions $[a,b] \to \mathbb{R}$, which with the norm $\|\cdot\|_\infty$ is a Banach algebra.

A **partition of** $[a,b]$ is a set $P = \{t_0, \dots, t_n\}$ such that $a = t_0 < \cdots < t_n = b$. For example, $P = \{a, b\}$ is a partition of $[a,b]$. If $Q$ is a partition of $[a,b]$ and $P \subset Q$, we say that $Q$ is a **refinement** of $P$. For $f : [a,b] \to \mathbb{R}$, we define

$$V(f,P) = \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})|.$$

It is straightforward to show using the triangle inequality that if $Q$ is a refinement of $P$ then

$$V(f,P) \leq V(f,Q).$$

In particular, any partition $P$ is a refinement of $\{a,b\}$, so

$$|f(b) - f(a)| \leq V(f,P).$$

The **total variation** of $f : [a,b] \to \mathbb{R}$ is

$$V_a^b f = \sup\{V(f,P) : P \text{ is a partition of } [a,b]\},$$

[1] In this note we speak about functions that take values in $\mathbb{R}$, because this makes it simpler to talk about monotone functions. Once the machinery is established we can then apply it to the real and imaginary parts of a function that takes values in $\mathbb{C}$.

and if $V_a^b f < \infty$ we say that $f$ is of **bounded variation**. We denote by $BV[a,b]$ the set of functions $[a,b] \to \mathbb{R}$ of bounded variation. For a function $f \in BV[a,b]$, we define $v : [a,b] \to \mathbb{R}$ by $v(x) = V_a^x f$ for $x \in [a,b]$, called the **variation of** $f$.

If $f : [a,b] \to \mathbb{R}$ is monotone, it is straightforward to check that $V_a^b f = |f(b) - f(a)|$, hence that $f$ is of bounded variation.

We first show that $BV[a,b] \subset B[a,b]$.

**Lemma 1.** *If $f : [a,b] \to \mathbb{R}$ is of bounded variation, then*

$$\|f\|_\infty \leq |f(a)| + V_a^b f.$$

*Proof.* Let $x \in [a,b]$ If $x = a$ the result is immediate. If $x = b$, then

$$|f(b)| \leq |f(a)| + |f(b) - f(a)| \leq |f(a)| + V_a^b f.$$

Otherwise, $P = \{a, x, b\}$ is a partition of $[a,b]$ and

$$|f(x) - f(a)| \leq V(f, P) \leq V_a^b f.$$

□

The total variation of functions has several properties. The following lemma and that fact that functions of bounded variation are bounded imply that $BV[a,b]$ is an algebra.[2]

**Lemma 2.** *If $f, g \in BV[a,b]$ and $c \in \mathbb{R}$, then the following statements are true.*

1. *$V_a^b f = 0$ if and only if $f$ is constant.*
2. *$V_a^b(cf) = |c| V_a^b(f)$.*
3. *$V_a^b(f+g) \leq V_a^b f + V_a^b g$.*
4. *$V_a^b(fg) \leq \|f\|_\infty V_a^b g + \|g\|_\infty V_a^b f$.*
5. *$V_a^b |f| \leq V_a^b f$.*
6. *$V_a^b f = V_a^x f + V_x^b$ for $a \leq x \leq b$.*

**Lemma 3.** *If $f : [a,b] \to \mathbb{R}$ is differentiable on $(a,b)$ and $\|f'\|_\infty < \infty$, then*

$$V_a^b f \leq \|f'\|_\infty (b-a).$$

*Proof.* Suppose that $P = \{a = t_0 < \cdots < t_n = b\}$ is a partition of $[a,b]$. By the mean value theorem, for each $j = 1, \ldots, n$ there is some $x_j \in (t_{j-1}, t_j)$ at which

$$f'(x_j) = \frac{f(t_j) - f(t_{j-1})}{t_j - t_{j-1}}.$$

[2] N. L. Carothers, *Real Analysis*, p. 204, Lemma 13.3.

Then

$$\begin{aligned}
V(f,P) &= \sum_{j=1}^{n} |f(t_j) - f(t_{j-1})| \\
&= \sum_{j=1}^{n} (t_j - t_{j-1})|f'(x_j)| \\
&\leq \|f'\|_\infty \sum_{j=1}^{n} (t_j - t_{j-1}) \\
&= \|f'\|_\infty (b-a).
\end{aligned}$$

□

**Lemma 4.** *If $f \in C^1[a,b]$, then*

$$V_a^b f \leq \int_a^b |f'(t)|dt.$$

*Proof.* Let $P = \{t_0, \ldots, t_n\}$ be a partition of $[a,b]$. Then, by the fundamental theorem of calculus,

$$\begin{aligned}
V(f,P) &= \sum_{j=1}^{n} |f(t_j) - f(t_{j-1})| \\
&\leq \sum_{j=1}^{n} \left| \int_{t_{j-1}}^{t_j} f'(t) \right| \\
&\leq \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} |f'(t)|dt \\
&= \int_a^b |f'(t)|dt.
\end{aligned}$$

Therefore

$$V_a^b f = \sup_P V(f,P) \leq \int_a^b |f'(t)|dt.$$

□

**Lemma 5.** *If $f : [a,b] \to \mathbb{R}$ is a polynomial, then*

$$V_a^b f = \int_a^b |f'(t)|dt.$$

*Proof.* Because $f$ is a polynomial, $f$ is also, so $f'$ is piecewise monotone, say $f' = c_j|f'|$ on $(t_{j-1}, t_j)$ for $j = 1, \ldots, n$, for some $c_j \in \{+1, -1\}$ and $a = t_0 < \cdots < t_n = b$. Then

$$\int_{t_{j-1}}^{t_j} |f'(t)|dt = c_j \int_{t_{j-1}}^{t_j} f'(t)dt = c_j(f(t_j) - f(t_{j-1})),$$

giving, because $t_0 < \cdots < t_n$ is a partition of $[a, b]$,

$$\begin{aligned}\int_a^b |f'(t)|dt &= \sum_{j=1}^n \int_{t_{j-1}}^{t_j} |f'(t)|dt \\ &= \sum_{j=1}^n c_j(f(t_j) - f(t_{j-1})) \\ &\leq \sum_{j=1}^n |f(t_j) - f(t_{j-1})| \\ &\leq V_a^b f.\end{aligned}$$

□

**Lemma 6.** *If $f_m$ is a sequence of functions $[a, b] \to \mathbb{R}$ that converges pointwise to some $f : [a, b] \to \mathbb{R}$ and $P$ is some partition of $[a, b]$, then*

$$V(f_m, P) \to V(f, P).$$

*If $f_m$ is a sequence in $BV[a, b]$ that converges pointwise to some $f : [a, b] \to \mathbb{R}$, then*

$$V_a^b f \leq \liminf_{m\to\infty} V_a^b f_m.$$

*Proof.* Say $P = \{t_0, \ldots, t_n\}$. Then, because taking the limit of convergent sequences is linear,

$$\begin{aligned}\lim_{m\to\infty} V(f_m, P) &= \lim_{m\to\infty} \sum_{j=1}^n |f_m(t_j) - f_m(t_{j-1})| \\ &= \sum_{j=1}^n \lim_{m\to\infty} |f_m(t_j) - f_m(t_{j-1})| \\ &= \sum_{j=1}^n |f(t_j) - f(t_{j-1})| \\ &= V(f, P).\end{aligned}$$

Let $P = \{t_0, \ldots, t_n\}$ be a partition of $[a, b]$. Then

$$\begin{aligned} V(f,P) &= \sum_{j=1}^{n} |f(t_j) - f(t_{j-1})| \\ &= \sum_{j=1}^{n} \lim_{m\to\infty} |f_m(t_j) - f_m(t_{j-1})| \\ &= \lim_{m\to\infty} V(f_m, P) \\ &\leq \liminf_{m\to\infty} V_a^b f_m. \end{aligned}$$

This is true for any partition $P$ of $[a, b]$, which yields

$$V_a^b f \leq \liminf_{m\to\infty} V_a^b f_m.$$

$\square$

We now prove that $BV[a, b]$ is a Banach space.[3]

**Theorem 7.** *With the norm*

$$\|f\|_{BV} = |f(a)| + V_a^b f.$$

*$BV[a, b]$ is a Banach space.*

*Proof.* Using Lemma 2, it is straightforward to check that $BV[a, b]$ is a normed linear space. Suppose that $f_m$ is a Cauchy sequence in $BV[a, b]$. By Lemma 1 it follows that $f_m$ is a Cauchy sequence in $B[a, b]$, and thus converges in $B[a, b]$ to some $f \in B[a, b]$.

Let $P$ be a partition of $[a, b]$ and let $\epsilon > 0$. Because $f_n$ is a Cauchy sequence in $BV[a, b]$, there is some $N$ such that if $n, m \geq N$ then $\|f_m - f_n\|_{BV} < \epsilon$. For $n \geq N$, Lemma 6 yields

$$\begin{aligned} \|f - f_n\|_{BV} &\leq |f(a) - f_n(a)| + V(f - f_n, P) \\ &= \lim_{m\to\infty} (|f_m(a) - f_n(a)| + V(f_m - f_n, P)) \\ &\leq \sup_{m \geq N} (|f_m(a) - f_n(a)| + V(f_m - f_n, P)) \\ &= \sup_{m\geq N} \|f_m - f_n\|_{BV} \\ &\leq \epsilon. \end{aligned}$$

Because $f - f_N \in BV[a, b]$ and $f_N \in BV[a, b]$ and $BV[a, b]$ is an algebra, $f = (f - f_N) + f_N \in BV[a, b]$. That is, the Cauchy sequence $f_n$ converges in $BV[a, b]$ to $f \in BV[a, b]$, showing that $BV[a, b]$ is a complete metric space and thus a Banach space. $\square$

[3] N. L. Carothers, *Real Analysis*, p. 206, Theorem 13.4.

The following theorem shows that a function of bounded of variation can be written as the difference of nondecreasing functions.[4]

**Theorem 8.** *Let $f \in BV[a,b]$ and let $v$ be the variation of $f$. Then $v - f$ and $v$ are nondecreasing.*

*Proof.* If $x, y \in [a,b]$, $x < y$, then, using Lemma 2,

$$\begin{aligned} v(y) - v(x) &= V_a^y f - V_a^x f \\ &= V_x^y f \\ &\geq |f(y) - f(x)| \\ &\geq f(y) - f(x). \end{aligned}$$

That is, $v(y) - f(y) \geq v(x) - f(x)$, showing that $v - f$ is nondecreasing, and because $f$ is nondecreasing we have $f(y) - f(x) \geq 0$ and so $v(y) - v(x) \geq 0$. $\square$

The following theorem tells us that a function of bounded variation is right or left continuous at a point if and only if its variation is respectively right or left continuous at the point.[5]

**Theorem 9.** *Let $f \in BV[a,b]$ and let $v$ be the variation of $f$. For $x \in [a,b]$, $f$ is right (respectively left) continuous at $x$ if and only if $v$ is right (respectively left) continuous at $x$.*

*Proof.* Assume that $v$ is right continuous at $x$. If $\epsilon > 0$, there is some $\delta > 0$ such that $x \leq y < x + \delta$ implies that $v(y) - v(x) = |v(y) - v(x)| < \epsilon$. If $x \leq y < x + \delta$, then

$$|f(y) - f(x)| \leq v(y) - v(x) < \epsilon,$$

showing that $f$ is right continuous at $x$.

Assume that $f$ is right continuous at $x$, with $a \leq x < b$. Let $\epsilon > 0$. There is some $\delta > 0$ such that $x \leq y < x + \delta$ implies that $|f(y) - f(x)| < \frac{\epsilon}{2}$. Because $V_x^b f$ is a supremum over partitions of $[x,b]$, there is some partition $P = \{t_0, t_1, \ldots, t_n\}$ of $[x,b]$ such that $V_x^b f - \frac{\epsilon}{2} \leq V(f,P)$. Let $x \leq y < \min\{\delta, t_1 - x\}$. Then $Q = \{t_0, y, t_1, \ldots, t_n\}$ is a refinement of $P$, so

$$\begin{aligned} V_x^b f - \frac{\epsilon}{2} &\leq V(f,P) \\ &\leq V(f,Q) \\ &= |f(y) - f(t_0)| + V(f, \{y, t_1, \ldots, t_n\}) \\ &< \frac{\epsilon}{2} + V_y^b f. \end{aligned}$$

Hence

$$\epsilon > V_x^b f - V_y^b f = V_x^y f = v(y) - v(x) = |v(y) - v(x)|,$$

showing that $v$ is right continuous at $x$. $\square$

[4] N. L. Carothers, *Real Analysis*, p. 207, Theorem 13.5.
[5] N. L. Carothers, *Real Analysis*, p. 207, Theorem 13.9.

For $f \in BV[a,b]$ and for $v$ the variation of $f$, we define the **positive variation of** $f$ as

$$p(x) = \frac{v(x) + f(x) - f(a)}{2}, \qquad x \in [a,b],$$

and the **negative variation of** $f$ as

$$n(x) = \frac{v(x) - f(x) + f(a)}{2}, \qquad x \in [a,b].$$

We can write the variation as $v = p + n$. We now establish properties of the positive and negative variations.[6]

**Theorem 10.** *Let $f \in BV[a,b]$, let $v$ be its variation, let $p$ be its positive variation, and let $n$ be its negative variation. Then $0 \leq p \leq v$ and $0 \leq n \leq v$, and $p$ and $n$ are nondecreasing.*

*Proof.* For $x \in [a,b]$, $v(x) = V_a^x f \geq |f(x) - f(a)|$. Because $v(x) \geq -(f(x) - f(a))$, we have $p(x) \geq 0$, and because $v(x) \geq f(x) - f(a)$ we have $n(x) \geq 0$. And then $v = p + n$ implies that $p \leq v$ and $n \leq v$.

For $x < y$,

$$\begin{aligned} p(y) - p(x) &= \frac{v(y) + f(y) - v(x) - f(x)}{2} \\ &= \frac{1}{2}\left(V_x^y f + (f(y) - f(x))\right) \\ &\geq \frac{1}{2}\left(|f(y) - f(x)| + (f(y) - f(x))\right) \\ &\geq 0 \end{aligned}$$

and

$$\begin{aligned} n(y) - n(x) &= \frac{v(y) - f(y) - v(x) + f(x)}{2} \\ &= \frac{1}{2}\left(V_x^y f - (f(y) - f(x))\right) \\ &\geq \frac{1}{2}\left(|f(y) - f(x)| - (f(y) - f(x))\right) \\ &\geq 0. \end{aligned}$$

□

We now prove that $BV[a,b]$ is a Banach algebra.[7]

**Theorem 11.** *$BV[a,b]$ is a Banach algebra.*

[6] N. L. Carothers, *Real Analysis*, p. 209, Proposition 13.11.
[7] N. L. Carothers, *Real Analysis*, p. 209, Proposition 13.12.

*Proof.* For $f_1, f_2 \in BV[a,b]$, let $v_1, v_2$, $p_1, p_2$, $n_1, n_2$ be their variations, positive variations, and negative variations, respectively. Then

$$\begin{aligned} f_1 f_2 &= (f_1(a) + p_1 - n_1)(f_2(a) + p_2 - n_2) \\ &= f_1(a) f_2(a) + p_1 p_2 + n_1 n_2 - n_1 p_2 - n_2 p_1 \\ &+ f_1(a) p_2 + f_2(a) p_1 - f_1(a) n_2 - f_2(a) n_1. \end{aligned}$$

Using this and the fact that if $f$ is nondecreasing then $V_a^b f = f(b) - f(a)$,

$$\begin{aligned} \|f_1 f_2\|_{BV} &= |f_1(a)||f_2(a)| + V_a^b(f_1 f_2) \\ &\le |f_1(a)||f_2(a)| + V_a^b(p_1 p_2) + V_a^b(n_1 n_2) + V_a^b(n_1 p_2) + V_a^b(n_2 p_1) \\ &+ |f_1(a)| V_a^b p_2 + |f_2(a)| V_a^b p_1 + |f_1(a)| V_a^b n_2 + |f_2(a)| V_a^b n_1 \\ &= |f_1(a)||f_2(a)| + p_1(b) p_2(b) + n_1(b) n_2(b) + n_1(b) p_2(b) + n_2(b) p_1(b) \\ &+ |f_1(a)| p_2(b) + |f_2(a)| p_1(b) + |f_1(a)| n_2(b) + |f_2(a)| n_1(b) \\ &= (|f_1(a)| + p_1(b) + n_1(b))(|f_2(a)| + p_2(b) + n_2(b)) \\ &= (|f_1(a) + v_1(b))(|f_2(a)| + v_2(b)) \\ &= \|f_1\|_{BV} \|f_2\|_{BV}, \end{aligned}$$

which shows that $BV[a,b]$ is a normed algebra. And $BV[a,b]$ is a Banach space, so $BV[a,b]$ is a Banach algebra. ☐

**Theorem 12.** *If $f \in C^1[a,b]$, then*

$$V_a^b f = \int_a^b |f'(t)| dt.$$

*Let $(f')^+$ and $(f')^-$ be the positive and negative parts of $f'$ and let $p$ and $n$ be the positive and negative variations of $f$. Then, for $x \in [a,b]$,*

$$p(x) = \int_a^x (f')^+(t) dt, \qquad n(x) = \int_a^x (f')^-(t) dt.$$

*Proof.* Lemma 4 states that $V_a^b \le \int_a^b |f'(t)| dt$. Because $f'$ is continuous it is Riemann integrable, hence for any $\epsilon > 0$ there is some partition $P = \{t_0, \dots, t_n\}$ of $[a,b]$ such that if $x_j \in [t_{j-1}, t_j]$ for $j = 1, \dots, n$ then

$$\left| \int_a^b |f'(t)| dt - \sum_{j=1}^n |f'(x_j)|(t_j - t_{j-1}) \right| < \epsilon.$$

By the mean value theorem, for each $j = 1, \dots, n$ there is some $x_j \in (t_{j-1}, t_j)$ such that $f'(x_j) = \frac{f(t_j) - f(t_{j-1})}{t_j - t_{j-1}}$. Then

$$V(f, P) = \sum_{j=1}^n |f(t_j) - f(t_{j-1})| = \sum_{j=1}^n |f'(x_j)|(t_j - t_{j-1}),$$

so

$$\left| \int_a^b |f'(t)|dt - V(f,P) \right| < \epsilon,$$

and thus

$$\int_a^b |f'(t)|dt < V(f,P) + \epsilon \leq V_a^b f + \epsilon.$$

This is true for all $\epsilon > 0$, therefore

$$\int_a^b |f'(t)|dt \leq V_a^b f,$$

which is what we wanted to show.

Write

$$g(t) = (f')^+(t) = \max\{f'(t), 0\}, \qquad h(t) = (f')^-(t) = -\min\{f'(t), 0\}.$$

These satisfy $g + h = |f'|$ and $g - h = f'$. Using the fundamental theorem of calculus,

$$\begin{aligned} p(x) &= \frac{1}{2}\left(v(x) + f(x) - f(a)\right) \\ &= \frac{1}{2}\left(V_a^x f + \int_a^x f'(t)dt\right) \\ &= \frac{1}{2}\left(\int_a^b |f'(t)|dt + \int_a^b f'(t)dt\right) \\ &= \int_a^b g(t)dt \end{aligned}$$

and

$$\begin{aligned} n(x) &= \frac{1}{2}\left(v(x) - f(x) + f(a)\right) \\ &= \frac{1}{2}\left(V_a^x f - \int_a^x f'(t)dt\right) \\ &= \frac{1}{2}\left(\int_a^x |f'(t)|dt - \int_a^x f'(t)dt\right) \\ &= \int_a^x h(t)dt. \end{aligned}$$

□

## 2 Helly's selection theorem

We will use the following lemmas in the proof of the Helly selection theorem.[8]

[8] N. L. Carothers, *Real Analysis*, p. 210, Theorem 13.13; p. 211, Lemma 13.14; p. 211, Lemma 13.15.

**Lemma 13.** *Suppose that $X$ is a set, that $f_n : X \to \mathbb{R}$ is a sequence of functions, and that there is some $K$ such that $\|f_n\|_\infty \leq K$ for all $n$. If $D$ is a countable subset of $X$, then there is a subsequence of $f_n$ that converges pointwise on $D$ to some $\phi : D \to \mathbb{R}$, which satisfies $\|\phi\|_\infty \leq K$.*

*Proof.* Say $D = \{x_k : k \geq 1\}$. Write $f_n^0 = f_n$. The sequence of real numbers $f_n^0(x_1)$ satisfies $f_n^0(x_1) \in [-K, K]$ for all $n$, and since the set $[-K, K]$ is compact there is a subsequence $f_n^1(x_1)$ of $f_n^0(x_1)$ that converges, say to $\phi(x_1) \in [-K, K]$. Suppose that $f_n^m(x_m)$ is a subsequence of $f_n^{m-1}(x_m)$ that converges to $\phi(x_m) \in [-K, K]$. Then the sequence of real numbers $f_n^m(x_{m+1})$ satisfies $f_n^m(x_{m+1}) \in [-K, K]$ for all $n$, and so there is a subsequence $f_n^{m+1}(x_{m+1})$ of $f_n^m(x_{m+1})$ that converges, say to $\phi(x_{m+1}) \in [-K, K]$. Let $k \geq 1$. Then one checks that $f_n^n(x_k) \to \phi(x_k)$ as $n \to \infty$, namely, $f_n^n$ is a subsequence of $f_n$ that converges pointwise on $D$ to $\phi$, and for each $k$ we have $\phi(x_k) \in [-K, K]$. $\square$

**Lemma 14.** *Let $D \subset [a, b]$ with $a \in D$ and $b = \sup D$. If $\phi : D \to \mathbb{R}$ is nondecreasing, then $\Phi : [a, b] \to \mathbb{R}$ defined by*

$$\Phi(x) = \sup\{\phi(t) : t \in [a, x] \cap D\}$$

*is nondecreasing and the restriction of $\Phi$ to $D$ is equal to $\phi$.*

**Lemma 15.** *If $f_n : [a, b] \to \mathbb{R}$ is a sequence of nondecreasing functions and there is some $K$ such that $\|f_n\|_\infty \leq K$ for all $n$, then there is a nondecreasing function $f : [a, b] \to \mathbb{R}$, satisfying $\|f\|_\infty \leq K$, and a subsequence of $f_n$ that converges pointwise to $f$.*

*Proof.* Let $D = (\mathbb{Q} \cap [a, b]) \cup \{a\}$. By Lemma 13, there is a function $\phi : D \to \mathbb{R}$ and a subsequence $f_{a_n}$ of $f_n$ that converges pointwise on $D$ to $\phi$, and $\|\phi\|_\infty \leq K$. Because each $f_n$ is nondecreasing, if $x, y \in D$ and $x < y$ then

$$\phi(x) = \lim_{n\to\infty} f_{a_n}(x) \leq \lim_{n\to\infty} f_{a_n}(y) = \phi(y),$$

namely, $\phi$ is nondecreasing. $D$ is a dense subset of $[a, b]$ and $a \in D$, so applying Lemma 14, there is a nondecreasing function $\Phi : [a, b] \to \mathbb{R}$ such that for $x \in D$,

$$\Phi(x) = \phi(x) = \lim_{n\to\infty} f_{a_n}(x).$$

Suppose that $\Phi$ is continuous at $x \in [a, b]$ and let $\epsilon > 0$. Using the fact that $\Phi$ is continuous at $x$, there are $p, q \in \mathbb{Q} \cap [a, b]$ such that $p < x < q$ and $\Phi(q) - \Phi(p) = |\Phi(q) - \Phi(p)| < \frac{\epsilon}{2}$. Because $p, q \in D$, $f_{a_n}(p) \to \Phi(p)$ and $f_{a_n}(q) \to \Phi(q)$, so there is some $N$ such that $n \geq N$ implies that both $|f_{a_n}(p) - \Phi(p)| < \frac{\epsilon}{2}$ and $|f_{a_n}(q) - \Phi(q)| < \frac{\epsilon}{2}$. Then for $n \geq N$, because each function $f_{a_n}$ is nondecreasing,

$$\begin{aligned} f_{a_n}(x) &\geq f_{a_n}(p) \\ &\geq \Phi(p) - \frac{\epsilon}{2} \\ &\geq \Phi(q) - \epsilon \\ &\geq \Phi(x) - \epsilon. \end{aligned}$$

Likewise, for $n \geq N$,

$$\begin{aligned} f_{a_n}(x) &\leq f_{a_n}(q) \\ &\leq \Phi(q) + \frac{\epsilon}{2} \\ &< \Phi(p) + \epsilon \\ &\leq \Phi(x) + \epsilon. \end{aligned}$$

This shows that if $\Phi$ is continuous at $x \in [a,b]$ then $f_{a_n}(x) \to \Phi(x)$.

Let $D(\Phi)$ be the collection of those $x \in [a,b]$ such that $\Phi$ is not continuous at $x$. Because $\Phi$ is monotone, $D(\Phi)$ is countable. So we have established that if $x \in [a,b] \setminus D(\Phi)$ then $f_{a_n}(x) \to \Phi(x)$. Because $f_{a_n} : [a,b] \to \mathbb{R}$ satisfies $\|f_{a_n}\|_\infty \leq K$ and $D(\Phi)$ is countable, Lemma 13 tells us that there is a function $F : D \to \mathbb{R}$ and a subsequence $f_{b_n}$ of $f_{a_n}$ such that $f_{b_n}$ converges pointwise on $D$ to $F$, and $\|F\|_\infty \leq K$. We define $f : [a,b] \to \mathbb{R}$ by $f(x) = \Phi(x)$ for $x \notin D(\Phi)$ and $f(x) = F(x)$ for $x \in D(\Phi)$. $\|f\|_\infty \leq K$. For $x \notin D(\Phi)$, $f_{a_n}(x)$ converges to $\Phi(x) = f(x)$, and $f_{b_n}(x)$ is a subsequence of $f_{a_n}(x)$ so $f_{b_n}(x)$ converges to $f(x)$. For $x \in D(\Phi)$, $f_{b_n}(x)$ converges to $F(x) = f(x)$. Therefore, for any $x \in [a,b]$ we have that $f_{b_n}(x) \to f(x)$, namely, $f_{b_n}$ converges pointwise to $f$. Because each function $f_{b_n}$ is nondecreasing, it follows that $f$ is nondecreasing. $\square$

Finally we prove the **pointwise Helly selection theorem**.[9]

**Theorem 16.** *Let $f_n$ be a sequence in $BV[a,b]$ and suppose there is some $K$ with $\|f_n\|_{BV} \leq K$ for all $n$. There is some subsequence of $f_n$ that converges pointwise to some $f \in BV[a,b]$, satisfying $\|f\|_{BV} \leq K$.*

*Proof.* Let $v_n$ be the variation of $f_n$. This satisfies, for any $n$,

$$\|v_n\|_\infty = V_a^b f_n \leq K$$

and

$$\|v_n - f_n\|_\infty \leq \|v_n\|_\infty + \|f_n\|_\infty \leq K + \|f_n\|_{BV} \leq 2K.$$

Theorem 8 tells us that $v_n - f_n$ and $v_n$ are nondecreasing, so we can apply Lemma 15 to get that there is a nondecreasing function $g : [a,b] \to \mathbb{R}$ and a subsequence $v_{a_n} - f_{a_n}$ of $v_n - f_n$ that converges pointwise to $g$. Then we use Lemma 15 again to get that there is a nondecreasing function $h : [a,b] \to \mathbb{R}$ and a subsequence $v_{b_n}$ of $v_{a_n}$ that converges pointwise to $h$. Because $g$ and $h$ are pointwise limits of nondecreasing functions, they are each nondecreasing and so belong to $BV[a,b]$. We define $f = h - g \in BV[a,b]$. For $x \in [a,b]$,

$$\begin{aligned} \lim_{n\to\infty} f_{b_n}(x) &= \lim_{n\to\infty} v_{b_n}(x) - \lim_{n\to\infty} (v_{b_n}(x) - f_{b_n}(x)) \\ &= h(x) - g(x) \\ &= f(x), \end{aligned}$$

[9] N. L. Carothers, *Real Analysis*, p. 212, Theorem 13.16.

namely the subsequence $f_{b_n}$ of $f_n$ converges pointwise to $f$. By Lemma 6, because $f_{b_n}$ is a sequence in $BV[a,b]$ that converges pointwise to $f$ we have

$$\begin{aligned}
\|f\|_{BV} &= |f(a)| + V_a^b f \\
&\leq |f(a)| + \liminf_{n\to\infty} V_a^b f_{b_n} \\
&= \liminf_{n\to\infty} \left(|f_{b_n}(a)| + V_a^b f_{b_n}\right) \\
&= \liminf_{n\to\infty} \|f_{b_n}\|_{BV} \\
&\leq K,
\end{aligned}$$

completing the proof. □