The Hilbert transform on $\mathbb{R}$

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1 The principal value integral

Let $A_\epsilon = \{x \in \mathbb{R} : |x| \geq \epsilon\}$. For $f \in \bigcap_{\epsilon>0} L^1(A_\epsilon)$, if $\int_{A_\epsilon} f(x)dx$ has a limit as $\epsilon \to 0$, we denote it by

$$PV \int_{\mathbb{R}} f(x)dx = \lim_{\epsilon \to 0} \int_{|x| \geq \epsilon} f(x)dx.$$

Let $\mathscr{S}$ be the collection of Schwartz functions $\mathbb{R} \to \mathbb{C}$ and let $\mathscr{S}'$ be its dual space, whose elements are called tempered distributions. For $\phi \in \mathscr{S}$, for $k > j$,

$$\begin{aligned}
\left| \int_{A_{1/j}} \frac{\phi(x)}{x} dx - \int_{A_{1/k}} \frac{\phi(x)}{x} dx \right| &= \left| \int_{1/k \leq |x| < 1/j} \frac{\phi(x)}{x} dx \right| \\
&= \left| \int_{1/k \leq |x| < 1/j} \frac{\phi(x) - \phi(0)}{x} dx \right| \\
&\leq \int_{1/k \leq |x| < 1/j} \|\phi'\|_b \\
&= \|\phi'\|_b \cdot \left(\frac{1}{j} - \frac{1}{k} \right) \\
&= \|\phi'\|_b \cdot \frac{k-j}{kj} \\
&\leq \frac{\|\phi'\|_b}{j}.
\end{aligned}$$

Therefore $\int_{A_{1/j}} \frac{\phi(x)}{x} dx$ is a Cauchy sequence in $\mathbb{C}$ and hence converges. Then the following limit exists:

$$\langle \phi, W \rangle = PV \int_{\mathbb{R}} \frac{\phi(x)}{x} dx = \lim_{\epsilon \to 0} \int_{|x| \geq \epsilon} \frac{\phi(x)}{x} dx.$$

It is apparent that $W : \mathscr{S} \to \mathbb{C}$ is linear, and one proves that $W \in \mathscr{S}'$.

For $\phi \in \mathscr{S}$, by Hadamard's lemma, there is a C^∞ function $\psi : \mathbb{R} \to \mathbb{C}$ such that $\phi(x) = \phi(0) + x\psi(x)$ for all x. For $\epsilon > 0$,

$$\begin{aligned}\int_\epsilon^1 \phi'(x) \log x dx &= x\psi(x) \log x \Big|_\epsilon^1 - \int_\epsilon^1 (\phi(x) - \phi(0)) \frac{1}{x} dx \\ &= -\epsilon\psi(\epsilon) \log \epsilon - \int_\epsilon^1 \frac{\phi(x)}{x} dx - \phi(0) \log \epsilon\end{aligned}$$

and

$$\begin{aligned}\int_{-1}^{-\epsilon} \phi'(x) \log |x| dx &= x\psi(x) \log(-x) \Big|_{-1}^{-\epsilon} - \int_{-1}^{-\epsilon} (\phi(x) - \phi(0)) \frac{1}{x} dx \\ &= -\epsilon\psi(-\epsilon) \log \epsilon - \int_{-1}^{-\epsilon} \frac{\phi(x)}{x} dx + \phi(0) \log \epsilon.\end{aligned}$$

Hence

$$\int_{\epsilon \le |x| \le 1} \phi'(x) \log |x| dx = -\epsilon \cdot (\psi(\epsilon) + \psi(-\epsilon)) \cdot \log \epsilon - \int_{\epsilon \le |x| \le 1} \frac{\phi(x)}{x} dx.$$

On the other hand,

$$\int_{|x| \ge 1} \phi'(x) \log |x| dx = -\int_{|x| \ge 1} \frac{\phi(x)}{x} dx.$$

Therefore

$$PV \int_{\mathbb{R}} \phi'(x) \log |x| dx = -PV \int_{\mathbb{R}} \frac{\phi(x)}{x} dx.$$

Let $\mu\phi(x) = \phi(-x)$ and $\tau_y\phi(x) = \phi(x - y)$. Then

$$\tau_y \mu\phi(x) = \mu\phi(x - y) = \phi(y - x) = \tau_x\phi(y).$$

Write

$$\begin{aligned}\phi * \psi(x) &= \int_{\mathbb{R}} \phi(y)\psi(x - y) dy \\ &= \int_{\mathbb{R}} \phi(y)(\tau_y\psi)(x) dy \\ &= \int_{\mathbb{R}} \phi(y)(\tau_x\mu\psi)(y) dy.\end{aligned}$$

For $u \in \mathscr{S}'$ and for $\phi \in \mathscr{S}$, define $\phi * u : \mathbb{R} \to \mathbb{C}$ by

$$(\phi * u)(x) = \langle \tau_x\mu\phi, u \rangle .$$

One proves that $\phi * u$ is a tempered distribution, and satisfies

$$\langle \psi, \phi * u \rangle = \langle (\mu\phi) * \psi, u \rangle .$$

Define $\tau_x u \in \mathscr{S}'$ by

$$\langle \phi, \tau_x u \rangle = \langle \tau_{-x}\phi, u \rangle .$$

2 The Hilbert transform

For $\epsilon > 0$, for $\phi \in \mathscr{S}$ let

$$\begin{aligned} H_\epsilon \phi(x) &= \frac{1}{\pi} \int_{|y| \geq \epsilon} \frac{\phi(x-y)}{y} dy \\ &= \frac{1}{\pi} \int_{|y| \geq \epsilon} \frac{\tau_y \phi(x)}{y} dy \\ &= \frac{1}{\pi} \int_{|y| \geq \epsilon} \frac{\tau_x \mu \phi(y)}{y} dy. \end{aligned}$$

Define

$$\begin{aligned} H\phi(x) &= \lim_{\epsilon \to 0} H_\epsilon \phi(x) \\ &= \frac{1}{\pi} \cdot PV \int_{\mathbb{R}} \frac{\tau_x \mu \phi(y)}{y} dy \\ &= \frac{1}{\pi} \cdot PV \int_{\mathbb{R}} \frac{\phi(x-y)}{y} dy \\ &= \frac{1}{\pi} \cdot PV \int_{\mathbb{R}} \frac{\phi(y)}{x-y} dy. \end{aligned}$$

For $x \in \mathbb{R}$,

$$\begin{aligned} \phi * W(x) &= \langle \tau_x \mu \phi, W \rangle \\ &= PV \int_{\mathbb{R}} \frac{\tau_x \mu \phi(y)}{y} dy \\ &= PV \int_{\mathbb{R}} \frac{\phi(x-y)}{y} dy \\ &= PV \int_{\mathbb{R}} \frac{\phi(y)}{x-y} dy \\ &= \lim_{\epsilon \to 0} \int_{|x| \geq \epsilon} \frac{\phi(y)}{x-y} dy. \end{aligned}$$

Thus

$$H\phi = \frac{1}{\pi} \phi * W = \phi * \left(\frac{W}{\pi} \right).$$

We calculate

$$\begin{aligned}
\left\langle \phi, \widehat{W} \right\rangle &= \left\langle \widehat{\phi}, W \right\rangle \\
&= \lim_{\epsilon \to 0} \int_{\epsilon \le |\xi| \le 1/\epsilon} \frac{\widehat{\phi}(\xi)}{\xi} d\xi \\
&= \lim_{\epsilon \to 0} \int_{|\xi| \ge \epsilon} \left(\int_{\mathbb{R}} \phi(x) e^{-2\pi i x \xi} dx \right) \frac{1}{\xi} d\xi \\
&= \lim_{\epsilon \to 0} \int_{\mathbb{R}} \phi(x) \left(\int_{\epsilon \le |\xi| \le 1/\epsilon} \frac{e^{-2\pi i x \xi}}{\xi} d\xi \right) dx \\
&= \lim_{\epsilon \to 0} \int_{\mathbb{R}} \phi(x) \left(\int_{\epsilon \le |\xi| \le 1/\epsilon} -i \frac{\sin 2\pi x \xi}{\xi} d\xi \right) dx.
\end{aligned}$$

Check that

$$\lim_{\epsilon \to 0} \int_{\epsilon \le |\xi| \le 1/\epsilon} \frac{\sin 2\pi x \xi}{\xi} d\xi = \pi \cdot \operatorname{sgn} x.$$

Then, using the dominated convergence theorem,

$$\left\langle \phi, \widehat{W} \right\rangle = \int_{\mathbb{R}} \phi(x) \cdot -i\pi \cdot \operatorname{sgn} x dx.$$

Thus, $\widehat{W} = -\pi i \cdot \operatorname{sgn}$.

Now,

$$\widehat{\phi * u} = \widehat{\phi} \cdot \widehat{u}.$$

Then

$$\widehat{H\phi}(\xi) = \frac{1}{\pi} \widehat{\phi * W}(\xi) = \frac{1}{\pi} \widehat{\phi}(\xi) \cdot \widehat{W}(\xi) = \widehat{\phi}(\xi) \cdot -i \cdot \operatorname{sgn}(\xi).$$

Let

$$m_H(\xi) = -i \cdot \operatorname{sgn}(\xi),$$

with which

$$\widehat{H\phi} = m_H \cdot \widehat{\phi}.$$

Writing $F\phi = \widehat{\phi}$,

$$FH\phi = m_H \cdot F\phi.$$

So

$$H\phi = F^{-1}(m_H \cdot F\phi),$$

and hence

$$H^2\phi = F^{-1}(m_H \cdot FH\phi) = F^{-1}(m_H \cdot m_H F\phi).$$

For $\xi \neq 0$, $m_H(\xi)^2 = -1$, which yields

$$H^2\phi = F^{-1}(-F\phi) = -\phi.$$

Thus $H^2 = -\mathrm{id}$. Therefore $\|H\phi\|_{L^2} = \|\phi\|_{L^2}$.

Thus it makes sense to define $H : L^2(\mathbb{R}) \to L^2(\mathbb{R})$. For $f, g \in L^2(\mathbb{R})$, by Plancherel's theorem, and as $\overline{m_H} = -m_H$,

$$\begin{aligned}
\langle Hf, g\rangle &= \left\langle \widehat{Hf}, \widehat{g} \right\rangle \\
&= \int_{\mathbb{R}} \widehat{Hf}(\xi) \cdot \overline{\widehat{g}(\xi)} d\xi \\
&= \int_{\mathbb{R}} m_H(\xi) \cdot \widehat{f}(\xi)\overline{\widehat{g}(\xi)} d\xi \\
&= -\int_{\mathbb{R}} \widehat{f}(\xi) \cdot \overline{m_H(\xi) \cdot \widehat{g}(\xi)} d\xi \\
&= -\int_{\mathbb{R}} \widehat{f}(\xi) \cdot \overline{\widehat{Hg}(\xi)} d\xi \\
&= -\langle f, Hg\rangle .
\end{aligned}$$

But $\langle Hf, g\rangle = \langle f, H^* g\rangle$, so

$$\langle f, H^* g\rangle = \langle f, -Hg\rangle ,$$

which implies that $H^* g = -Hg$ and thus $H^* = -H$. Furthermore,

$$\widehat{H^* g}(\xi) = -\widehat{Hg}(\xi) = -m_H(\xi) \cdot \widehat{g}(\xi) = i \cdot \operatorname{sgn}(\xi) \cdot \widehat{g}(\xi).$$

3 The Poisson kernel

For $y > 0$, calculate

$$\begin{aligned}
\int_{\mathbb{R}} e^{2\pi i\xi x} e^{-2\pi y|\xi|} d\xi &= \frac{1}{2\pi i x + 2\pi y} - \frac{1}{2\pi i x - 2\pi y} \\
&= \frac{2\pi i x - 2\pi y - 2\pi i x - 2\pi y}{-4\pi^2 x^2 - 4\pi^2 y^2} \\
&= \frac{y}{\pi(x^2 + y^2)}
\end{aligned}$$

and

$$\begin{aligned}
-i\int_{\mathbb{R}} e^{2\pi i\xi x} \operatorname{sgn}(\xi) e^{-2\pi y|\xi|} d\xi &= \frac{i}{2\pi i x + 2\pi y} + \frac{i}{2\pi i x - 2\pi y} \\
&= \frac{i(2\pi i x - 2\pi y + 2\pi i x + 2\pi y)}{-4\pi^2 x^2 - 4\pi^2 y^2} \\
&= \frac{-4\pi x}{-4\pi^2 x^2 - 4\pi^2 y^2} \\
&= \frac{x}{\pi(x^2 + y^2)}.
\end{aligned}$$

For $y > 0$ let

$$P_y(x) = \frac{1}{\pi} \frac{y}{x^2 + y^2}$$

and

$$Q_y(x) = \frac{1}{\pi} \frac{x}{x^2 + y^2}.$$

Then

$$\widehat{P}_y(\xi) = e^{-2\pi y |\xi|}$$

and

$$\widehat{Q}_y(\xi) = -i \cdot \operatorname{sgn}(\xi) e^{-2\pi y |\xi|}.$$

Also,

$$P_y(x) + iQ_y(x) = \frac{1}{\pi} \frac{y + ix}{x^2 + y^2} = \frac{1}{\pi} \frac{1}{y - ix}.$$

For a Borel measurable function $f : \mathbb{R} \to \mathbb{C}$ for which the integral exists,

$$\begin{aligned}
(P_y * f)(x) &= \int_{\mathbb{R}} P_y(x - t) f(t) dt \\
&= \int_{\mathbb{R}} f(t) \frac{y}{\pi((x - t)^2 + y^2)} dt \\
&= \int_{\mathbb{R}} f(x - t) \frac{y}{\pi(t^2 + y^2)} dt
\end{aligned}$$

and

$$\begin{aligned}
(Q_y * f)(x) &= \int_{\mathbb{R}} Q_y(x - t) f(t) dt \\
&= \int_{\mathbb{R}} f(t) \frac{x - t}{\pi((x - t)^2 + y^2)} dt \\
&= \int_{\mathbb{R}} f(x - t) \frac{t}{\pi(t^2 + y^2)} dt.
\end{aligned}$$

Then

$$\begin{aligned}
P_y * f(x) + iQ_y * f(x) &= \int_{\mathbb{R}} f(x - t) \frac{1}{\pi} \frac{1}{y - it} dt \\
&= \int_{\mathbb{R}} f(t) \frac{1}{\pi} \frac{1}{y - ix + it} dt \\
&= \frac{i}{\pi} \int_{\mathbb{R}} \frac{f(t)}{x + iy - t} dt.
\end{aligned}$$

For $y_1, y_2 > 0$,

$$\begin{aligned}
\widehat{P_{y_1} * P_{y_2}}(\xi) &= \widehat{P_{y_1}}(\xi) \cdot \widehat{P_{y_2}}(\xi) \\
&= e^{-2\pi y_1 |\xi|} \cdot e^{-2\pi y_2 |\xi|} \\
&= e^{-2\pi (y_1 + y_2) |\xi|} \\
&= \widehat{P_{y_1 + y_2}}(\xi).
\end{aligned}$$

Therefore $(P_y)_{y>0}$ is a semigroup using convolution: for $y_1, y_2 > 0$,

$$P_{y_1} * P_{y_2} = P_{y_1+y_2}.$$

Let $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ and for $\phi \in \mathscr{S}$ let

$$F_\phi(z) = \frac{i}{\pi} \int_{\mathbb{R}} \frac{\phi(t)}{z-t} dt, \qquad z \in \mathbb{H},$$

which is a complex analytic function. For $z = x + iy \in \mathbb{H}$,

$$P_y * \phi(x) + iQ_y * \phi(x) = \frac{i}{\pi} \int_{\mathbb{R}} \frac{\phi(t)}{x + iy - t} dt = F_\phi(z).$$

It is proved that for $1 \leq p < \infty$ and $f \in L^p(\mathbb{R})$, $Q_\epsilon * f - H_\epsilon f \to 0$ in L^p as $\epsilon \to 0$, and that for almost all $x \in \mathbb{R}$, $Q_\epsilon * f(x) - H_\epsilon f(x) \to 0$ as $\epsilon \to 0$.[1]

For $1 < p < \infty$, it can be proved that there is some C_p such that

$$\|H\phi\|_{L^p} \leq C_p \|\phi\|_{L^p}$$

for all $\phi \in \mathscr{S}$, with $C_p \leq 2p$ for $2 \leq p < \infty$ and $C_p \leq \frac{2p}{p-1}$ for $1 < p \leq 2$.[2]

[1] Loukas Grafakos, *Classical Fourier Analysis*, second ed., p. 254, Theorem 4.1.5.
[2] Loukas Grafakos, *Classical Fourier Analysis*, second ed., p. 255, Theorem 4.1.7.