The Fourier transform of holomorphic functions

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For $f \in L^1(\mathbb{R})$, define

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} e^{-2\pi i \xi x} f(x) dx, \qquad \xi \in \mathbb{R}.$$

For $a > 0$, write

$$S_a = \{z \in \mathbb{C} : |\operatorname{Im} z| < a\}.$$

We define $\mathfrak{F}_a$ to be the set of functions f that are holomorphic on S_a and for which there is some $A > 0$ such that

$$|f(x+iy)| \leq \frac{A}{1+x^2}, \qquad x + iy \in S_a. \tag{1}$$

For example, for $f(z) = e^{-\pi z^2}$,

$$|f(z)| = |e^{-\pi(x+iy)^2}| = |e^{-\pi x^2 - 2\pi i xy + \pi y^2}| = e^{-\pi x^2} e^{\pi y^2},$$

and for any $a > 0$, $f \in S_a$.

The following is from Stein and Shakarchi.[1]

Theorem 1. *If $a > 0$ and $f \in \mathfrak{F}_a$, then for any $0 \leq b < a$,*

$$\widehat{f}(\xi) = e^{-2\pi|\xi|b} \int_{-\infty}^{\infty} e^{-2\pi i \xi x} f(x - i \cdot \operatorname{sgn} \xi \cdot b) dx, \qquad \xi \in \mathbb{R}.$$

Proof. If $b = 0$ then the claim is immediate. If $0 < b < a$, we define $g(z) = e^{-2\pi i \xi z} f(z)$. Because $f \in \mathfrak{F}_a$ there is some $A > 0$ such that f satisfies (1). We

[1] Elias M. Stein and Rami Shakarchi, *Complex Analysis*, p. 114, Theorem 2.1.

prove the claim separately for $\xi \geq 0$ and $\xi \leq 0$. For $\xi \geq 0$, with $R > 0$,

$$\begin{aligned}
\left| \int_{-R-ib}^{-R} g(z)dz \right| &\leq \int_{-R-ib}^{-R} |e^{-2\pi i \xi z} f(z)| dz \\
&= \int_{-b}^{0} |e^{-2\pi i \xi(-R+iy)} f(-R+iy)| dy \\
&= \int_{-b}^{0} e^{2\pi \xi y} |f(-R+iy)| dy \\
&\leq \int_{-b}^{0} e^{2\pi \xi y} \frac{A}{1+R^2} dy \\
&= O(R^{-2})
\end{aligned}$$

and likewise

$$\left| \int_{R}^{R-ib} g(z)dz \right| = O(R^{-2}).$$

g is holomorphic on S_a, so by Cauchy's integral theorem, taking $R \to \infty$,

$$\int_{-\infty}^{\infty} g(z)dz = \int_{-\infty-ib}^{\infty-ib} g(z)dz,$$

i.e.,

$$\begin{aligned}
\widehat{f}(\xi) &= \int_{-\infty-ib}^{-\infty-ib} e^{-2\pi i \xi z} f(z) dz \\
&= \int_{-\infty}^{\infty} e^{-2\pi i \xi(x-ib)} f(x-ib) dx \\
&= e^{-2\pi \xi b} \int_{-\infty}^{\infty} e^{-2\pi i \xi x} f(x-ib) dx.
\end{aligned}$$

For $\xi \leq 0$, with $R > 0$,

$$\begin{aligned}
\left| \int_{-R+ib}^{-R} g(z)dz \right| &\leq \int_{-R}^{-R+ib} |e^{-2\pi i \xi z} f(z)| dz \\
&= \int_{0}^{b} |e^{-2\pi i \xi(-R+iy)} f(-R+iy)| dy \\
&= \int_{0}^{b} e^{2\pi \xi y} |f(-R+iy)| dy \\
&\leq \int_{0}^{b} e^{2\pi \xi y} \frac{A}{1+R^2} dy \\
&= O(R^{-2}),
\end{aligned}$$

and likewise

$$\left| \int_R^{R+ib} g(z)dz \right| = O(R^{-2}).$$

By Cauchy's integral theorem, taking $R \to \infty$,

$$\int_{-\infty}^{\infty} g(z)dz = \int_{-\infty+ib}^{\infty+ib} g(z)dz,$$

i.e.,

$$\begin{aligned}
\widehat{f}(\xi) &= \int_{-\infty+ib}^{\infty+ib} e^{-2\pi i \xi z} f(z)dz \\
&= \int_{-\infty}^{\infty} e^{-2\pi i \xi (x+ib)} f(x+ib)dx \\
&= e^{2\pi \xi b} \int_{-\infty}^{\infty} e^{-2\pi i \xi x} f(x+ib)dx.
\end{aligned}$$

□

Corollary 2. *If $a > 0$ and $f \in \mathfrak{F}_a$, then for any $0 \le b < a$ there is some B such that*

$$|\widehat{f}(\xi)| \le B e^{-2\pi b |\xi|}, \qquad \xi \in \mathbb{R}.$$

Proof. Because $f \in \mathfrak{F}_a$ there is some $A > 0$ such f satisfies (1). Put

$$B = A \int_{-\infty}^{\infty} \frac{1}{1+x^2} dx = \pi A.$$

By Theorem 1,

$$\begin{aligned}
|\widehat{f}(\xi)| &\le e^{-2\pi |\xi| b} \int_{-\infty}^{\infty} |e^{-2\pi i \xi x} f(x - i \cdot \operatorname{sgn} \xi \cdot b)| dx \\
&= e^{-2\pi |\xi| b} \int_{-\infty}^{\infty} |f(x - i \cdot \operatorname{sgn} \xi \cdot b)| dx \\
&\le e^{-2\pi |\xi| b} \int_{-\infty}^{\infty} \frac{A}{1+x^2} dx \\
&= e^{-2\pi |\xi| b} \cdot B.
\end{aligned}$$

□

Define

$$\mathfrak{F} = \bigcup_{a>0} \mathfrak{F}_a.$$

We now prove the **Fourier inversion formula** for functions belonging to $\mathfrak{F}$.[2]

[2] Elias M. Stein and Rami Shakarchi, *Complex Analysis*, p. 115, Theorem 2.2.

Theorem 3. *If $f \in \mathfrak{F}$, then*

$$f(x) = \int_{-\infty}^{\infty} e^{2\pi i x\xi} \widehat{f}(\xi) d\xi, \qquad x \in \mathbb{R}.$$

Proof. Say $f \in \mathfrak{F}_a$, write

$$\int_{-\infty}^{\infty} e^{2\pi i x\xi} d\xi = \int_{0}^{\infty} e^{2\pi i x\xi} d\xi + \int_{-\infty}^{0} e^{2\pi i x\xi} d\xi = I_1 + I_2,$$

and take $0 < b < a$. First we handle I_1. By Theorem 1, for $\xi > 0$,

$$\widehat{f}(\xi) = e^{-2\pi\xi b} \int_{-\infty}^{\infty} e^{-2\pi i\xi u} f(u - ib) du = \int_{-\infty}^{\infty} e^{-2\pi i\xi(u-ib)} f(u - ib) du,$$

with which, because $\xi b > 0$,

$$\begin{aligned}
\int_{0}^{\infty} e^{2\pi i x\xi} \widehat{f}(\xi) d\xi &= \int_{0}^{\infty} e^{2\pi i x\xi} \left(\int_{-\infty}^{\infty} e^{-2\pi i\xi(u-ib)} f(u - ib) du \right) d\xi \\
&= \int_{-\infty}^{\infty} f(u - ib) \int_{0}^{\infty} e^{-2\pi i\xi(u-ib-x)} d\xi du \\
&= \int_{-\infty}^{\infty} f(u - ib) \frac{1}{2\pi i(u - ib - x)} du \\
&= \frac{1}{2\pi i} \int_{L_1} \frac{f(\zeta)}{\zeta - x} d\zeta.
\end{aligned}$$

where $L_1 = \{u - ib : u \in \mathbb{R}\}$ traversed left to right. Now we handle I_2. By Theorem 1, for $\xi < 0$,

$$\widehat{f}(\xi) = e^{2\pi\xi b} \int_{-\infty}^{\infty} e^{-2\pi i\xi u} f(u + ib) dx = \int_{-\infty}^{\infty} e^{-2\pi i\xi(u+ib)} f(u + ib) dx,$$

with which, because $\xi b < 0$,

$$\begin{aligned}
\int_{-\infty}^{0} e^{2\pi i x\xi} \widehat{f}(\xi) d\xi &= \int_{-\infty}^{0} e^{2\pi i x\xi} \left(\int_{-\infty}^{\infty} e^{-2\pi i\xi(u+ib)} f(u + ib) du \right) d\xi \\
&= \int_{-\infty}^{\infty} f(u + ib) \int_{-\infty}^{0} e^{-2\pi i\xi(u+ib-x)} d\xi du \\
&= \int_{-\infty}^{\infty} f(u + ib) \frac{-1}{2\pi i(u + ib - x)} du \\
&= -\frac{1}{2\pi i} \int_{L_2} \frac{f(\zeta)}{\zeta - x} d\xi,
\end{aligned}$$

where $L_2 = \{u + ib : u \in \mathbb{R}\}$ traversed left to right. Thus

$$\int_{-\infty}^{\infty} e^{2\pi i x\xi} \widehat{f}(\xi) d\xi = \frac{1}{2\pi i} \int_{L_1} \frac{f(\zeta)}{\zeta - x} d\zeta - \frac{1}{2\pi i} \int_{L_2} \frac{f(\zeta)}{\zeta - x} d\xi. \tag{2}$$

Let $\gamma - R$ be the rectangle starting at $-R-ib$, going to $R-ib$, going to $R+ib$, going to $-R+ib$, going to $-R-ib$. Because this rectangle and its interior are contained in S_a, on which f is holomorphic, by the residue theorem we have, for $R > |x|$,

$$\int_{\gamma_R} \frac{f(\zeta)}{\zeta - x} d\zeta = 2\pi i \cdot \mathrm{Res}_{\zeta=x} \frac{f(\zeta)}{\zeta - x} = 2\pi i \cdot f(x).$$

We estimate the integrand on the vertical sides of γ_R. For the left side, taking A such that f satisfies (1),

$$\left| \int_{-R+ib}^{-R-ib} \frac{f(\zeta)}{\zeta - x} d\zeta \right| \le \int_{-b}^{b} \left| \frac{f(-R+iy)}{-R+iy-x} \right| dy \le \int_{-b}^{b} \frac{A}{1+R^2} \cdot \frac{1}{R-|x|} dy = O(R^{-3}).$$

For the right side,

$$\left| \int_{R-ib}^{R+ib} \frac{f(\zeta)}{\zeta - x} d\zeta \right| \le \int_{-b}^{b} \left| \frac{f(R+iy)}{R+iy-x} \right| dy \le \int_{-b}^{b} \frac{A}{1+R^2} \cdot \frac{1}{R-|x|} dy = O(R^{-3}).$$

Thus, taking $R \to \infty$ we get

$$\int_{L_1} \frac{f(\zeta)}{\zeta - x} d\zeta - \int_{L_2} \frac{f(\zeta)}{\zeta - x} d\zeta = 2\pi i \cdot f(x),$$

which by (2) is

$$\int_{-\infty}^{\infty} e^{2\pi i x \xi} \widehat{f}(\xi) d\xi = f(x),$$

proving the claim. □

We now prove the **Poisson summation formula**.[3]

Theorem 4. *If $f \in \mathfrak{F}$, then*

$$\sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} \widehat{f}(n).$$

Proof. Say $f \in \mathfrak{F}_a$, take $0 < b < a$, and for N a positive integer let γ_N be the rectangle starting at $-N - \frac{1}{2} - ib$, going to $N + \frac{1}{2} - ib$, going to $N + \frac{1}{2} + ib$, going to $-N - \frac{1}{2} + ib$, going to $-N - \frac{1}{2} - ib$. Because $f \in \mathfrak{F}_a$, $\frac{f(z)}{e^{2\pi i z} - 1}$ is meromorphic on a region containing γ_N and its interior, and has poles at $z = -N, \dots, N$, with residues

$$\mathrm{Res}_{z=n} \frac{f(z)}{e^{2\pi i z} - 1} = \frac{f(n)}{2\pi i e^{2\pi i n}} = \frac{f(n)}{2\pi i}.$$

Thus the residue theorem gives us

$$\int_{\gamma_N} \frac{f(z)}{e^{2\pi i z} - 1} dz = 2\pi i \sum_{|n| \le N} \frac{f(n)}{2\pi i} = \sum_{|n| \le N} f(n). \tag{3}$$

[3]Elias M. Stein and Rami Shakarchi, *Complex Analysis*, p. 118, Theorem 2.4.

For the left side of γ_N, with $z = -N - \frac{1}{2} + iy$, $-b \le y \le b$,

$$|e^{2\pi iz} - 1| = |e^{-2\pi iN - \pi i - 2\pi y} - 1| = | - e^{-2\pi y} - 1| \ge 1,$$

so, taking $A > 0$ such that f satisfies (1),

$$\begin{aligned}\left| \int_{-N-\frac{1}{2}+ib}^{-N-\frac{1}{2}-ib} \frac{f(z)}{e^{2\pi iz} - 1} dz \right| &\le \int_{-b}^{b} \left| f\left(-N - \frac{1}{2} + iy \right) \right| dy \\ &\le \int_{-b}^{b} \frac{A}{1 + \left(-N - \frac{1}{2}\right)^2} dy \\ &= O(N^{-2}).\end{aligned}$$

Likewise,

$$\left| \int_{N+\frac{1}{2}-ib}^{N+\frac{1}{2}+ib} \frac{f(z)}{e^{2\pi iz} - 1} dz \right| = O(N^{-2}).$$

Therefore, taking $N \to \infty$, (3) becomes

$$\int_{L_1} \frac{f(z)}{e^{2\pi iz} - 1} dz - \int_{L_2} \frac{f(z)}{e^{2\pi iz} - 1} dz = \sum_{n \in \mathbb{Z}} f(n),$$

where $L_1 = \{x - ib : x \in \mathbb{R}\}$, traversed left to right, and $L_2 = \{x + ib : x \in \mathbb{R}\}$, traversed left to right. Then, as $b > 0$,

$$\begin{aligned}\sum_{n \in \mathbb{Z}} f(n) &= \int_{L_1} f(z) \frac{e^{-2\pi iz}}{1 - e^{-2\pi iz}} dz + \int_{L_2} f(z) \frac{1}{1 - e^{2\pi iz}} dz \\ &= \int_{L_1} f(z) e^{-2\pi iz} \sum_{n=0}^{\infty} (e^{-2\pi iz})^n dz + \int_{L_2} f(z) \sum_{n=0}^{\infty} (e^{2\pi iz})^n \\ &= \sum_{n=0}^{\infty} \int_{L_1} e^{-2\pi i(n+1)z} f(z) dz + \sum_{n=0}^{\infty} \int_{L_2} e^{2\pi inz} f(z) dz \\ &= \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} e^{-2\pi in(x-ib)} f(x - ib) dx + \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} e^{2\pi in(x+ib)} f(x + ib) dx \\ &= \sum_{n=1}^{\infty} e^{-2\pi nb} \int_{-\infty}^{\infty} e^{-2\pi inx} f(x - ib) dx \\ &+ \sum_{n=0}^{\infty} e^{-2\pi nb} \int_{-\infty}^{\infty} e^{2\pi inx} f(x + ib) dx.\end{aligned}$$

Using Theorem 1 this becomes

$$\sum_{n \in \mathbb{Z}} f(n) = \sum_{n=1}^{\infty} \widehat{f}(n) + \sum_{n=0}^{\infty} \widehat{f}(-n) = \sum_{n \in \mathbb{Z}} \widehat{f}(n),$$

proving the claim. $\square$

Take as granted that

$$\int_{-\infty}^{\infty} e^{-2\pi i\xi x} e^{-\pi x^2} dx = e^{-\pi\xi^2}, \qquad \xi \in \mathbb{R}.$$

For $t > 0$ and $a \in \mathbb{R}$, with $y = t^{1/2}(x + a)$,

$$\begin{aligned}\int_{-\infty}^{\infty} e^{-2\pi i\xi x} e^{-\pi t(x+a)^2} dx &= \int_{-\infty}^{\infty} e^{-2\pi i\xi(t^{-1/2}y-a)} e^{-\pi y^2} t^{-1/2} dy \\ &= e^{2\pi i\xi a} t^{-1/2} \int_{-\infty}^{\infty} e^{-2\pi i\xi t^{-1/2} y} e^{-\pi y^2} dy \\ &= e^{2\pi i\xi a} t^{-1/2} e^{-\pi\xi^2 t^{-1}}.\end{aligned}$$

With $f(x) = e^{-\pi t(x+a)^2}$, this shows us that

$$\widehat{f}(\xi) = e^{2\pi i\xi a} t^{-1/2} e^{-\pi\xi^2 t^{-1}},$$

and applying the Poisson summaton gives

$$\sum_{n\in\mathbb{Z}} e^{-\pi t(n+a)^2} = \sum_{n\in\mathbb{Z}} e^{2\pi i n a} t^{-1/2} e^{-\pi n^2 t^{-1}}. \tag{4}$$

Define

$$\vartheta(t) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 t}, \qquad t > 0.$$

Using (4) with $a = 0$ gives

$$\vartheta(t) = t^{-1/2} \vartheta\left(\frac{1}{t}\right).$$