The infinite-dimensional torus

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1 Locally compact abelian groups

Let \( \mathbb{N} \) denote the positive integers.

If \( G_i, i \in I \), are compact abelian groups, we define their **direct product** to be the cartesian product

\[
\prod_{i \in I} G_i
\]

with the coarsest topology such that the projection maps \( \pi_i : \prod_{j \in I} G_j \to G_i \) are continuous (namely the product topology), with which the direct product is a compact abelian group. We write \( G^\omega = \prod_{\mathbb{N}} G \).

We shall be interested especially in the compact abelian group \( T = S^1 \), and we call \( T^\omega \) the **infinite-dimensional torus**.

If \( \Gamma_i, i \in I \), are discrete abelian groups, their **direct sum**, denoted by

\[
\bigoplus_{i \in I} \Gamma_i,
\]

consists of those elements \( x \) of the cartesian product \( \prod_{i \in I} \Gamma_i \) such that the set \( \{ i \in I : \pi_i(x) \neq 0 \} \) is finite. Let \( p_i : \bigoplus_{j \in I} \Gamma_j \to \Gamma_i \) be the restriction of \( \pi_i \) to \( \bigoplus_{j \in I} \Gamma_j \). We give the direct sum the finest topology such that the inclusion maps \( q_i : \Gamma_i \to \bigoplus_{j \in I} \Gamma_j \), defined by

\[
(p_j \circ q_i)(x) = \begin{cases} x & j = i \\ 0 & j \neq i \end{cases}, \quad x \in \Gamma_i,
\]

are continuous. With this topology, the direct sum is a discrete abelian group. We write

\[
\Gamma^\omega = \bigoplus_{\mathbb{N}} \Gamma.
\]
We shall be interested especially in the discrete abelian group \( \mathbb{Z} \), and in the infinite direct sum \( \mathbb{Z}^\infty \). (I don’t know how significant an object it is, but I mention that the abelian group \( \prod \mathbb{Z} \) is called the Baer-Specker group.)

When speaking about 0 or 1 in a locally compact abelian group, it is unambiguous that this symbol denotes the identity element of the group, because there is only one distinguished element in a locally compact abelian group. Often we denote the identity element of a compact abelian group by 1 and the identity element of a discrete abelian group by 0.

If \( G_1, \ldots, G_n \) are locally compact abelian groups, it is straightforward to check that the cartesian product

\[
\prod_{k=1}^n G_k
\]

with the product topology is a locally compact abelian group. We call this both the direct product and the direct sum and write

\[
G_1 \oplus \cdots \oplus G_n = \bigoplus_{k=1}^n G_k = \prod_{k=1}^n G_k = G_1 \times \cdots \times G_n.
\]

## 2 Dual groups

If \( G \) is a locally compact abelian group, denote by \( \hat{G} \) its dual group, that is, the set of continuous group homomorphisms \( G \to S^1 \). For \( g \in G \) and \( \phi \in \hat{G} \) we write

\[
\langle x, \phi \rangle = \phi(x).
\]

\( \hat{G} \) has the initial topology induced by \( \{ \phi \mapsto \langle x, \phi \rangle : x \in G \} \), with which it is a locally compact abelian group. If \( G \) is compact then \( \hat{G} \) is discrete, and if \( G \) is discrete then \( \hat{G} \) is compact.

**Theorem 1.** Suppose that \( G_1, \ldots, G_n \) are locally compact abelian groups. Then the dual group of \( G_1 \oplus \cdots \oplus G_n \) is isomorphic as a topological group to \( \hat{G}_1 \oplus \cdots \oplus \hat{G}_n \).

We prove in the following theorem that for discrete abelian groups, the dual group of a direct sum is the direct product of the dual groups.\(^1\) In particular, this shows that the dual group of \( \mathbb{Z}^\infty \) is \( T^\infty \). Then by the Pontryagin duality theorem\(^2\) we get that the dual group of \( T^\infty \) is \( \mathbb{Z}^\infty \).

**Theorem 2.** Suppose that \( \Gamma_i, i \in I \), are discrete abelian groups and let

\[
\Gamma = \bigoplus_{i \in I} \Gamma_i, \quad G = \prod_{i \in I} \hat{\Gamma}_i.
\]


\(^2\) Walter Rudin, *Fourier Analysis on Groups*, p. 28, Theorem 1.7.2.
Then $\Phi : G \to \hat{\Gamma}$, defined by

$$(\Phi g)(\gamma) = \prod_{i \in I} \langle p_i(\gamma), \pi_i(g) \rangle, \quad g \in G, \gamma \in \Gamma,$$

is an isomorphism of topological groups. Here, $\pi_i : \Gamma \to \hat{\Gamma}_i$ and $p_i : \Gamma \to \Gamma_i$ are the projection maps.

**Proof.** The definition of $(\Phi g)(\gamma)$ makes sense because $\{i \in I : p_i(\gamma) \neq 0\}$ is finite and hence $\{i \in I : \langle p_i(\gamma), \pi_i(g) \rangle \neq 1\}$ is finite. For $g, h \in G$ and $\gamma \in \Gamma$,

$$(\Phi(gh))(\gamma) = \prod_{i \in I} \langle p_i(\gamma), \pi_i(gh) \rangle$$

$$= \prod_{i \in I} \langle p_i(\gamma), \pi_i(g) \rangle \langle p_i(\gamma), \pi_i(h) \rangle$$

$$= (\Phi g)(\gamma)(\Phi h)(\gamma)$$

$$= ((\Phi g)(\Phi h))(\gamma),$$

showing that $\Phi(gh) = \Phi(g)\Phi(h)$ and hence that $\Phi$ is a homomorphism. Suppose that $g \in \ker \Phi$. For each $i \in I$ and each $\gamma \in \Gamma_i$,

$$((\Phi g) \circ q_i)(\gamma) = (\Phi g)(q_i(\gamma)) = 1,$$

where $q_i : \Gamma_i \to \Gamma$ is the inclusion map. This is true for all $\gamma \in \Gamma_i$, so $(\Phi g) \circ q_i$ is the identity element of $\hat{\Gamma}_i$. And this is true for all $i \in I$, so $\Phi g$ is the identity element of $G$. Therefore $\Phi$ is one-to-one. Suppose that $\alpha \in \hat{\Gamma}$. Define $g \in G$ as follows: for each $i \in I$, take $\pi_i(g) = \alpha \circ q_i \in \hat{\Gamma}_i$. Then $g$ satisfies $\Phi g = \alpha$, hence $\Phi$ is onto and is therefore a group isomorphism.

A continuous bijection from a compact topological space to a Hausdorff space is a homeomorphism, so to prove that $\Phi$ is a homeomorphism it suffices to prove that $\Phi$ is continuous. $\hat{\Gamma}$ has the initial topology induced by $\{\alpha \mapsto \langle \gamma, \alpha \rangle : \gamma \in \Gamma\}$, which are maps $\hat{\Gamma} \to S^1$, so by the universal property of the initial topology, to prove that $\Phi$ is continuous it suffices to prove that for each $\gamma \in \Gamma$,

$$g \mapsto \langle \gamma, \Phi g \rangle$$

is continuous $G \to S^1$. For $\gamma \in \Gamma$, let $J_\gamma = \{i \in I : p_i(\gamma) \neq 0\}$, which is a finite set. For each $i \in J_\gamma$, it is straightforward to check that the map $g \mapsto \langle p_i(\gamma), \pi_i(g) \rangle$ is continuous $G \to S^1$. Hence the map

$$g \mapsto (\Phi g)(\gamma) = \prod_{i \in J_\gamma} \langle p_i(\gamma), \pi_i(g) \rangle$$

is continuous $G \to S^1$, being a product of finitely many continuous functions $G \to S^1$, and this completes the proof. \qed
Let $G$ be a locally compact abelian group. If $\Gamma_0$ is a finite subset of $\hat{G}$ and $a_\gamma \in \mathbb{C}$ for each $\gamma \in \Gamma_0$, we call the function $G \to \mathbb{C}$ defined by

$$x \mapsto \sum_{\gamma \in \Gamma_0} a_\gamma \langle x, \gamma \rangle$$

a **trigonometric polynomial** on $G$. Suppose that $G$ is a compact abelian group. Its dual group $\hat{G}$ separates points in $G$; this is not immediate and is proved using the inversion theorem for the Fourier transform. The set of trigonometric polynomials on $G$ is a self-adjoint algebra that contains the constant functions, so the Stone-Weierstrass theorem then tells us that it is dense in the Banach algebra $C(G)$. Because $\mathbb{C}$ is separable, it follows that if $\hat{G}$ is countable then $C(G)$ is separable. In particular, any closed subgroup $G$ of $T^\omega$ is a compact abelian group whose dual group one checks to be countable, so $C(G)$ is separable.

A compact Hausdorff space $X$ is metrizable if and only if the Banach algebra $C(X)$ is separable. We established in the previous paragraph that if $G$ is a compact abelian group with countable dual group then the trigonometric polynomials are dense in the Banach algebra $C(G)$. Therefore, every compact abelian group with countable dual group is metrizable. In particular, $T^\omega$ and all its closed subgroups are metrizable. In fact, it is proved in Rudin that for a compact abelian group, (i) being metrizable, (ii) having a countable dual group, and (iii) being isomorphic as a topological group to a closed subgroup of $T^\omega$ are equivalent.

## 3 $T^\omega$ and $Z^\infty$

Let $\pi_n : T^\omega \to S^1$ and $p_n : Z^\infty \to Z$ be the projection maps and let $q_n : Z \to Z^\infty$ be the inclusion map.

For $x \in T^\omega$ and $\gamma \in Z^\infty$,

$$\langle x, \gamma \rangle = \prod_{n \in \mathbb{N}} (\pi_n(x), p_n(\gamma)) = \prod_{n \in \mathbb{N}} \pi_n(x)^{p_n(\gamma)},$$

where for each $n$, $\pi_n(x) \in S^1$ and $p_n(\gamma) \in Z$.

Let $m$ be the Haar measure on $T^\omega$ such that $m(T^\omega) = 1$. Because the dual group of $T^\omega$ is $Z^\infty$, for any $f \in L^1(m)$ the Fourier transform of $f$ is the function $\hat{f} \in C_0(Z^\infty)$ defined by

$$\hat{f}(\gamma) = \int_{T^\omega} f(x) \langle -x, \gamma \rangle dm(x) = \int_{T^\omega} f(x) \prod_{n \in \mathbb{N}} \pi_n(\gamma)^{-p_n(x)} dm(x), \quad \gamma \in Z^\infty.$$
4 Kronecker sets

Suppose that $G$ is a locally compact abelian group and that $E$ is a subset of $G$, which we give the subspace topology. $E$ is called a Kronecker set if for every continuous $f : E \to S^1$ and every $\epsilon > 0$, there is some $\gamma \in \hat{G}$ such that
\[
\sup_{x \in E} |f(x) - \langle x, \gamma \rangle| < \epsilon.
\]

We first prove the following lemma from Rudin\textsuperscript{6}.

**Lemma 3.** If $0 < \alpha < \beta < 1$, then the set of polynomials with integer coefficients and 0 constant term is dense in the real Banach algebra $C([\alpha, \beta])$ of continuous functions $[\alpha, \beta] \to \mathbb{R}$.

**Proof.** Let $R$ be the closure in $C([\alpha, \beta])$ of the set of polynomials with integer coefficients and 0 constant term. Because $x \in R$, $R$ separates points in $[\alpha, \beta]$ and for every $a \in [\alpha, \beta]$ there is some $f \in R$ such that $f(a) \neq 0$. It is straightforward to check that $R$ is closed under addition and multiplication. If we show that $\mathbb{R} \subset R$, it will follow that $R$ is an algebra over $\mathbb{R}$, and then by the Stone-Weierstrass theorem we will get that $R$ is dense in $C([\alpha, \beta])$, and hence equal to $C([\alpha, \beta])$ as $R$ is closed.

Let $c \in \mathbb{R}$, let $p$ be prime, and define
\[
S_p(x) = \frac{1 - x^p - (1 - x)^p}{p}, \quad x \in [\alpha, \beta].
\]

Using that $p$ is prime, by the binomial theorem it follows that $S_p$ is a polynomial with integer coefficients and 0 constant term. Partitioning $\mathbb{R}$ into intervals of length $p$, $c$ lies in one of these intervals and hence there is some integer $q_p$ such that $|c - \frac{q_p}{p}| < \frac{1}{p}$. For $x \in [\alpha, \beta]$,
\[
|q_p S_p(x) - c| \leq |c - \frac{q_p}{p}| + \frac{|q_p|}{p} (\beta^p + (1 - \alpha)^p)
\leq \frac{1}{p} + (\frac{|c| + \frac{1}{p}}{p} (\beta^p + (1 - \alpha)^p).
\]

Hence $\|q_p S_p - c\|_\infty \to 0$ as $p \to \infty$. $q_p$ is an integer so for each $p$, $q_p S_p$ is a polynomial with integer coefficients and 0 constant term, so this shows that $c \in R$, completing the proof. \hfill \Box

An arc in a topological space is a homeomorphic image of a compact subset of $\mathbb{R}$ of nonzero length. The following theorem shows that there is an arc in $T^\omega$ that is a Kronecker set\textsuperscript{7}.

**Theorem 4.** $T^\omega$ contains an arc that is a Kronecker set.

\textsuperscript{6}Walter Rudin, *Fourier Analysis on Groups*, p. 104, Lemma 5.2.8.
\textsuperscript{7}Walter Rudin, *Fourier Analysis on Groups*, p. 103, Theorem 5.2.7.
Proof. Let $0 < \alpha < \beta < 1$, define $x : \left[\alpha, \beta\right] \to \mathbb{T}^\omega$ by

$$(\pi_n \circ x)(t) = \exp(2\pi it^n), \quad t \in [\alpha, \beta], \quad n \in \mathbb{N},$$

and let $L$ be the image of $[\alpha, \beta]$ under $x$. Assign $L$ the subspace topology inherited from $\mathbb{T}^\omega$, and suppose that $f : L \to S^1$ is continuous. One proves that there is a continuous function $h : [\alpha, \beta] \to \mathbb{R}$ that satisfies

$$(f \circ x)(t) = \exp(2\pi ith(t)), \quad \alpha \leq t \leq \beta.$$ 

Let $\epsilon > 0$, and by Lemma 3, let $S_m(x) = \sum_{j=1}^{m} a_j x^j$ be a polynomial with integer coefficients such that $\|S_m - h]\|_\infty < \epsilon$. Define $\gamma \in \mathbb{Z}^\omega$ by $p_j(\gamma) = a_j$ for $1 \leq j \leq m$ and $p_j(\gamma) = 0$ otherwise. For $t \in [\alpha, \beta],$

$$|f(x(t)) - \langle x(t), \gamma \rangle| = \left| \exp(2\pi ih(t)) - \prod_{n \in \mathbb{N}} (\pi_n(x(t)), p_n(\gamma)) \right|$$

$$= \left| \exp(2\pi ih(t)) - \prod_{n=1}^{m} (\pi_n(x(t)), a_n) \right|$$

$$= \left| \exp(2\pi ih(t)) - \prod_{n=1}^{m} \exp(2\pi ia_n t^n) \right|$$

$$= \left| \exp(2\pi ih(t)) - \exp \left( \sum_{n=1}^{m} 2\pi i a_n t^n \right) \right|$$

$$\leq 2\pi h(t) - \sum_{n=1}^{m} 2\pi a_n t^n$$

$$= 2\pi |h(t) - S_m(t)|$$

$$< 2\pi \epsilon,$$

using the fact that $|\exp(iA) - \exp(iB)| \leq |A - B|$ for $A, B \in \mathbb{R}$. Hence, for every $\epsilon > 0$ there is some $\gamma \in \mathbb{Z}^\omega$ such that

$$\sup_{y \in L} |f(y) - \langle y, \gamma \rangle| < \epsilon,$$

showing that $L$ is a Kronecker set. \qed

5 Subgroups

Suppose that $G$ is a locally compact abelian group. For each $x \in G$, let $t_x : G \to G$ be defined by $t_x(y) = x + y$, which is a homeomorphism, and let $\sigma : G \to G$ be defined by $\sigma(x) = -x$, which is also a homeomorphism. If $A$ is an open set in $G$ and $B$ is a subset of $G$, then

$$A + B = \bigcup_{x \in B} t_x(A),$$

6
which is open because \( t_x(A) \) is open for each \( x \in B \). Furthermore, if \( A \) and \( B \) are both compact sets in \( G \) then \( A \times B \) is compact in \( G \times G \) and \( A + B \) is the image of \( A \times B \) under the continuous map \((x,y) \mapsto x + y \) hence is compact.

By a neighborhood of a point \( x \) in a topological space we mean a set such that \( x \) lies in the interior of the set, in other words, a set that contains an open neighborhood of the point. The collection of all neighborhoods of a point \( x \) is a filter, and a neighborhood base at \( x \) is a filter base for the neighborhood filter of \( x \). In a locally compact Hausdorff space, every point \( x \) has a neighborhood base consisting of compact neighborhoods of \( x \).

Let \( A : G \times A \to G \) be \( A(x,y) = x + y \), which is continuous. If \( W \) is a neighborhood of 0 in \( G \), then \( A^{-1}(W) \) is a neighborhood of \((0,0) \) in \( G \times G \). A base for the product topology on \( G \times G \) consists of sets of the form \( U_1 \times U_2 \) where \( U_1, U_2 \) are open sets in \( G \), so there are open sets \( U_1, U_2 \) in \( G \) such that \((0,0) \in U_1 \times U_2 \subset A^{-1}(W) \). Each of \( U_1 \) and \( U_2 \) are then open neighborhoods of 0 in \( G \), so \( V = U_1 \cap U_2 \) is also an open neighborhood of 0 in \( G \), and then \( V \times V \) is open in \( G \times G \) and

\[
(0,0) \in V \times V \subset U_1 \times U_2 \subset A^{-1}(W).
\]

Hence \( A(0,0) \subset A(V \times V) \subset W \), i.e. \( 0 \in V + V \subset W \), and \( V + V \) is open because \( V \) is open. Therefore, for every neighborhood \( W \) of 0 in a locally compact abelian group, there is some \( V \) that is an open neighborhood of 0 and that satisfies \( V + V \subset W \).

Suppose that \( G \) is a locally compact abelian group. A subset \( E \) of \( G \) is called symmetric if \( E = -E \). If \( N \) is a compact neighborhood of 0 then \( N \) contains an open neighborhood \( U \) of 0. The set \( U \cap \sigma(U) \) is an open neighborhood of 0 and the set \( N \cap \sigma(N) \) is compact (an intersection of compact sets in a Hausdorff space is compact) and contains \( U \cap \sigma(U) \), hence \( N \cap \sigma(N) \) is a compact symmetric neighborhood of 0 that is contained in \( N \). It follows that in a locally compact abelian group, there is a neighborhood base at 0 consisting of compact symmetric neighborhoods of 0.

Suppose that \( G \) is an abelian group and that \( H \) is a subgroup of \( G \). We define the quotient group \( G/H \) be the collection of cosets of \( H \), which is an abelian group where we define

\[
(x + H) + (y + H) = (x + y) + H, \quad x, y \in G.
\]

Let \( \pi : G \to G/H \) be the projection map, which is a homomorphism with \( \ker \pi = H \).

We are now equipped to define quotient groups in the category of locally compact abelian groups. Suppose that \( G \) is a locally compact abelian group and that \( H \) is a closed subgroup of \( G \). We assign \( G/H \) the final topology induced by the projection map \( \pi \) (namely, the quotient topology). For \( x + H \in G/H \), there is a compact neighborhood \( N \) of \( x \) in \( G \); that is, there is a compact set \( N \) and an open set \( U \) such that \( x \in U \subset N \). Because \( \pi \) is continuous, \( \pi(N) \) is compact, and because \( \pi \) is open, \( \pi(U) \) is open, so \( \pi(N) \) is a compact neighborhood of \( x + H \) in \( G/H \). Therefore \( G/H \) is locally compact. It remains to prove that
$G/H$ is Hausdorff and that addition and negation are continuous to prove that $G/H$ is a locally compact abelian group. Suppose that $x + H, y + H$ are distinct elements of $G/H$, i.e. $x - y \notin H$. The set $y + H = t_y(H)$ is closed because $H$ is closed, and $x \notin y + H$ so $G \setminus t_y(H)$ is an open neighborhood of $x$, and hence $W = t_w(G \setminus t_y(H))$ is an open neighborhood of 0 such that $x + W$ is disjoint from $y + H$. Because $W$ is an open neighborhood of 0 there is an open neighborhood $V$ of 0 such that $V + V \subset W$. Furthermore, there is a compact symmetric neighborhood of 0, $N$, contained in $V$. If $(x + H + N) \cap (y + H + N) \neq \emptyset$ then there are $h_1, h_2 \in H$ and $n_1, n_2 \in N$ such that $x + h_1 + n_1 = y + h_2 + n_2$, and then $x + (n_1 - n_2) = y + (h_2 - h_1)$. But $-n_2 \in N$ because $N$ is symmetric and so $n_1 - n_2 \in N + N \subset V + V \subset W$, so $x + (n_1 - n_2) \in x + W$, and $h_2 - h_1 \in H$, so $y + (h_2 - h_1) \in y + H$, contradicting that $x + W$ and $y + H$ are disjoint. Therefore $x + H + N$ and $y + H + N$ are disjoint, and their images under $\pi$ are then disjoint neighborhoods of $x + H$ and $y + H$ in $G/H$, showing that $G/H$ is Hausdorff. It is straightforward to prove that addition and negation are continuous in $G/H$, and therefore $G/H$ is a locally compact abelian group.

If $H$ is a closed subgroup of a locally compact abelian group $G$, the annihilator of $H$, denoted $\Lambda_H$, is the set of all $\gamma \in \hat{G}$ such that

$$\langle x, \gamma \rangle = 1, \quad x \in H.$$ 

For each $x \in H$, the map $\gamma \mapsto \langle x, \gamma \rangle$ is continuous $\hat{G} \rightarrow S^1$ so the inverse image of $\{1\}$ under this map is closed. $\Lambda_H$ is the intersection of all these inverse images hence is closed, and is a closed subgroup because it is apparent that $\Lambda_H$ is a subgroup of $\hat{G}$. It can be proved that $\Lambda_H$ is the dual of the quotient group $G/H$ and that the quotient group $\hat{G}/\Lambda_H$ is the dual of $H$.

The following lemma shows that we can extend continuous characters on a closed subgroup to the entire group.

**Lemma 5.** Suppose that $H$ is a closed subgroup of a locally compact abelian group $G$. If $\phi \in \hat{H}$, then there is some $\gamma \in \hat{G}$ whose restriction to $H$ is equal to $\phi$.

**Proof.** $\phi \in \hat{H} = \hat{G}/\Lambda_H$, so there is some $\gamma \in \hat{G}$ such that for all $x \in H$, $\gamma(x) = \phi(x)$. \hfill \Box

Suppose that $G$ is a locally compact abelian group. It can be proved that if $E$ is a compact open set in $G$ and $0 \in E$, then $E$ contains a compact open subgroup of $G$.

We are now equipped to prove the following theorem.

**Theorem 6.** Suppose that $G$ is a compact group. $G$ is connected if and only if $\gamma \in \hat{G}$ having finite order implies that $\gamma = 0$.

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8Walter Rudin, *Fourier Analysis on Groups*, p. 35, Theorem 2.1.2.
9Walter Rudin, *Fourier Analysis on Groups*, p. 36, Theorem 2.1.4.
10Walter Rudin, *Fourier Analysis on Groups*, p. 41, Lemma 2.4.3.
11Walter Rudin, *Fourier Analysis on Groups*, p. 47, Theorem 2.5.6.
Proof. Assume that $G$ is not connected. Then there is a clopen subset $A$ that is neither $G$ nor $\emptyset$. Because $G$ is compact, both $A$ and $G \setminus A$ are compact and open, and one of them, call it $E$, contains 0. Because $E$ is a compact open set containing 0, $E$ contains a compact open subgroup $H$ of $G$, and $H \neq G$ because $E \neq G$. Because $H$ is open, the singleton $\{0+H\}$ in the quotient group $G/H$ is an open set, and therefore $G/H$ is discrete. But $G$ is compact and $G/H$ is the image of $G$ under the projection map, so $G/H$ is compact. Hence $G/H$ is finite.

The dual of $G/H$ is $\Lambda_H$, which is a subgroup of $\hat{G}$. Because $G/H$ contains more than one element (as $H \neq G$), $\Lambda_H$ contains some $\gamma \neq 0$, and $\gamma$ has finite order because it is contained in the finite subgroup $\Lambda_H$.

Assume that $\gamma \in \hat{G}$ has order finite order and that $\gamma \neq 0$. Every element of $\gamma(G)$ has finite order and $\gamma(G) \neq \{1\}$, so $\gamma(G)$ is not connected. But if $G$ were connected then $\gamma(G)$, a continuous image of $G$, would be connected, hence $G$ is not connected.  

**Lemma 7.** Suppose that $G$ is a locally compact abelian group. If $A$ is an open subgroup of $G$, then $A$ is closed.

Proof. $A$ is a subgroup of $G$, which gives us

$$A = G \setminus \bigcup_{x \in G \setminus A} (x + A).$$

Because each set $x + A$ is open, this shows that $A$ is closed.  

6 Measures

Suppose that $\mathcal{M}$ is a $\sigma$-algebra on a set $X$. If $\mu$ is a complex measure on $\mathcal{M}$ we denote by $|\mu|$ its total variation, which is a finite positive measure on $\mathcal{M}$. The total variation norm of $\mu$ is $||\mu|| = |\mu|(X)$.

Suppose that $X$ is a Hausdorff space with Borel $\sigma$-algebra $\mathcal{B}_X$ and that $\mu$ is a complex Borel measure on $X$. We say that $\mu$ is outer regular if for each $E \in \mathcal{B}_X$,

$$|\mu|(E) = \inf\{|\mu|(V) : E \subset V \text{ and } V \text{ is open}\}$$

inner regular if for each $E \in \mathcal{B}_X$,

$$|\mu|(E) = \sup\{|\mu|(F) : F \subset E \text{ and } F \text{ is closed}\},$$

and tight if for each $E \in \mathcal{B}_X$,

$$|\mu|(E) = \sup\{|\mu|(K) : K \subset E \text{ and } K \text{ is compact}\}.$$  

(Because we demand that $X$ be Hausdorff, a compact set is closed and hence belongs to the Borel $\sigma$-algebra of $X$; compact sets need not belong to the Borel

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\(\sigma\)-algebra of a topological space that is not Hausdorff.) We remark that the words “inner regular” often means what we call tight. We say that \(\mu\) is regular if it is both outer regular and tight, and we also remark that calling a measure regular often means being outer regular and what we call inner regular. What we call a regular complex Borel measure means precisely what Rudin means by these words in *Fourier Analysis on Groups*, and using Rudin’s notation we define

\[
M(X) = \{\mu : \mu \text{ is a regular complex Borel measure on } X\}.
\]

It is a fact that a complex Borel measure on a metrizable space is outer regular and inner regular\(^{13}\) and that a complex Borel measure on a Polish space is regular\(^{14}\).

Suppose that \(X\) and \(Y\) are locally compact Hausdorff spaces and that \(\mu \in M(X)\) and \(\lambda \in M(Y)\). It is a fact that there is a unique element of \(M(X \times Y)\), denoted \(\mu \times \lambda\), such that for any \(A \in \mathcal{B}_X\) and \(B \in \mathcal{B}_Y\),

\[
(\mu \times \lambda)(A \times B) = \mu(A)\lambda(B).
\]

We call \(\mu \times \lambda\) the **product measure** of \(\mu\) and \(\lambda\).

Suppose that \(G\) is a locally compact abelian group with addition \(A : G \times G \to G\). For \(\mu, \lambda \in M(G)\), we define the **convolution** of \(\mu\) and \(\lambda\) to be the pushforward of the product \(\mu \times \lambda\) by \(A\),

\[
\mu \ast \lambda = A_*(\mu \times \lambda),
\]

and it can be proved that \(\mu \ast \lambda \in M(G)\), that convolution is commutative and associative, and that \(\|\mu \ast \lambda\| \leq \|\mu\|\|\lambda\|\)\(^{15}\). Then, with convolution as multiplication and using the total variation norm, \(M(G)\) is a unital commutative Banach algebra, with unity \(\delta_0\).

For \(\mu \in M(G)\), the **Fourier transform of** \(\mu\) is the function \(\hat{\mu} : \hat{G} \to \mathbb{C}\) defined by

\[
\hat{\mu}(\gamma) = \int_G \langle -x, \gamma \rangle d\mu(x), \quad \gamma \in \hat{G}.
\]

One proves that \(\hat{\mu}\) is bounded and uniformly continuous, and we define

\[
B(\hat{G}) = \{\hat{\mu} : \mu \in M(G)\}.
\]

### 7 Idempotent measures

If \(G\) is a locally compact abelian group and \(\mu \in M(G)\), we say that \(\mu\) is **idempotent** if \(\mu \ast \mu = \mu\), and we denote the set of idempotent elements of \(M(G)\)

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by \( J(G) \). Because the Fourier transform of a convolution is the product of the Fourier transforms, for \( \mu \in M(G) \) we have \( \mu * \mu = \mu \) if and only if \( \hat{\mu}^2 = \hat{\mu} \). But \( \hat{\mu}^2 = \hat{\mu} \) is equivalent to \( \hat{\mu} \) having range contained in \( \{0, 1\} \), so for \( \mu \in M(G) \), we have that \( \mu \in J(G) \) if and only if \( \hat{\mu} \) is the characteristic function of some subset of \( \hat{G} \). For \( \mu \in J(G) \), we write

\[
S(\mu) = \{ \gamma \in \hat{G} : \hat{\mu}(\gamma) = 1 \}.
\]

Suppose that \( \Lambda \) is an open subgroup of \( \hat{G} \). Then \( \Lambda \) is closed, and the fact that \( \Lambda \) is open implies that the singleton containing the identity in \( \hat{G}/\Lambda \) is open and hence that \( \hat{G}/\Lambda \) is a discrete abelian group. Denoting the annihilator of \( \Lambda \) by \( H \), which is a closed subgroup of \( G \), the quotient group \( \hat{G}/\Lambda \) is the dual group of \( H \) and hence \( H \) is compact. Let \( m_H \) be the Haar measure on \( H \) such that \( m_H(H) = 1 \). Taking \( m_H(E) = m_H(E \cap H) \), if \( \gamma \in \Lambda \) then

\[
\hat{m}_H(\gamma) = \int_G \langle -x, \gamma \rangle dm_H(x) = \int_H \langle -x, \gamma \rangle dm_H(x) = \int_H dm_H(x) = m_H(H) = 1.
\]

If \( \gamma \in \hat{G} \setminus \Lambda \) then there is some \( x_0 \in H \) such that \( \langle x_0, \gamma \rangle \neq 1 \), and then

\[
\int_H \langle -x, \gamma \rangle dm_H(x) = \langle x_0, \gamma \rangle \int_H \langle -x_0 - x, \gamma \rangle dm_H(x) = \langle x_0, \gamma \rangle \int_H \langle -x, \gamma \rangle dm_H(x),
\]

showing that \( \hat{m}_H(\gamma) = \langle x_0, \gamma \rangle \hat{m}_H(\gamma) \), and because \( \langle x_0, \gamma \rangle \neq 1 \) this implies that \( \hat{m}_H(\gamma) = 0 \). Therefore, \( \Lambda = S(m_H) \).

If \( E = \gamma_0 + \Lambda \), then with

\[
d\mu(x) = \langle x, \gamma_0 \rangle dm(H)
\]

we have \( \mu \in J(G) \) and \( E = S(\mu) \).

### 8 Sidon sets

Let \( G \) be a compact abelian group and let \( E \subset \hat{G} \). A function \( f \in L^1(G) \) is called an \textbf{E-function} if \( \gamma \in \hat{G} \setminus E \) implies that \( \hat{f}(\gamma) = 0 \). An \textbf{E-polynomial} is a trigonometric polynomial \( f \) on \( G \) that is an \( E \)-function.

We call a subset \( E \) of \( \hat{G} \) a \textbf{Sidon set} if there is some \( B_E \geq 0 \) such that for every \( E \)-polynomial \( f \) on \( G \),

\[
\sum_{\gamma \in E} |\hat{f}(\gamma)| \leq B_E \|f\|_{\infty}.
\]

We shall use the following lemma later\[16\]

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\[16\]Walter Rudin, *Fourier Analysis on Groups*, p. 121, Theorem 5.7.3.
Lemma 8. Suppose that $\Gamma$ is a discrete abelian group that is the dual group of a compact abelian group $G$. If $E \subset \Gamma$ is a Sidon set with constant $B_E$, then every bounded $E$-function $f$ on $G$ satisfies
\[ \sum_{\gamma \in E} |\hat{f}(\gamma)| \leq B_E \|f\|_\infty. \]

9 Dirichlet series

Define $\sigma : \mathbb{Z}^\infty \to \mathbb{Z}$ by $\sigma(\gamma) = \sum_{n \in \mathbb{N}} p_n(\gamma)$, i.e. the sum of the entries of $\gamma$, which makes sense because any element of $\mathbb{Z}^\infty$ has only finitely many nonzero entries.

Let $Y$ be those $\gamma \in \mathbb{Z}^\infty$ such that $p_n(\gamma) \geq 0$ for all $n \in \mathbb{N}$, and let $E = Y \cap \sigma^{-1}(1)$. In other words, the elements of $E$ are those $\gamma \in \mathbb{Z}^\infty$ one coordinate of which is 1 and all other coordinates of which are 0. The proof of the following theorem is from Rudin\(^{17}\).

Theorem 9. If $f \in L^\infty(\mathbb{T}^\omega)$ and $\hat{f}(\gamma) = 0$ for all $\gamma \in X \setminus Y$, then
\[ \sum_{\gamma \in E} |\hat{f}(\gamma)| \leq \|f\|_\infty. \]

Proof. $\sigma : \mathbb{Z}^\infty \to \mathbb{Z}$ is a continuous group homomorphism, and $\ker \sigma$ is an open subgroup of $\mathbb{Z}^\infty$, because $\mathbb{Z}^\infty$ is discrete. Because $\sigma^{-1}(1)$ is a coset of this open subgroup, there is some $\mu \in J(\mathbb{T}^\omega)$ such that $\hat{\mu}$ is the characteristic function of $\sigma^{-1}(1)$, and this $\mu$ satisfies $\|\mu\| = 1$. Define $g : \mathbb{T}^\omega \to \mathbb{C}$ by
\[ g(x) = (f \ast \mu)(x) = \int_{\mathbb{T}^\omega} f(x-y) d\mu(y), \quad x \in \mathbb{T}^\omega, \]
whose Fourier transform is $\hat{g}(\gamma) = \hat{f}(\gamma) \hat{\mu}(\gamma)$. If $\gamma \notin E$ then $\gamma \notin Y$ or $\gamma \notin \sigma^{-1}(1)$.

In the first case $\hat{f}(\gamma) = 0$ and in the second case $\hat{\mu}(\gamma) = 0$, and hence $\gamma \notin E$ implies that $\hat{g}(\gamma) = 0$, namely, $g$ is an $E$-function. Also, it is apparent from the definition of $g$ that $\|g\|_\infty \leq \|f\|_\infty$.

Suppose that $P$ is an $E$-polynomial. Hence there is a finite subset $E_0$ of $E$ such that $\gamma \notin E_0$ implies that $\hat{P}(\gamma) = 0$, and thus there are $c_\gamma \in \mathbb{C}$, $\gamma \in E_0$, such that
\[ P(x) = \sum_{\gamma \in E_0} c_\gamma \langle x, \gamma \rangle = \sum_{\gamma \in E_0} c_\gamma \prod_{n \in \mathbb{N}} \langle \pi_n(x), p_n(\gamma) \rangle, \quad x \in \mathbb{T}^\omega. \]

$E_0 \subset E$, so any element of $E_0$ has one entry 1, say $p_{n_0}(\gamma) = 1$, and all other entries 0, so
\[ P(x) = \sum_{\gamma \in E_0} c_\gamma \pi_{n_0}(x). \]

\(^{17}\)Walter Rudin, Fourier Analysis on Groups, p. 224, Theorem 8.7.9.
Define \( x \in \mathbb{T}^{\omega} \) by taking \( c_\gamma \cdot \pi_{n_\gamma}(x) = |c_\gamma| \) for each \( \gamma \in E_0 \), and all other entries of \( x \) to be 1 \( \in S_1 \); this makes sense because if \( \gamma_1, \gamma_2 \in E_0 \) and \( n_{\gamma_1} = n_{\gamma_2} \) then \( \gamma_1 = \gamma_2 \). For this \( x \), \( P(x) = \sum_{\gamma \in E_0} |c_\gamma| \). But it is apparent that \( \|P\|_\infty \leq \sum_{\gamma \in E_0} |c_\gamma| \), so 

\[
\|P\|_\infty = \sum_{\gamma \in E_0} |c_\gamma|.
\]

This shows that \( E \) is a Sidon set with \( B_E = 1 \). Therefore by Lemma \(^8\) because \( g \) is a bounded \( E \)-function on \( \mathbb{T}^{\omega} \) we get \( \sum_{\gamma \in E} |\hat{g}(\gamma)| \leq \|g\|_\infty \). But \( \hat{\mu} \) is the characteristic function of \( \sigma^{-1}(1) \) and \( E = Y \cap \sigma^{-1}(1) \), so 

\[
\sum_{\gamma \in E} \hat{f}(\gamma) = \sum_{\gamma \in E} \hat{f}(\gamma) \hat{\mu}(\gamma) = \sum_{\gamma \in E} \hat{g}(\gamma) \leq \|g\|_\infty \leq \|f\|_\infty,
\]

proving the claim.

Following Rudin, we use the above theorem to prove a theorem about Dirichlet series due to Bohr\(^{18}\).

**Theorem 10 (Bohr).** If 

\[
\phi(s) = \sum_{k=1}^{\infty} \frac{c_k}{k^s}
\]

and \( |\phi(s)| \leq 1 \) for all \( s \) such that \( \text{Re } s > 0 \), then

\[
\sum_{p} |c_p| \leq 1.
\]

**Proof.** For \( k \in \mathbb{N} \), let \( \gamma(k) \in Y \) such that \( k = \prod_{n=1}^{\infty} p_n^{h_n(\gamma(k))} \), where \( p_n \) are the primes and where \( h_n : \mathbb{Z}^\infty \to \mathbb{Z} \) are the projection maps; so far we have denoted these projection maps by \( p_n \), rather than using \( h_n \), but the symbol \( p_n \) has such a strong association with the primes that we change notation here. The map \( k \mapsto \gamma(k) \) is a bijection \( \mathbb{N} \to Y \), and we write \( c_\gamma = c_k \). We shall use the fact that the image of the primes under this bijection is \( E \).

Let \( s \) be a complex number in the half-plane of convergence of \( \phi \) and write

\[ z_n(s) = p_n^{-s} = \exp(-s \log p_n). \] Then,

\[
\phi(s) = \sum_{k=1}^{\infty} c_k k^{-s}
\]

\[
= \sum_{\gamma \in Y} c_{\gamma} \left( \prod_{n=1}^{\infty} p_n h_n(\gamma) \right)^{-s}
\]

\[
= \sum_{\gamma \in Y} c_{\gamma} \prod_{n=1}^{\infty} p_n^{-s h_n(\gamma)}
\]

\[
= \sum_{\gamma \in Y} c_{\gamma} \prod_{n=1}^{\infty} z_n(s)^{h_n(\gamma)}
\]

Defining \( T : \mathbb{R} \to \mathbb{T}^\omega \) by

\[
(\pi_n \circ T)(\sigma) = \exp(-i\sigma \log p_n), \quad n \in \mathbb{N}, \sigma \in \mathbb{R},
\]

we have, as \( z_n(i\sigma) = \exp(-i\sigma \log p_n), \)

\[
\phi(i\sigma) = \sum_{\gamma \in Y} c_{\gamma} \prod_{n=1}^{\infty} (\pi_n(T(\sigma)), h_n(\gamma)) = \sum_{\gamma \in Y} c_{\gamma} \langle T(\sigma), \gamma \rangle.
\]

One checks that the function \( f : \mathbb{T}^\omega \to \mathbb{C} \) defined by \( f(x) = \sum_{\gamma \in Y} c_{\gamma} \langle x, \gamma \rangle \) satisfies the conditions of Theorem 33.22 and thus gets

\[
\sum_p |c_p| = \sum_{\gamma \in E} |c_{\gamma}| = \sum_{\gamma \in E} |\hat{f}(\gamma)| \leq \|f\|_\infty
\]

I do not see why \( \|f\|_\infty \leq 1 \). However, granted this, the claim follows. \( \square \)

10 Descriptive set theory

If \((X,d)\) is a compact metric space, \(C(X,X)\) is a Polish space with the uniform metric \( d(f,g) \to \sup_{x \in X} d(f(x), g(x)) \). We denote by \( H(X) \) the group of homeomorphisms of \( X \), which one proves is a \( G_\delta \) set in \( C(X,X) \). Because \( H(X) \) is a \( G_\delta \) set in a Polish space, it is a Polish space with the subspace topology. A homeomorphism \( h \) of \( X \) is said to be minimal if there is no proper closed subset of \( X \) that is invariant under \( h \), and is called distal if \( x \neq y \) implies that there is some \( \epsilon > 0 \) such that for all \( n \in \mathbb{N} \), \( d(h^n(x), h^n(y)) > \epsilon \). It has been proved (Beleznay-Foreman) that the collection of minimal distal homeomorphisms of \( \mathbb{T}^\omega \) is a Borel \( \Sigma^1_1 \)-complete set in \( H(\mathbb{T}^\omega) \).\(^{19}\)

\(^{19}\)Alexander S. Kechris, Classical Descriptive Set Theory, p 262, Theorem 33.22.
11 Further reading


Aoki, Nobuo and Totoki, Haruo, *Ergodic automorphisms of $T^\infty$ are Bernoulli automorphisms*, Publ. RIMS, Kyoto Univ. 10 (1975), 535–544.


Biroli, Marco and Maheux, Patrick, *Logarithmic Sobolev inequalities and Nash-type inequalities for sub-markovian symmetric semigroups*, https://hal.inria.fr/hal-00465177v1/document


