Laguerre polynomials and Perron-Frobenius operators

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1 Laguerre polynomials

1.1 Definition and generating functions

Let $D = \frac{d}{dx}$. For $\alpha > -1$ and $n \geq 0$ let

$$L_n^\alpha(x) = e^x \frac{x^{-\alpha}}{n!} D^n(e^{-x} x^{n+\alpha}),$$

called the **Laguerre polynomials**. Using the Leibniz rule for $D^n(f \cdot g)$ yields

$$L_n^\alpha(x) = \sum_{k=0}^n \frac{\Gamma(n+\alpha+1)}{\Gamma(k+\alpha+1)} \frac{(-x)^k}{k!(n-k)!}.$$

The generating function for the Laguerre polynomials is[1]

$$w(x,z) = (1-z)^{-\alpha-1} e^{-xz/(1-z)} = \sum_{n=0}^\infty L_n^\alpha(x) z^n, \qquad |z| < 1.$$

Define

$$W(x,y,z) = (1-z)^{-1} e^{-(x+y)z/(1-z)} (xyz)^{-\alpha/2} I_\alpha\left(\frac{2\sqrt{xyz}}{1-z}\right), \qquad |z| < 1,$$

where

$$I_\alpha(x) = i^{-\alpha} J_\alpha(ix) = \sum_{m=0}^\infty \frac{1}{m!\Gamma(m+\alpha+1)} (x/2)^{2m+\alpha}$$

and

$$J_\alpha(x) = \sum_{m=0}^\infty \frac{(-1)^m}{m!\Gamma(m+\alpha+1)} (x/2)^{2m+\alpha}.$$

[1]N. N. Lebedev, *Special Functions and Their Applications*, p. 77, §4.17.

W satisfies

$$W(x,y,z) = \sum_{n=0}^{\infty} \frac{n! L_n^\alpha(x) L_n^\alpha(y)}{\Gamma(n+\alpha+1)} z^n.$$

1.2 Differential equations satisfied by Laguerre polynomials

w satisfies the ordinary differential equation

$$(1-z^2)\partial_z w + (x - (1-z)(1+\alpha))w = 0.$$

This yields, for $n \geq 1$,

$$(n+1)L_{n+1}^\alpha(x) + (x - \alpha - 2n - 1)L_n^\alpha(x) + (n+\alpha)L_{n-1}^\alpha(x) = 0. \tag{1}$$

w also satisfies the ordinary differential equation

$$(1-t)\partial_x w + tw = 0,$$

which yields, for $n \geq 1$,

$$DL_n^\alpha - DL_{n-1}^\alpha + L_{n-1}^\alpha = 0. \tag{2}$$

Using (1) and (2) gives

$$xDL_n^\alpha = nL_n^\alpha - (n+\alpha)L_{n-1}^\alpha, \qquad n \geq 1. \tag{3}$$

Using (3) and (2) we get, for $n \geq 0$,

$$xD^2 L_n^\alpha(x) + (\alpha + 1 - x)DL_n^\alpha(x) + nL_n^\alpha(x) = 0. \tag{4}$$

1.3 Integral formulas for Laguerre polynomials

For $\nu > -1$, $a > 0$, $b > 0$, using the series for J_ν one calculates[2]

$$\int_0^\infty e^{-a^2x^2} J_\nu(bx) x^{\nu+1} dx = \frac{b^\nu}{(2a^2)^{\nu+1}} e^{-\frac{b^2}{4a^2}}. \tag{5}$$

Applying this with $\nu = n + \alpha$, $a = 1$, $b = 2\sqrt{x}$, $x = \sqrt{t}$ yields

$$\int_0^\infty e^{-t} J_{n+\alpha}(2\sqrt{xt})(\sqrt{t})^{n+\alpha+1} \cdot \frac{1}{2\sqrt{t}} dt = \frac{(2\sqrt{x})^{n+\alpha}}{2^{n+\alpha+1}} e^{-x},$$

i.e.

$$\int_0^\infty e^{-t} J_{n+\alpha}(2\sqrt{xt})(\sqrt{xt})^{n+\alpha} dt = e^{-x} x^{n+\alpha}. \tag{6}$$

[2] N. N. Lebedev, *Special Functions and Their Applications*, p. 132, §5.15, Example 2.

Now, it is a fact that

$$\frac{d}{du} u^{\nu/2} J_\nu(2\sqrt{u}) = u^{(\nu-1)/2} J_{\nu-1}(2\sqrt{u}),$$

and using this and (6), we get that for $\alpha > 1$ and $n \geq 0$,

$$L_n^\alpha(x) = \frac{e^x x^{-\alpha/2}}{n!} \int_0^\infty t^{n+\alpha/2} J_\alpha(2\sqrt{xt}) e^{-t} dt. \tag{7}$$

We remind ourselves that for $\alpha > -1$ and $|z| < 1$,

$$(1-z)^{-\alpha-1} e^{-yt/(1-t)} = \sum_{n=0}^\infty L_n^\alpha(y) z^n.$$

For $|z| < \frac{1}{3}$, using this and $e^{-\frac{yt}{1-t} - \frac{y}{2}} = e^{-\frac{2yt+y-yt}{2(1-t)}} = e^{-\frac{y(1+t)}{2(1-t)}}$ one checks that

$$\begin{aligned} &(1-z)^{-\alpha-1} \int_0^\infty e^{-\frac{y(1+t)}{2(1-t)}} y^{\alpha/2} J_\alpha(\sqrt{xy}) dy \\ =& \sum_{n=0}^\infty z^n \int_0^\infty e^{-y/2} y^{\alpha/2} J_\alpha(\sqrt{xy}) L_n^\alpha(y) dy. \end{aligned}$$

Then one gets, for $|z| < 1$,

$$2e^{-x/2} x^{\alpha/2} \sum_{n=0}^\infty (-1)^n L_n^\alpha(x) z^n = \sum_{n=0}^\infty z^n \int_0^\infty e^{-y/2} y^{\alpha/2} J_\alpha(\sqrt{xy}) L_n^\alpha(y) dy.$$

Therefore for $\alpha > -1$ and $n \geq 0$,

$$e^{-x/2} x^{\alpha/2} L_n^\alpha(x) = \frac{(-1)^n}{2} \int_0^\infty J_\alpha(\sqrt{xy}) e^{-y/2} y^{\alpha/2} L_n^\alpha(y) dy. \tag{8}$$

1.4 Orthogonality of Laguerre polynomials

Let

$$\rho_\alpha(x) = e^{-x} x^\alpha.$$

Let

$$u_n = \rho_\alpha^{1/2} L_n^\alpha, \qquad n \geq 0.$$

u_n satisfies the differential equation

$$(xu_n')' + \left(n + \frac{\alpha+1}{2} - \frac{x}{4} - \frac{\alpha^2}{4x}\right) u_n = 0.$$

Using this we get

$$x(u_n' u_m - u_m' u_n)\Big|_0^\infty + (n-m) \int_0^\infty u_m u_n dx = 0.$$

Then

$$(n-m)\int_0^\infty u_m u_n dx = 0. \tag{9}$$

Using (1) yields for $n \geq 2$,

$$n(L_n^\alpha)^2 - (n+\alpha)(L_{n-1}^\alpha)^2 - (n+1)L_{n+1}^\alpha L_{n-1}^\alpha + 2L_n^\alpha L_{n-1}^\alpha + (n+\alpha-1)L_n^\alpha L_{n-2}^\alpha = 0.$$

Using this and (9), for $n \geq 2$,

$$n\int_0^\infty e^{-x}x^\alpha L_n^\alpha(x)^2 dx = (n+\alpha)\int_0^\infty e^{-x}x^\alpha L_{n-1}^\alpha(x)^2 dx.$$

Iterating this, for $n \geq 2$,

$$\begin{aligned}\int_0^\infty e^{-x}x^\alpha L_n^\alpha(x)^2 dx &= \frac{(n+\alpha)(n+\alpha-1)\cdots(\alpha+2)}{n(n-2)\cdots 3\cdot 2}\int_0^\infty e^{-x}x^\alpha L_1^\alpha(x)^2 dx \\ &= \frac{\Gamma(n+\alpha+1)}{n!}.\end{aligned}$$

1.5 Asymptotics for Laguerre polynomials

It can be proved that for $\alpha > -1$, with $N = n + \frac{\alpha+1}{2}$,[3] for $x \in \mathbb{R}_{\geq 0}$,

$$L_n^\alpha(x) \sim \frac{\Gamma(n+\alpha+1)}{n!}e^{x/2}(Nx)^{-\alpha/2}J_\alpha(2\sqrt{Nx}), \qquad n \to \infty.$$

1.6 Laguerre expansions

Suppose that $f : \mathbb{R}_{>0} \to \mathbb{R}$ is piecewise smooth in every interval $[x_1, x_2]$, $0 < x_1 < x_2 < \infty$, and $f \in L^2(d\rho_\alpha)$. Let

$$c_n(f) = \frac{n!}{\Gamma(n+\alpha+1)}\int_0^\infty f(x)L_n^\alpha(x)\rho_\alpha(x)dx,$$

$\rho_\alpha(x) = e^{-x}x^\alpha$. It can be proved that[4] if f is continuous at x then

$$\sum_{n=0}^N c_n(f)L_n^\alpha(x) \to f(x), \qquad N \to \infty,$$

and if f is not continuous at x then

$$\sum_{n=0}^N c_n(f)L_n^\alpha(x) \to \frac{f(x+0)}{2} + \frac{f(x-0)}{2}, \qquad N \to \infty,$$

which makes sense because f is a priori piecewise continuous.

[3] N. N. Lebedev, *Special Functions and Their Applications*, p. 87, §4.22.

[4] N. N. Lebedev, *Special Functions and Their Applications*, p. 88, §4.23, Theorem 3.

Let $\nu > -\frac{1}{2}(\alpha+1)$ and $f(x) = x^\nu$. Integrating by parts,

$$\begin{aligned} c_n(f) &= \frac{n!}{\Gamma(n+\alpha+1)} \int_0^\infty x^{\nu+\alpha} L_n^\alpha(x) e^{-x} \\ &= \frac{1}{\Gamma(n+\alpha+1)} \int_0^\infty x^\nu D^n(e^{-x} x^{n+\alpha}) dx \\ &= (-1)^n \frac{\Gamma(\nu+\alpha+1)\Gamma(\nu+1)}{\Gamma(n+\alpha+1)\Gamma(\nu-n+1)}. \end{aligned}$$

Thus

$$x^\nu = \Gamma(\nu+\alpha+1)\Gamma(\nu+1) \sum_{n=0}^\infty \frac{(-1)^n L_n^\alpha(x)}{\Gamma(n+\alpha+1)\Gamma(\nu-n+1)}.$$

For p a positive integer,

$$x^p = \Gamma(p+\alpha+1) \cdot p! \sum_{n=0}^p \frac{(-1)^n L_n^\alpha(x)}{\Gamma(n+\alpha+1) \cdot (p-n)!}.$$

Define

$$f(x) = (ax)^{-\alpha/2} J_\alpha(2\sqrt{ax}), \qquad \alpha > -1, \quad a > 0, \quad x > 0.$$

Using

$$(1-z)^{-\alpha-1} e^{-xz/(1-z)} = \sum_{n=0}^\infty L_n^\alpha(x) z^n, \qquad |z| < 1,$$

we obtain, as $e^{-\frac{xz}{1-z}-x} = e^{-x/(1-z)}$,

$$\begin{aligned} &(1-z)^{-\alpha-1} \int_0^\infty e^{-x/(1-z)} (x/a)^{\alpha/2} J_\alpha(2\sqrt{ax}) dx \\ &= \int_0^\infty e^{-x} (x/a)^{\alpha/2} J_\alpha(2\sqrt{ax}) \sum_{n=0}^\infty L_n^\alpha(x) z^n dx \\ &= \sum_{n=0}^\infty \left(\int_0^\infty f(x) L_n^\alpha(x) \rho_\alpha(x) dx \right) z^n. \end{aligned}$$

Doing the change of variable $2\sqrt{ax} = by$ with $b > 0$ and then applying (6) with

$A^2 = \frac{b^2}{4a(1-z)}$ and $\nu = \alpha$,

$$
\begin{aligned}
&(1-z)^{-\alpha-1}\int_0^\infty e^{-x/(1-z)}(x/a)^{\alpha/2}J_\alpha(2\sqrt{ax})dx\\
=&(1-z)^{-\alpha-1}(2a)^{-\alpha-1}b^{\alpha+2}\int_0^\infty e^{-\frac{b^2y^2}{4a(1-z)}}J_\alpha(by)y^{\alpha+1}dy\\
=&(1-z)^{-\alpha-1}(2a)^{-\alpha-1}b^{\alpha+2}\cdot\frac{b^\alpha}{(2A^2)^{\alpha+1}}e^{-\frac{b^2}{4A^2}}\\
=&(1-z)^{-\alpha-1}(2a)^{-\alpha-1}b^{\alpha+2}\cdot b^{-\alpha-2}(2a(1-z))^{\alpha+1}e^{-a(1-z)}\\
=&e^{-a(1-z)}\\
=&e^{-a}\sum_{n=0}^\infty\frac{(az)^n}{n!}.
\end{aligned}
$$

Therefore

$$e^{-a}\sum_{n=0}^\infty\frac{a^n}{n!}z^n = \sum_{n=0}^\infty\left(\int_0^\infty f(x)L_n^\alpha(x)\rho_\alpha(x)dx\right)z^n,$$

whence, for $n \geq 0$,

$$c_n(f) = \frac{n!}{\Gamma(n+\alpha+1)}\int_0^\infty f(x)L_n^\alpha(x)\rho_\alpha(x)dx = \frac{n!}{\Gamma(n+\alpha+1)}e^{-a}\frac{a^n}{n!}.$$

Therefore, for $\alpha > -1$, $a > 0$, $x > 0$,

$$(ax)^{-\alpha/2}J_\alpha(2\sqrt{ax}) = \sum_{n=0}^\infty c_n(f)L_n^\alpha(x) = e^{-a}\sum_{n=0}^\infty\frac{a^n}{\Gamma(n+\alpha+1)}L_n^\alpha(x).$$

2 Integral operators

We remind ourselves that, for $\alpha = 1$,

$$u_n(x) = \rho_1(x)^{1/2}L_n^1(x) = e^{-x/2}x^{1/2}L_n^1(x).$$

$\{u_n : n \geq 0\}$ is an orthonormal basis for $L^2(\mathbb{R}_{\geq 0})$.

For $x, y \in \mathbb{R}_{>0}$ define

$$k(x,y) = k_x(y) = k^x(y) = \frac{J_1(2\sqrt{xy})}{((e^x-1)(e^y-1))^{1/2}}.$$

For $\phi \in L^2(\mathbb{R}_{\geq 0})$ and $y \in \mathbb{R}_{>0}$, define

$$K\phi(y) = \int_{\mathbb{R}_{\geq 0}} k_y(x)\phi(x)dx.$$

We have established, with $\alpha = 1$,

$$J_1(2\sqrt{xy}) = (xy)^{1/2} e^{-x} \sum_{n=0}^{\infty} \frac{x^n}{(n+1)!} L_n^1(y).$$

Hence

$$\begin{aligned}\int_0^\infty k_y(x)\phi(x)dx &= \int_0^\infty \phi(x)(e^x-1)^{-1/2}(e^y-1)^{-1/2}(xy)^{1/2}e^{-x} \\ &\cdot \sum_{n=0}^{\infty} \frac{x^n}{(n+1)!} L_n^\alpha(y)dx \\ &= \sum_{n=0}^{\infty} \frac{(e^y-1)^{-1/2} y^{1/2} L_n^1(y)}{(n+1)!} \int_0^\infty \phi(x)(e^x-1)^{-1/2} x^{1/2} e^{-x} x^n dx \\ &= \sum_{n=0}^{\infty} q_n(y) \langle \phi, p_n \rangle ,\end{aligned}$$

for

$$p_n(x) = \frac{1}{(n+1)!}(e^x-1)^{-1/2} e^{-x} x^{n+\frac{1}{2}} = \frac{1}{(n+1)!} e^{-x/2}(e^x-1)^{-1/2} x^n u_n(x)$$

and

$$q_n(y) = (e^y-1)^{-1/2} y^{1/2} L_n^1(y) = (1-e^{-y})^{-1/2} u_n(y).$$

Then

$$K\phi = \sum_{n=0}^{\infty} q_n \langle \phi, p_n \rangle .$$

The following states the trace of the operator $K : L^2(\mathbb{R}_{\geq 0}) \to L^2(\mathbb{R}_{\geq 0})$.[5]

Theorem 1. $\operatorname{tr} K = \int_0^\infty k(x,x)dx = \int_0^\infty \frac{J_1(2x)}{(e^x-1)} dx = 0.7711\ldots$.

3 Hardy spaces

For $x \in \mathbb{R}$ let $P_x = \{z \in \mathbb{C} : \operatorname{Re} z > x\}$. Let H be the collection of holomorphic functions $f : P_{-1/2} \to \mathbb{C}$ such that for any $x > -\frac{1}{2}$, $f|P_x$ is bounded and such that

$$\int_{\mathbb{R}} \left| f\left(-\frac{1}{2} + iy\right)\right|^2 dy < \infty.$$

Define $M : L^2(\mathbb{R}_{\geq 0}) \to H$, for $\phi \in L^2(\mathbb{R}_{\geq 0})$, by

$$M\phi(z) = \int_{\mathbb{R}_{\geq 0}} e^{-zs-s/2}\phi(s)ds.$$

[5] cf. A. A. Kirillov, *Elements of the Theory of Representations*, p. 211, §13, Theorem 2.

For $f \in H$ define

$$P_\lambda f(z) = \sum_{k \geq 1} \frac{1}{(z+k)^2} f\left(\frac{1}{z+k}\right) \qquad \operatorname{Re} z > -\frac{1}{2},$$

called a **Perron-Frobenius operator**. λ denotes Lebesgue measure.

Let

$$h(s) = \left(\frac{1-e^{-s}}{s}\right)^{1/2}$$

for $s \in \mathbb{R}_{>0}$, with $h(0) = 1$. Because $h \in L^\infty(\mu)$, it makes sense to define $S : L^2(\mathbb{R}_{\geq 0}) \to L^2(\mathbb{R}_{\geq 0})$ by

$$S\phi(s) = h\phi, \qquad \phi \in L^2(\mathbb{R}_{\geq 0}).$$

Define $A : H \to L^2(\mathbb{R}_{\geq 0})$ by

$$A = S \circ M^{-1}.$$

We prove that P_λ and K are conjugate.[6]

Theorem 2. $P_\lambda = A^{-1}KA$.

Proof. Let $\phi \in L^2(\mathbb{R}_{\geq 0})$ and set $f = M\phi$. Then

$$A^{-1}KAf = A^{-1}KS\phi.$$

We calculate

$$\begin{aligned}
(S^{-1}KS\phi)(x) &= h(x)^{-1} \int_{\mathbb{R}_{\geq 0}} k_x(y) \cdot h(y) \cdot \phi(y)dy \\
&= \left(\frac{x}{1-e^{-x}}\right)^{1/2} \int_0^\infty \frac{J_1(2\sqrt{xy})}{((e^x-1)(e^y-1))^{1/2}} \cdot \left(\frac{1-e^{-y}}{y}\right)^{1/2} \cdot \phi(y)dy \\
&= \int_0^\infty \left(\frac{x}{y}\right)^{1/2} \frac{e^{x/2}}{(e^x-1)^{1/2}} \frac{(e^y-1)^{1/2}}{e^{y/2}} \frac{J_1(2\sqrt{xy})}{((e^x-1)(e^y-1))^{1/2}} \phi(y)dy \\
&= \int_0^\infty \left(\frac{x}{y}\right)^{1/2} \frac{e^{(x-y)/2}}{e^x-1} J_1(2\sqrt{xy})\phi(y)dy.
\end{aligned}$$

Then

$$\begin{aligned}
&(MS^{-1}KS\phi)(z) \\
&= \int_{\mathbb{R}_{\geq 0}} e^{-zx-x/2}(S^{-1}KS\phi)(x)dx \\
&= \int_0^\infty e^{-zx-x/2} \left(\left(\frac{x}{y}\right)^{1/2} \frac{e^{(x-y)/2}}{e^x-1} J_1(2\sqrt{xy})\phi(y)dy\right) dx \\
&=
\end{aligned}$$

[6] Marius Iosifescu and Cor Kraaikamp, *Metrical Theory of Continued Fractions*, p. 9, Proposition 1.1.1.

It is a fact that for $\operatorname{Re} z > -1$ and for $t \geq 0$,

$$\sum_{k\geq 0}(z+k)^{-2}\exp\left(-\frac{t}{z+k}\right) = \int_0^\infty (st^{-1})^{1/2}e^{-zs}\frac{J_1(2\sqrt{st})}{e^s-1}ds.$$

Using this,

$$\begin{aligned}(MS^{-1}KS\phi)(z) &= \int_0^\infty e^{-y/2}\left(\int_0^\infty (xy^{-1})^{1/2}e^{-zx}\frac{J_1(2\sqrt{xy})}{e^x-1}dx\right)\phi(y)dy \\ &= \int_0^\infty e^{-y/2}\sum_{k\geq 1}(z+k)^{-2}\exp\left(-\frac{y}{z+k}\right)\cdot\phi(y)dy \\ &= \sum_{k\geq 1}(z+k)^{-2}\left(\int_0^\infty \exp\left(-\frac{y}{z+k}-\frac{y}{2}\right)\phi(y)dy\right) \\ &= \sum_{k\geq 1}(z+k)^{-2}\cdot M\phi\left(\frac{1}{z+k}\right).\end{aligned}$$

Thus, as $f = M\phi$,

$$(MS^{-1}KSM^{-1}f)(z) = \sum_{k\geq 1}(z+k)^{-2}f\left(\frac{1}{z+k}\right) = P_\lambda f(z),$$

that is,

$$A^{-1}KAf(z) = P_\lambda f(z).$$

□