The Laplace operator is essentially self-adjoint

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1 Operators in $L^2(\mathbb{R}^d)$

Write $H = L^2(\mathbb{R}^d)$, which is a Hilbert space with inner product

$$\langle f, g\rangle = \int_{\mathbb{R}^d} f\overline{g}, \qquad f, g \in H.$$

An **operator in** H is a linear subspace $\mathscr{D}(T)$ of H and a linear map $T : \mathscr{D}(T) \to H$. We write

$$\mathscr{R}(T) = T(\mathscr{D}(T)).$$

The **graph of** T is

$$\mathscr{G}(T) = \{(f, Tf) : f \in \mathscr{D}(T)\},$$

which is a linear subspace of $H \times H$. For operators S and T in H, when $\mathscr{G}(S) \subset \mathscr{G}(T)$ we write

$$S \subset T.$$

$S = T$ is equivalent with $S \subset T$ and $T \subset S$.

We call T **densely defined** if $\mathscr{D}(T)$ is dense in H. When T is densely defined, we define $\mathscr{D}(T^*)$ to be the set of those $g \in H$ such that

$$f \mapsto \langle Tf, g\rangle$$

is continuous $\mathscr{D}(T) \to \mathbb{C}$. Because T is densely defined, for each $g \in \mathscr{D}(T^*)$ there is a unique $T^*g \in H$ such that[1]

$$\langle Tf, g\rangle = \langle f, T^*g\rangle, \qquad f \in \mathscr{D}(T).$$

$\mathscr{D}(T^*)$ is a linear subspace of H and $T^* : \mathscr{D}(T^*) \to H$ is linear, and thus T^* is an operator in H, called the **adjoint of** T.

We say that an operator T in H is **symmetric** if

$$\langle Tf, g\rangle = \langle f, Tg\rangle, \qquad f, g \in \mathscr{D}(T).$$

[1] http://individual.utoronto.ca/jordanbell/notes/trotter.pdf, §1.

When T is densely defined, this implies that $T^*g = Tg$ for all $g \in \mathscr{D}(T)$, which means that

$$T \subset T^*.$$

An operator T in H is called **positive** when it is symmetric and satisfies

$$\langle Tf, f \rangle \geq 0, \qquad v \in \mathscr{D}(T).$$

If T is densely defined and $T = T^*$, then T is called **self-adjoint**. The **Friedrichs extension theorem** states that if T is a densely defined positive operator in H, then there is a positive self-adjoint operator S in H such that $T \subset S$, that is, that a densely defined positive operator has a positive self-adjoint extension.[2]

An operator T in H is called **closed** if $\mathscr{G}(T)$ is closed in $H \times H$. This is equivalent with $\mathscr{D}(T)$ when assigned the inner product

$$\langle f, g \rangle_T = \langle f, g \rangle + \langle Tf, Tg \rangle, \qquad f, g \in \mathscr{D}(T). \tag{1}$$

being itself a Hilbert space. An operator T in H is called **closable** if there is some operator S in H such that

$$\overline{\mathscr{G}(T)} = \mathscr{G}(S),$$

namely $\overline{\mathscr{G}(T)}$ is a graph. If T is closable, we define

$$\mathscr{D}(\overline{T}) = \{f \in H : \text{there is some } g \in H \text{ such that } (f, g) \in \overline{\mathscr{G}(T)}\}.$$

Because $\overline{\mathscr{G}(T)}$ is a graph, for each $f \in \mathscr{D}(\overline{T})$ there is a unique $\overline{T}f \in H$ such that $(f, \overline{T}f) \in \overline{\mathscr{G}(T)}$. It is straightforward to check that $\overline{T}$ is linear, and thus is an operator in H. Then

$$\mathscr{G}(\overline{T}) = \overline{\mathscr{G}(T)},$$

and so $\overline{T}$ is a closed operator, called the **closure of** T. Now, if $f \in \mathscr{D}(\overline{T})$ then $(f, \overline{T}f) \in \overline{\mathscr{G}(T)}$ and so is a sequence $(f_k, Tf_k) \in \mathscr{G}(T)$ that tends to $(f, \overline{T}f)$ in $H \times H$, and in particular $\overline{T}f = \lim_{k \to \infty} Tf_k$.

An operator T in H is called **essentially self-adjoint** if it is densely defined, symmetric, and there is a unique self-adjoint operator S in H such that $T \subset S$. This is equivalent with T being densely defined, symmetric, and its closure $\overline{T}$ being self-adjoint.[3] For a densely defined symmetric operator T, it is further proved that T is essentially self-adjoint if and only if $\mathscr{R}(T + i)$ is dense in H and $\mathscr{R}(T - i)$ is dense in H.[4]

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[2] `http://www.math.umn.edu/~garrett/m/v/friedrichs.pdf`; `https://people.math.ethz.ch/~kowalski/spectral-theory.pdf`

[3] Michael Reed and Barry Simon, *Methods of Modern Mathematical Physics, volume I: Functional Analysis*, revised and enlarged edition, p. 256, §VIII.2.

[4] Michael Reed and Barry Simon, *Methods of Modern Mathematical Physics, volume I: Functional Analysis*, revised and enlarged edition, p. 257, Corollary to Theorem VIII.3; `http://www.math.umn.edu/~garrett/m/fun/adjointness_crit.pdf`

2 Schwartz functions

Let $\mathscr{S} = \mathscr{S}(\mathbb{R}^d)$ be the Fréchet space of Schwartz functions $\mathbb{R}^d \to \mathbb{C}$ and let $\mathscr{D}(T) = \mathscr{S}$, which is a dense linear subspace of H. Define $T : \mathscr{D}(T) \to H$ for $\phi \in \mathscr{D}(T)$ by

$$(Tf)(x) = -\sum_{j=1}^{d} (\partial_j^2 f)(x), \qquad x \in \mathbb{R}^d.$$

T is a densely defined operator in H, and thus we may also speak about its adjoint T^*.

For $f, g \in \mathscr{D}(T)$, integrating by parts and because a Schwartz function tends to 0 at ∞,

$$\begin{aligned}\langle Tf, g \rangle &= \int_{\mathbb{R}^d} \left(-\sum_{j=1}^{d} \partial_j^2 f \right) \overline{g} \\ &= \sum_{j=1}^{d} \int_{\mathbb{R}^d} (\partial_j f) \overline{(\partial_j g)} \\ &= \sum_{j=1}^{d} - \int_{\mathbb{R}^d} f \overline{\partial_j^2 g} \\ &= \langle f, Tg \rangle ,\end{aligned}$$

which shows that T is symmetric. Because T is densely defined,

$$T \subset T^*.$$

For $f \in \mathscr{D}(T)$,

$$\langle Tf, f \rangle = \sum_{j=1}^{d} \int_{\mathbb{R}^d} (\partial_j f) \overline{(\partial_j f)} = \sum_{j=1}^{d} \int_{\mathbb{R}^d} |\partial_j f|^2 \geq 0,$$

showing that T is positive. Thus, the Friedrichs extension theorem tells us that there is a positive self-adjoint extension of T.

3 Fourier transform

For $f \in \mathscr{S}$ we define

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i \xi \cdot x} f(x) dx, \qquad \xi \in \mathbb{R}^d,$$

and $\hat{f} \in \mathscr{S}$. Then there is a unique **unitary operator** $\mathscr{F} : H \to H$ such that $\mathscr{F}(f) = \hat{f}$ for $f \in \mathscr{S}$.[5] That $\mathscr{F}$ is a unitary operator means that $\mathscr{F} \circ \mathscr{F}^* = I$ and $\mathscr{F}^* \circ \mathscr{F} = I$.

[5] John B. Conway, *A Course in Functional Analysis*, second ed., p. 341, Theorem 6.17.

For $f \in \mathscr{D}(T)$ and $\xi \in \mathbb{R}^d$, integrating by parts,

$$\begin{aligned}
(\mathscr{F}(Tf))(\xi) &= \sum_{j=1}^{d} -\int_{\mathbb{R}^d} e^{-2\pi i\xi\cdot x}(\partial_j^2 f)(x)dx \\
&= \sum_{j=1}^{d} \int_{\mathbb{R}^d} (2\pi i\xi_j)e^{-2\pi i\xi\cdot x}(\partial_j f)(x)dx \\
&= \sum_{j=1}^{d} -\int_{\mathbb{R}^d} (2\pi i\xi_j)^2 e^{-2\pi i\xi\cdot x} f(x)dx \\
&= \mathscr{F}(f)(\xi)\cdot \sum_{j=1}^{d} -(2\pi i\xi_j)^2,
\end{aligned}$$

so

$$(\mathscr{F}(Tf))(\xi) = \mathscr{F}(f)(\xi)\cdot |2\pi\xi|^2. \tag{2}$$

We define[6]

$$\mathscr{D}(A) = \left\{ f \in H : \int_{\mathbb{R}^d} |2\pi x|^4 |f(x)|^2 dx < \infty \right\}$$

and for $f \in \mathscr{D}(A)$ we define

$$(Af)(x) = |2\pi x|^2 f(x), \qquad x \in \mathbb{R}^d.$$

$A : \mathscr{D}(A) \to H$ is a linear map. It is apparent that $\mathscr{S} \subset \mathscr{D}(A)$, so A is densely defined. We have established above in (2) that

$$(\mathscr{F} \circ T)(f) = (A \circ \mathscr{F})(f), \qquad f \in \mathscr{D}(T),$$

which we can write as

$$T(f) = (\mathscr{F}^* \circ A \circ \mathscr{F})(f), \qquad f \in \mathscr{D}(T). \tag{3}$$

Theorem 1. *A is self-adjoint.*

Proof. For $f, g \in \mathscr{D}(A)$,

$$\langle Af, g\rangle = \int_{\mathbb{R}^d} (Af)(x)\overline{g(x)}dx = \int_{\mathbb{R}^d} |2\pi x|^2 f(x)\overline{g(x)}dx = \int_{\mathbb{R}^d} f(x)\overline{|2\pi x|^2 g(x)}dx,$$

so $\langle Af, g\rangle = \langle f, Ag\rangle$, namely A is symmetric: $A \subset A^*$.

Let $g \in \mathscr{D}(A^*)$. That is, $g \in H$ and there is some C_g such that

$$|\langle Af, g\rangle| \le C_g \|f\|, \qquad f \in \mathscr{D}(A).$$

Let B_r be the open ball of radius r with center 0, and define

$$f_r(x) = |2\pi x|^2 g(x) 1_{B_r}(x) 1_{B_r}(g(x)).$$

[6] Michael Loss, *The Laplace operator as a self adjoint operator*, `http://people.math.gatech.edu/~loss/14SPRINGTEA/laplacian.pdf`

Because f_r has compact support and is bounded, it belongs to H and further to $\mathscr{D}(A)$. On the one hand,

$$|\langle f_r, Ag\rangle| = |\langle Af_r, g\rangle| \leq C_g \|f_r\|.$$

On the other hand,

$$\begin{aligned}\langle f_r, Ag\rangle &= \int_{\mathbb{R}^d} |2\pi x|^2 g(x) 1_{B_r}(x) 1_{B_r}(g(x)) \overline{|2\pi x|^2 g(x)} dx \\ &= \int_{\mathbb{R}^d} |2\pi x|^4 |g(x)|^2 1_{B_r}(x) 1_{B_r}(g(x)) dx \\ &= \|f_r\|^2.\end{aligned}$$

Hence $\|f_r\| \leq C_g$. Now, for $x \in \mathbb{R}^d$,

$$|f_r(x)|^2 = |2\pi x|^4 |g(x)|^2 1_{B_r}(x) 1_{B_r}(g(x)) \to |2\pi x|^4 |g(x)|^2$$

as $r \to \infty$, and therefore applying the monotone convergence theorem gives

$$\|f_r\|^2 = \int_{\mathbb{R}^d} |f_r(x)|^2 dx \to \int_{\mathbb{R}^d} |2\pi x|^4 |g(x)|^2 dx.$$

Because $\|f_r\| \leq C_g$ for each r,

$$\int_{\mathbb{R}^d} |2\pi x|^4 |g(x)|^2 dx \leq C_g^2,$$

which shows that $g \in \mathscr{D}(A)$ and thus establishes that $A^* \subset A$. □

Let

$$\mathscr{D}(L) = \mathscr{F}^{-1}(\mathscr{D}(A)),$$

and because $\mathscr{F}$ is a unitary operator and $\mathscr{D}(A)$ is dense in H, $\mathscr{D}(L)$ is dense in H. Define

$$L(f) = (\mathscr{F}^* \circ A \circ \mathscr{F})(f), \qquad f \in \mathscr{D}(A),$$

which by (3) satisfies

$$L(f) = T(f), \qquad \mathscr{D}(T),$$

namely

$$T \subset L.$$

By Theorem 1, A is self-adjoint, and because $\mathscr{F} \in \mathscr{L}(H; H)$,[7]

$$((\mathscr{F}^* \circ A) \circ \mathscr{F}))^* = \mathscr{F}^* \circ (\mathscr{F}^* \circ A)^* = \mathscr{F}^* \circ A^* \circ \mathscr{F} = \mathscr{F}^* \circ A \circ \mathscr{F},$$

and thus L is self-adjoint. Because L is self-adjoint, L is closed,[8] and so by (1), with the inner product

$$\langle f, g\rangle_L = \langle f, g\rangle + \langle Lf, Lg\rangle, \qquad f, g \in \mathscr{D}(L),$$

[7] Walter Rudin, *Functional Analysis*, second ed., p. 348, Theorem 13.2.
[8] Walter Rudin, *Functional Analysis*, second ed., p. 352, Theorem 13.9.

$\mathscr{D}(L)$ is a Hilbert space. We remind ourselves that

$$\mathscr{D}(L) = \{f \in H : \mathscr{F}(f) \in \mathscr{D}(A)\} = \{f \in H : \int_{\mathbb{R}^d} |2\pi\xi|^4 |(\mathscr{F} f)(\xi)|^2 d\xi < \infty\}.$$

More explicitly, let $L^0(\mathbb{R}^d)$ be the collection of equvialence classes of Borel measurable functions $\mathbb{R}^d \to \mathbb{C}$, where two functions are deemed equivalent if the set of points on which they are not equal has Lebesgue measure 0. Then $H = L^2(\mathbb{R}^d)$ is equal to the set of those $f \in L^0(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d} |f|^2 < \infty.$$

Thus

$$\mathscr{D}(L) = \{f \in L^0(\mathbb{R}^d) : \int_{\mathbb{R}^d} |f|^2 < \infty \text{ and } \int_{\mathbb{R}^d} |2\pi\xi|^4 |(\mathscr{F} f)(\xi)|^2 d\xi < \infty\}.$$

We have established that L is a self-adjoint extension of T. We now prove that T is essentially self-adjoint, which means that L is the only self-adjoint extension of T and also that L is the closure of T. Moreover, because T is densely defined and positive its Friedrichs extension S is positive and self-adjoint, and if T is essentially self-adjoint then $L = S$, showing that L is positive. (This can be showed directly.) As we stated in §1, for T to be essentially self-adjoint it is equivalent that $\mathscr{R}(T + i)$ and $\mathscr{R}(T - i)$ each be dense in H.

Theorem 2. *T is essentially self-adjoint.*

Proof. If V is a linear subspace of H, then $V^{\perp\perp} = \overline{V}$, hence for V to be dense is equivalent to $V^\perp = \{0\}$.[9] Let $j \in \{-i, i\}$ and let $g \in \mathscr{R}(T + j)^\perp$, that is, $g \in H$ and

$$\langle (T + j)f, g\rangle = 0, \qquad f \in \mathscr{D}(T).$$

Then for any $f \in \mathscr{D}(T)$, because $\mathscr{F}$ is unitary and using (2),

$$\begin{aligned} 0 &= \left\langle \mathscr{F}^{-1} \circ \mathscr{F}((T + j)f), g\right\rangle \\ &= \langle \mathscr{F}(Tf) + j\mathscr{F} f, \mathscr{F} g\rangle \\ &= \left\langle (|2\pi \cdot|^2 + j)\mathscr{F} f, \mathscr{F} g\right\rangle \\ &= \left\langle \mathscr{F} f, (|2\pi \cdot|^2 - j)\mathscr{F} g\right\rangle. \end{aligned}$$

Because $\mathscr{D}(T)$ is dense in H and U is unitary, $U(\mathscr{D}(T))$ is dense in H, and therefore the above implies that

$$(|2\pi \cdot|^2 - j)\mathscr{F} g = 0.$$

Because $||2\pi \cdot|^2 - j| \geq 1$, this implies that

$$\mathscr{F} g = 0,$$

and $\mathscr{F}$ being unitary yields in particular that it is one-to-one, so $g = 0$. Therefore $\mathscr{R}(T + j)^\perp = \{0\}$ and so $\mathscr{R}(T + j)$ is dense in H, which implies that T is essentially self-adjoint. ☐

[9] See `http://individual.utoronto.ca/jordanbell/notes/pvm.pdf`

For people who work on partial differential equations, the games they play seem to value showing that solutions exist rather than showing that they are unique. But in fact, in any area of mathematics it is usually better to know that at most one thing exists than that at least one thing exists. If I know that a function can be extended from one set to a larger set in several ways, then I ought only to think about what those extensions have in common. Thus it is probably more important to know that the Laplace operator defined on the Schwartz functions is essentially self-adjoint than to have explicitly constructed a self-adjoint extension, since if there could be other self-adjoint extensions we would have no reason to care about our extension except where it is equal to the originally defined Laplace operator.

4 Spectral theorem

If T is an operator in H, the **resolvent set of** T is the set of those $\lambda \in \mathbb{C}$ such that there is some $S \in \mathscr{L}(H;H)$ satisfying

$$S \circ (T - \lambda I) \subset (T - \lambda I) \circ S = I.$$

The **spectrum of** T, denoted $\sigma(T)$, is the complement in $\mathbb{C}$ of the resolvent set of T.

If A is a self-adjoint operator in H, then the **spectral theorem**[10] says that there is a unique **resolution of the identity**[11] $E : \mathscr{B}_{\mathbb{R}} \to \mathscr{L}(H;H)$ such that

$$\langle Af, g\rangle = \int_{\mathbb{R}} \lambda dE_{f,g}(\lambda), \qquad f \in \mathscr{D}(A), \quad g \in H.$$

$E_{f,g}(B) = \langle E(B)f, g\rangle$ for $B \in \mathscr{B}_{\mathbb{R}}$, and for each $f \in \mathscr{D}(A)$ and $g \in H$, $E_{f,g}$ is a complex measure on $\mathscr{B}_{\mathbb{R}}$. Furthermore, $E(\sigma(A)) = I$.

Suppose that A is a self-adjoint operator in H. Let $\{D_i\}$ be a countable collection of open discs that are a base for the topology of $\mathbb{C}$. For a Borel measurable function $h : \mathbb{R} \to \mathbb{C}$, let V_h be the union of those D_i for which $Ehf^{-1}(D_i)) = 0$. Then $E(h^{-1}(V_f)) = 0$. We define the **essential range of** h to be the complement of V_h in $\mathbb{C}$,

$$R_h = V_h^c = \bigcap_i D_i^c.$$

For $\lambda \in \mathbb{R}$, if $h(\lambda) \notin R_h$ then there is some i for which $h(\lambda) \in D_i$, i.e. $\lambda \in h^{-1}(D_i) \subset h^{-1}(V_f)$. Therefore,

$$E(\{\lambda \in \mathbb{R} : h(\lambda) \notin R_h\}) = 0.$$

We say that h is **essentially bounded** if R_h is a bounded subset of $\mathbb{C}$, and we write

$$\|h\|_\infty = \sup_{\lambda \in R_h} |\lambda|,$$

[10] Walter Rudin, *Functional Analysis*, second ed., p. 368, Theorem 13.30.

[11] `http://individual.utoronto.ca/jordanbell/notes/trotter.pdf`, §5.

the **essential supremum of** h. Now, let B be the Banach algebra of bounded Borel measurable functions $\mathbb{R} \to \mathbb{C}$, with the supremum norm

$$\|h\| = \sup_{\lambda \in \mathbb{R}} |h(\lambda)|.$$

Any element of B is essentially bounded, and we take

$$N = \{h \in B : \|h\|_\infty = 0\}.$$

This is a closed ideal of B, and

$$L^\infty(E) = B/N = \{h + N : h \in B\}$$

is a Banach algebra, with norm

$$\|h + N\| = \inf_{g \in N} \|h - g\| = \|h\|_\infty,$$

which makes sense because for $h + N = g + N$, $\|h\|_\infty = \|g\|_\infty$. Because $L^\infty(E)$ is a Banach algebra, it makes sense to talk about the spectrum of an element of it, and for $h + N \in L^\infty(E)$, $\sigma(h + N)$ is equal to the essential range of h. There is an isometric *-isomorphism Ψ from $L^\infty(E)$ to a closed normal subalgebra of $\mathscr{L}(H; H)$ such that[12]

$$\langle \Psi(h)f, g\rangle = \int_{\mathbb{R}} h(\lambda) dE_{f,g}(\lambda), \qquad f, g \in H, \quad h \in L^\infty(E).$$

It is a fact that if A is a self-adjoint operator in H then A is positive if and only if $\sigma(A) \subset [0, \infty)$.[13] Thus for the the positive self-adjoint extension L of the Laplace operator that we have constructed, $\sigma(L) \subset [0, \infty)$. For $t \geq 0$, define $h_t : \mathbb{R} \to \mathbb{R}$ by $h_t(\lambda) = e^{-t\lambda}$ for $\lambda \geq 0$ and $h_t(\lambda) = 0$ otherwise. Then $h_t \in L^\infty(E)$, where E is the resolution of the identity corresponding to L. Thus $\Psi(h_t)$ satisfies

$$\langle \Psi(h_t)f, g\rangle = \int_{\mathbb{R}} e^{-t\lambda} dE_{f,g}(\lambda) = \int_{[0,\infty)} e^{-t\lambda} dE_{f,g}(\lambda), \qquad f, g \in H,$$

because $E(\sigma(L)) = I$, i.e. $E([0, \infty)) = I$, i.e. $E((-\infty, 0)) = 0$. Moreover,

$$\Psi(h_0) = I.$$

We write

$$e^{t\Delta} = e^{-tL} = \Psi(h_t), \qquad t \geq 0.$$

Because Ψ is an algebra homomorphism, $(e^{t\Delta})_{t \geq 0}$ is a semigroup on H.

[12] Walter Rudin, *Functional Analysis*, second ed., p. 319, Theorem 12.21.

[13] Walter Rudin, *Functional Analysis*, second ed., p. 369, Theorem 13.31.